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THE GROWTH AND 1/4-THEOREMS FOR STARLIKE MAPPING IN B^p†

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Abstract

The Growth and 1/4-Theorems for the class of normalized biholomorphic starlike mappings on $B^p=\{Z=(Z_1, \cdots, Z_n)\in C^n|\sum_{i=1}^n|Z_i|^p<1\}$ for any $p>1$ are given.

Let $B^p=\{Z=(Z_1, \cdots, Z_n)\in C^n|\sum_{i=1}^n|Z_i|^p<1\}$ for any $p>1$, which is a class of Reinhardt domains. When p=2, B^2 is the ball in C^n. In [1], Carl H. FitzGerald, Gong Sheng and Roger W. Barnard have established the Growth and 1/4-Theorems for normalized biholomorphic starlike mappings in C^n, which generalized some results of geometric function theory of one complex variable to several complex variables. The main purpose of this paper is to establish the corresponding results for B^n, $p>1$. A holomorphic mapping f from B^p to C^n is normalized if $f(0)=0$ and its complex Jacobian at origin is identity matrix, i. e $\left\{\frac{\partial f_i}{\partial Z_j}\right\}\Big|_{z=0}=\delta_{ij}$. The normalized holomorphic mapping is starlike in C^n means that it is a biholomorphic mapping with a starlike image $f(B^p)$ with respect to the origin in C^n. A necessary and sufficient condition for a normalized holomorphic mapping to be starlike was given by F. J. Suffridge[2]. Here we rewrite the Suffridge's theorem without proof in terms of differential geometry as the following:

Theorem 1. *Let f: $B^p\to C^n$ be a normalized holomorphic mapping. Then it is starlike if and only if*

$$<f^{-1*}du\ d\rho>|_{w(z)}\geq 0 \quad (\text{for any } Z\in B^p\backslash\{0\}),$$

where $\langle\,,\rangle$ is the inner product, w is coordinate of $f(Z)$ in C^n, ρ is the distance from the origin in C^n, $u=\sum_{i=1}^n|Z_i|^p$.

We can prove the following

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Theorem 2. *Let f: $B^p \to C^n$ be a normalized biholomarphic starlike mapping. Then*

$$\frac{2pu(1-u^{\frac{1}{p}})}{1+u^{\frac{1}{p}}} \leq \langle f^{-1*}du,\ d\rho^2 \rangle \leq \frac{2pu(1+u^{\frac{1}{p}})}{1-u^{\frac{1}{p}}},$$

where $u=\sum_{i=1}^{n} |Z_i|^p$, $\rho = dist\ (0,\ w(Z))$.

Proof By using the Suffridge's theorem and noting $\langle dw^i,\ d\bar{w}^j\rangle = 2\delta_{ij}$, we have

$$\begin{aligned} 0 \leq \langle f^{-1*}du,\ d\rho^2\rangle &= \langle f^{-1*}(\partial u+\bar{\partial} u),\ \partial\rho^2+\bar{\partial}\rho^2\rangle \\ &= \langle f^{-1*}\partial u,\ \bar{\partial}\rho^2\rangle + \langle f^{-1*}\bar{\partial} u,\ \partial\rho^2\rangle \\ &= 2\mathrm{Re}\langle f^{-1*}\partial u,\ \bar{\partial}\rho^2\rangle \\ &= 2p\mathrm{Re} \sum_{i,j=1}^{n} \frac{|Z_i|^p}{Z_i} \cdot \frac{\partial Z_i}{\partial w_j} \cdot w_j, \end{aligned} \tag{1}$$

$$i.\ e. \quad \mathrm{Re} \sum_{i,j=1}^{n} \frac{|Z_i|^p}{Z_i} \cdot \frac{\partial Z_i}{\partial w_j} \cdot w_j \geq 0. \tag{2}$$

Let t be a complex number with $|t|<1$ and, for any fixed $Z \in B^p$, $Z^0 = Z/[u(Z)]^{\frac{1}{p}}$. Then

$$\sum_{i}^{n} |tZ_i^0|^p = |t|^p < 1.$$

That means $tZ^0 \in B^p$. Thus tZ^0 can be instead of Z in (2), and the inequality (2) is

$$\mathrm{Re} \sum_{i,j=1}^{n} \frac{|tZ_i^0|^p}{tZ_i^0} \cdot \frac{\partial Z_i}{\partial w_j} \cdot w_j(tZ^0) \geq 0,$$

i. e.

$$\mathrm{Re} \sum_{i,j=1}^{n} \frac{|Z_i^0|^p}{Z_i^*} \cdot \frac{\partial Z_i}{\partial w_j} \cdot \frac{w_j}{t}(tZ^0) \geq 0. \tag{3}$$

Now we define a new function of one complex variable t.

$$A(t) = \sum_{i,j=1}^{n} \frac{|Z_i^0|^p}{tZ_i^0} \cdot \frac{\partial Z_i}{\partial w_j} \cdot w_j(tZ^0). \tag{4}$$

It is clear that $A(t)$ is holomorphic in the unit disc $D=\{t \in C^1 \mid |t| < 1\}$ when $t \neq 0$ and $\mathrm{Re}A(t) \geq 0$. Because of the normalized condition of f, i. e. $\left.\frac{\partial w_i}{\partial Z_k}\right|_{z=0} = \delta_{ik}$, we have

$$\lim_{t\to 0} \frac{w_j(tZ^0)}{t} = \sum_{k=1}^{n} \frac{\partial w_j}{\partial Z_k} \cdot \left.\frac{\partial Z_k}{\partial t}\right|_{t=0} = \sum_{k=1}^{u} \left.\frac{\partial w_j}{\partial Z_k}\right|_{t=0} Z_k^0 = Z_j^0,$$

which implies $A(t)$ is also holomorphic at $t=0$ and $A(0)=1$, i. e. $A(t)$ is normalized holomorphic in the unit disc. So an argument of one complex variable can be used to obtain

$$\frac{1-|t|}{1+|t|} \leq \mathrm{Re}\{A(t)\} \leq \frac{1+|t|}{1-|t|} \tag{5}$$

for any $t \in D$. In particular, when $t = u(Z)^{\frac{1}{p}}$,

$$\frac{1-u^{\frac{1}{p}}(Z)}{1+u^{\frac{1}{p}}(Z)} \leq \mathrm{Re} \sum_{i,j}^{n} \frac{|Z_i^0|^p}{Z_i} \cdot \frac{\partial Z_i}{\partial w_j} \cdot w_j(Z) \leq \frac{1+u^{\frac{1}{p}}(Z)}{1-u^{\frac{1}{p}}(Z)}, \tag{6}$$

i. e.

$$\frac{u(Z)(1-u^{\frac{1}{p}}(Z))}{1+u^{\frac{1}{p}}(Z)}\le \operatorname{Re}\sum_{i,j}^{n}\frac{|Z_i|^p}{Z_i}\cdot\frac{\partial Z_i}{\partial w_j}\cdot w_j(Z)\le\frac{u(Z)(1+u^{\frac{1}{p}}(Z))}{1-u^{\frac{1}{p}}(Z)}. \tag{7}$$

Substituting (1) to (7), we obtain

$$\frac{2pu(1-u^{\frac{1}{p}})}{1+u^{\frac{1}{p}}}\le\langle f^{-1*}du,\ d\rho^2\rangle\le\frac{2pu(1+u^{\frac{1}{p}})}{1-u^{\frac{1}{p}}}. \tag{8}$$

Theorem 3. *Let $p>1$ and f: $B^p\to C^n$ be a normalized biholomarphic starlike mapping. Then*

$$\frac{|Z|}{(1-u^{\frac{1}{p}})^2}\ge\rho\ge\frac{|Z|}{(1+u^{\frac{1}{p}})^2},$$

where $|Z|=\left(\sum_{i=1}^{n}|Z_i|^2\right)^{\frac{1}{2}}$, $u=\sum_{i=1}^{n}|Z_i|^p$, $\rho=dist(0,\ w(Z))$.

Proof By Theorem 2

$$2\rho\langle f^{-1*}du,\ d\rho\rangle\le\frac{2pu(1+u^{\frac{1}{p}})}{1-u^{\frac{1}{p}}}. \tag{9}$$

In fact,

$$\langle f^{-1*}dn,\ d\rho\rangle=\langle d(u\circ f^{-1}),\ d\rho\rangle=\frac{\partial u}{\partial\rho}$$

is the directional derivative of u along the direction $d\rho$. So (9) is

$$\frac{\partial u}{\partial\rho}\cdot\frac{1-u^{\frac{1}{p}}}{pu(1+u^{\frac{1}{p}})}\le\frac{1}{\rho}, \tag{10}$$

for any $Z\in B^p\setminus\{0\}$.

Fixed a $Z\in B^p\setminus\{0\}$ and a positive s, the segment joining the origin O and the point $Q=f(Z)$ in C^n will be denoted by L and the plane determined by the three points O, Q and $Q_1=f(sZ)$ in C^n will be denoted by Σ. When the positive number s is small enough the small circle $B(s)=\{W=(W_1,\ \cdots,\ W_n)\in C^n\,|\,|W|=|f(sZ)|\}\cap\Sigma$ must lies in the image $f(B^p)$ and intersects the segment L at S. Now let us constract a path l on the plane Σ. It comsists of the short arc on the circle $B(s)$ from Q_1 to S and the segment joining the intersectional point S and Q. By the difinition of l, ρ is increasing on the path l. Integrating (10) along the path l from Q_1 to S, then to Q, we get

$$\int_{Q_1}^{Q}\frac{d\rho}{\rho}\ge\frac{1}{\rho}\int_{sZ}^{Z}\frac{1-u^{\frac{1}{p}}}{u(1+u^{\frac{1}{p}})}du. \tag{11}$$

For the right hand side of (11),

$$\frac{1}{p}\int_{sZ}^{Z}\frac{1-u^{\frac{1}{p}}}{u(1+u)^{\frac{1}{p}}}du=[\ln u^{\frac{1}{p}}-2\ln(1+u^{\frac{1}{p}})]_{sZ}^{Z}.$$

Therefore

$$\ln\rho(Q)-\ln\rho(Q_1)\ge\ln\frac{u^{\frac{1}{p}}(Z)}{[1+u^{\frac{1}{p}}(Z)]^2}+\ln[1+u^{\frac{1}{p}}(sZ)]^2-\ln u^{\frac{1}{p}}(sZ).$$

Thus

$$\rho(Q)\geq\frac{u^{\frac{1}{p}}(Z)}{[1+u^{\frac{1}{p}}(Z)]^2}\cdot[1+u^{\frac{1}{p}}(sZ)]^2\cdot\frac{\rho(Q_1)}{u^{\frac{1}{p}}(sZ)}. \tag{12}$$

Note that $Q_1=Q(s)=f(sZ)$ and the normalized condition of f,

$$\rho(Q_1)=\left(\sum_{i=1}^{n}|w_i(sZ)|^2\right)^{\frac{1}{2}}=\left(\sum_{i=1}^{n}|sZ_i+O(s^2)|^2\right)^{\frac{1}{2}}$$

$$=\left(\sum_{i=1}^{n}s^2|Z_i|^2+O(s^3)\right)^{\frac{1}{2}}=s\left(\sum_{i=1}^{n}|Z_i|^2+O(s)\right)^{\frac{1}{2}}.$$

But

$$u^{\frac{1}{p}}(sZ)=\left[\sum_{i=1}^{n}|sZ_i|^p\right]^{\frac{1}{p}}=s\left[\sum_{i=1}^{n}|Z_i|^p\right]^{\frac{1}{p}}.$$

So

$$\frac{\rho(Q_1)}{u^{\frac{1}{p}}(sZ)}=\frac{\left(\sum_{i=1}^{n}|Z_i|^2\right)^{\frac{1}{2}}+O(s)}{\left(\sum_{i=1}^{n}|Z_i|^p\right)^{\frac{1}{p}}}.$$

Let $s\to0$; then (12) becomes

$$\rho(Q)\geq\frac{u^{\frac{1}{p}}(Z)}{(1+u^{\frac{1}{p}}(Z))^2}\cdot\frac{(Z\bar{Z})^{\frac{1}{2}}}{u^{\frac{1}{p}}(Z)}=\frac{|Z|}{(1+u^{\frac{1}{p}}(Z))^2}. \tag{13}$$

Using the left side inequality in Theorem 2 and the procedure for getting (13), we can prove the inequality of the left hand side of the Theorem 3.

For any $i=1, 2, \cdots\cdots, n$,

$$\frac{\partial}{\partial|Z_i|^p}(u-|Z|^p)=1-\left(\frac{|Z_i|}{|Z|}\right)^{2-p}$$

such that

$$\begin{cases}\dfrac{\partial}{\partial|Z_i|^p}(u-|Z|^p)\geqq0, \text{ for } 1<p<2,\\ \dfrac{\partial}{\partial|Z_i|^p}(u-|Z|^p)\leq0, \text{ for } p\geq2.\end{cases}$$

Hence

$$\begin{aligned}&u^{\frac{1}{p}}\geq|Z|, \textit{ when } 1<p<2,\\ &u^{\frac{1}{p}}\leq|Z|, \textit{ wpen } p\geq2.\end{aligned} \tag{14}$$

Using Hörder's inequality, we have

$$\begin{aligned}&|Z|\geq n^{\frac{p-2}{2p}}u^{\frac{1}{p}}, \textit{ when } 1<p<2,\\ &|Z|\leq n^{\frac{p-2}{2p}}u^{\frac{1}{p}}, \textit{ when } p\geq2.\end{aligned} \tag{15}$$

Substituting (14) (15) into Theorem 3 we have

Coroliary. *Suppose that f: $B^p\to C^n$ is a starlike and normalized mapping Then*

$$\frac{u^{\frac{1}{p}}}{(1-u^{\frac{1}{p}})^2} \geq \rho \geq \frac{p-2}{n} \cdot \frac{u^{\frac{1}{p}}}{(1+u^{\frac{1}{p}})^2}, \text{ for } 2>p>1,$$

$$n^{\frac{p-2}{2p}} \cdot \frac{u^{\frac{1}{p}}}{(1-u^{\frac{1}{p}})^2} \geq \rho \geq \frac{u^{\frac{1}{p}}}{(1+u^{\frac{1}{p}})^2}, \text{ for } p \geq 2.$$

From the corollary above we immediately obtain the Koebe-type 1/4 Theorem for the class of normalized biholomorphic starlike mappings on B^p in C^n for any $p>1$.

Theorem 4. *Let f: $B^p \to C^n$ is a normalized biholomorphic starlike mapping. Then the image of f contains a ball centered at the origin in C^n with radius $1/4K$, where*

$$K=\begin{cases} n^{1/p-1/2}, & \text{for } 1<p<2, \\ 1, & \text{for } p \geq 2. \end{cases}$$

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