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ON A CLASS OF SINGULAR INTEGRAL EQUATIONS WITH T ANSLATIONS

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(Dedicated to the Tenth Anniversary of CAM)

Abstract

In this paper, singular integral equations with upper and lower translations are discussed and the equivalence for solving them in $\hat{H}$ and $\hat{H}_0$ are shown. Equations with a single pair of upper and lower translations are studied in detail. Their solutions as well as the conditions of solvability are obtained. The method used consists of transforring them to boundary value problems of analytic functions in upper and lower half-planes and then solving the latter.

§1. Equations with Translations

In practical engineering, certain mechanical problems (for example, cf. [1, 2]), are reduced to solve the singular integral equations (SIE) of the following type:

$$a\varphi(t)+\frac{b}{\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(\tau)}{\tau-t}\,d\tau+\sum_{k,j=1}^{n}\frac{c_{kj}}{2\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(\tau)}{(\tau-t-\alpha_j)^k}\,d\tau$$

$$-\sum_{k,j=1}^{n}\frac{d_{kj}}{2\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(\tau)}{(\tau-t-\beta_j)^k}\,d\tau=f(t),\ -\infty<t<+\infty, \tag{1.1}$$

where a, b, c_{kj}, d_{kj}, α_j, β_j are given constants satisfying the condition of normal type

$$a^2-b^2\neq 0 \tag{1.2}$$

and

$$\operatorname{Im}\alpha_j>0,\ \operatorname{Im}\beta_j<0,\ j=1,\cdots,n; \tag{1.3}$$

function f belongs to Hölder class on whole real axis, denoted as $f\in\hat{H}$; φ should be solved in $\hat{H}$. α_j and β_j are called upper and lower translations respectively. Equation (1.1) is called an SIE with translations α_j and β_j in $\hat{H}$.

For briefness, $\int_{-\infty}^{+\infty}$ is abbrieviated to symbol $\int$.

We define a subclass of $\hat{H}$:

$$\hat{H}_0=\{f\,|\,f\in\hat{H},\ f(\infty)=0\}. \tag{1.4}$$

If $f\in\hat{H}_0$ and φ should be solved in $\hat{H}_0$ also, then (1.1) is said to be an SIE in

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$\hat{H}_0$. We show that the discussion of SIE in $\hat{H}$ may be reduced to that in $\hat{H}_0$. For this purpose, we give the following lemma.

Lemma 1. *If $f\in\hat{H}^u(0<u<1)$, $y=\operatorname{Im}z\neq 0$ and $p>1$, then*

$$\int\frac{|f(\tau)-f(\infty)|}{|\tau-z|^p}d\tau\leqslant\frac{M}{|z|^u|y|^{p-1}},\tag{1.5}$$

where M is a constant independent of z. In particular, for natural number n, we have

$$\left|\int\frac{f(\tau)}{(\tau-z^{n+1})}d\tau\right|\leqslant\frac{M}{|z|^u|y|^n}.\tag{1.5$'$}$$

Proof Note that

$$\int\frac{|f(\tau)-f(\infty)|}{|\tau-z|^p}d\tau\leqslant A\int\frac{d\tau}{|\tau|^u|\tau-z|^p},$$

while

$$\int_{|\tau|\geqslant|z|/2}\frac{d\tau}{|\tau|^u|\tau-z|^p}\leqslant\frac{2^u}{|z|^u}\int\frac{d\tau}{|\tau-z|^p}=\frac{2^u}{|z|^u}\int\frac{d\tau}{|\tau-iy|^p}$$
$$=\frac{2^u}{|z|^u|y|^{p-1}}\int\frac{d\tau}{(\tau^2+1)^{p/2}},$$

$$\int_{|\tau|\leqslant|z|/2}\frac{d\tau}{|\tau|^u|\tau-z|^p}\leqslant\frac{4}{|z|\,|y|^{p-1}}\int_0^{|z|/2}\frac{d\tau}{\tau^u}=\frac{2^{1+u}}{(1-u)|z|^u|y|^{p-1}};$$

hence (1.5) is proved. Thence (1.5)$'$ is evident since $\int\frac{d\tau}{(\tau-z)^{n+1}}=0$.

Now, letting $t\to\infty$ in (1.1), we obtain

$$r\varphi(\infty)=f(\infty)\tag{1.6}$$

by the above lemma and the results in [3], where

$$r=a+\frac{1}{2}\sum_{j=1}^{\infty}(c_{1j}-d_{1j}).\tag{1.7}$$

If we consider the equation

$$a\varphi_0(t)+\frac{b}{\pi i}\int\frac{\varphi_0(\tau)}{\tau-t}d\tau+\sum_{k,j=1}^{n}\frac{c_{kj}}{2\pi i}\int\frac{\varphi_0(\tau)}{(\tau-t-\alpha_j)^k}d\tau$$
$$-\sum_{k,j=1}^{n}\frac{d_{kj}}{2\pi i}\int\frac{\varphi_0(\tau)}{(\tau-t-\beta_j)^k}d\tau=f_0(t),\ -\infty<t<+\infty,\tag{1.8}$$

in $\hat{H}_0$, where $f_0(t)=f(t)-f(\infty)$, then the solutions of (1.1) and (1.8) are related as follows.

Theorem 1. (i) *If $r\neq 0$, then equation (1.1) in $\hat{H}$ and equation (1.8) in $\hat{H}_0$ are both solvable or not, and if they are solvable, their solutions are related as*

$$\varphi(t)=\varphi_0(t)+\frac{1}{r}f(\infty).$$

(ii) *If $r=0$, then $f(\infty)=0$ is a necessary condition for the solvability of (1.1) in $\hat{H}$; if this condition is fulfiled, then (1.1) in $\hat{H}$ and (1.8) in $\hat{H}_0$ are both solvable or not, and if they are solvable, their solutions are related as*

$$\varphi(t)=\varphi_0(t)+C,$$

where C is an arbitrary constant.

In the sequel we always assume (1.1) is given in $\hat{H}_0$, which would give rise to great convenience in discussion.

The methods of solving equation (1.1) in some simple types were studied in [4], but the results obtained there have some limitation. In this paper, we investigate the case in which only one pair of upper and lower translations appears in (1.1) and obtain complete results. The equation considered here may be written as

$$a\varphi(t)+\frac{b}{\pi i}\int\frac{\varphi(\tau)}{\tau-d}t\tau+\frac{c}{2\pi i}\int\frac{\varphi(\tau)}{\tau-t-\alpha}d\tau-\frac{d}{2\pi i}\int\frac{\varphi(\tau)}{\tau-t-\beta}d\tau$$
$$=f(t),\ \operatorname{Im}\alpha>0,\ \operatorname{Im}\beta>0,\ -\infty<t<+\infty. \tag{1.9}$$

§ 2. Reduction to Boundary Value Problems of Analytic Functions

Introduce notations

$$(S\varphi)(z)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int\frac{\varphi(\tau)}{\tau-z}d\tau, & \operatorname{Im}z\neq 0,\\[2mm] \dfrac{1}{\pi i}\displaystyle\int\frac{\varphi(\tau)}{\tau-z}d\tau, & \operatorname{Im}z=0\end{cases}$$

and projection operators

$$(S^{\pm}\varphi)(z)=\begin{cases}(S\varphi)(z), & \operatorname{Im}z\neq 0,\\ \pm\frac{1}{2}\varphi(z)+\frac{1}{2}(S\varphi)(z), & \operatorname{Im}z=0.\end{cases}$$

Then (1.9) may be rewritten as

$$(a+b)(S^+\varphi)(t)-(a-b)(S^-\varphi)(t)+c(S^+\varphi)(t+\alpha)-d(S^-\varphi)(t+\beta)$$
$$=(S^+f)(t)-(S^-f)(t),\ -\infty<t<+\infty. \tag{2.1}$$

Let

$$\Delta(z)=\begin{cases}(a+b)(S^+\varphi)(z)+c(S^+\varphi)(z+\alpha)-(S^+f)(z), & \operatorname{Im}z\geqslant 0,\\ (a-b)(S^-\varphi)(z)+d(S^-\varphi)(z+\beta)-(S^-f)(z), & \operatorname{Im}z\leqslant 0.\end{cases}$$

By (2.1), $\Delta(z)$ is an entire function with $\lim\limits_{z\to\infty}\Delta(z)=0$, [3] and hence $\Delta(z)\equiv 0$. Thus,

$$(a+b)(S^+\varphi)(z)+c(S^+\varphi)(z+\alpha)=(S^+f)(z),\ \operatorname{Im}z\geqslant 0, \tag{2.2}$$

$$(a-b)(S^-\varphi)(z)+d(S^-\varphi)(z+\beta)=(S^-f)(z),\ \operatorname{Im}z\leqslant 0. \tag{2.3}$$

Denote the class of functions analytic in $\operatorname{Im}z=y>0$ (<0) and $\in\hat{H}_0$ on $y\geqslant 0$ $(\leqslant 0)$ by $A^+\hat{H}_0$ $(A^-\hat{H}_0)$. Then $S^{\pm}f$ and $S^{\pm}\varphi\in A^{\pm}\hat{H}_0$. Obviously, to solve (1.9) in $\hat{H}_0$ is equivalent to solve the following problems in $A^+\hat{H}_0$ and $A^-\hat{H}_0$ respectively:

$$\Phi^+(z)=\lambda\Phi^+(z+\alpha)+\frac{1}{a+b}(S^+f)(z),\ y\geqslant 0, \tag{2.4}$$

$$\Phi^-(z)=\mu\Phi^-(z+\beta)+\frac{1}{a-b}(S^-f)(z),\ y\leqslant 0, \tag{2.5}$$

where

$$\lambda=-c/(a+b),\ \mu=-d/(a-b). \tag{2.6}$$

Precisely, we have the following theorem.

Theorem 2. *If φ is a solution of (1.9) in $\hat{H}_0$, then $S^+\varphi$ and $S^-\varphi$ are solutions of*

(2.4) in $A^+\hat{H}_0$ and (2.5) in $A^-\hat{H}_0$ respectively; conversely, if $\Phi^+(z)$ and $\Phi^-(z)$ are solutions of (2.4) in $A^+\hat{H}_0$ and (2.5) in $A^-\hat{H}_0$ respectively, then $\varphi(t)=\Phi^+(t)-\Phi^-(t)$ is a solution of (1.9) in $\hat{H}_0$.

Thus, we need only discuss the methods of solving (2.4) and (2.5). Introduce translation operator T_α:

$$(T_\alpha g)(z)=g(z+\alpha).$$

Then, (2.4) and (2.5) may be written in more compact form:

$$\Phi^+(z)=\lambda(T_\alpha\Phi^+)(z)+\frac{1}{a+b}(S^+f)(z),\ y\geqslant 0, \tag{2.7}$$

$$\Phi^-(z)=\mu(T_\beta\Phi^-)(z)+\frac{1}{a-b}(S^-f)(z),\ y\leqslant 0. \tag{2.8}$$

We would discuss the methods of solving (2.7) (in $A^+\hat{H}_0$) and (2.8) (in $A^-\hat{H}_0$) for different cases.

§ 3. The Case $|\lambda|<1$ or $|\mu|<1$

By (2.7), we have immediately by iteration

$$\Phi^+(z)=\frac{1}{a+b}\sum_{j=0}^{n}\lambda^j(T_\alpha^jS^+f)(z)+\lambda^{n+1}(T_\alpha^{n+1}\Phi^+)(z),\ y\geqslant 0. \tag{3.1}$$

Assume $|\lambda|<1$ in this section. Therefore, if (2.4) has a solution in $A^+\hat{H}_0$, then, letting $n\to\infty$ in (3.1), we have

$$\Phi^+(z)=\frac{1}{a+b}\sum_{j=0}^{\infty}\lambda^j(T_\alpha^jS^+f)(z),\ y\geqslant 0. \tag{3.2}$$

In fact, we may verify it is actually a solution of (2.4), for, by denoting $\|S^+f\|=\max\limits_{y\geqslant 0}|(S^+f)(z)|$, then

$$|\lambda^j(T_\alpha^jS^+f)(z)|\leqslant\|S^+f\|\,|\lambda|^j.$$

The series in (3.2) then converges uniformly (and absolutely) on $y\geqslant 0$ and hence we easily know $\Phi^+(z)\in A^+\hat{H}_0$.

Similarly, we may obtain the solution of (2.5) in $A^-\hat{H}_0$:

$$\Phi^-(z)=\frac{1}{a-b}\sum_{j=0}^{\infty}\mu^j(T_\beta^jS^-f)(z),\quad y\leqslant 0, \tag{3.3}$$

when $|\mu|<1$.

Thus, we obtain

Theorem 3. *When $|\lambda|<1$, problem (2.4) has unique solution (3.2) in $A^+\hat{H}_0$, when $|\mu|<1$, problem (2.5) has unique solution (3.3) in $A^-\hat{H}_0$.*

These results are the same with those in [4] in fact.

§ 4. The Case $|\lambda|<1$ or $|\mu|<1$

Now we assume $\lambda=e^{i\theta}$ $(0\leqslant\theta<2\pi)$. As above, we know that, if (2.4) has a solution, it must be

$$\Phi^+(z)=\frac{1}{a+b}\sum_{j=0}^{\infty}e^{ij\theta}(T_\alpha^jS^+f)(z),\quad y\geqslant 0,\tag{4.1}$$

and the series appeared in the right-hand member ought to

(A) converges uniformly on $y\geqslant 0$,

(B) belongs to $A^+\hat{H}_0$.

Conversely, when condition (B) is fulfiled, it is not hard to verify that (4.1) is a solution of (2.4). Thereby (B) implies (A). We call (B) the condition of solvability for (2.4).

This condition of solvability is essential when $\lambda=1$. In fact, there exists function $f(t)=\frac{1}{t+i}$ which does not satisfy this condition, since $\sum_{j=0}^{\infty}(T_\alpha^jS^+f)(0)=\sum_{j=0}^{\infty}\frac{1}{i+j\alpha}$ diverges.

Now assume $\lambda=e^{i\theta}$, $0<\theta<2\pi$. We may show that (B) is fulfiled automatically. Let us verify condition (A) at first. Denote $a_n=\sum_{j=1}^{n}e^{ij\theta}$, the n

$$|a_n|\leqslant 2/(1-\cos\theta),\quad n=1,\ 2,\ \cdots.\tag{4.2}$$

We have

$$\lim_{n\to\infty}(T_\alpha^nS^+f)(z)=0,\quad y\geqslant 0,\tag{4.3}$$

uniformly and

$$\begin{aligned}&\sum_{j=1}^{\infty}|(T_\alpha^{j+1}S^+f)(z)-(T_\alpha^jS^+f)(z)|\\&\leqslant\frac{|\alpha|}{2\pi}\sum_{j=1}^{\infty}\int\left|\frac{f(\tau)}{(\tau-z-j\alpha)^2}\right|\cdot\left|\frac{\tau-z-j\alpha}{\tau-z-(j+1)\alpha}\right|d\tau\\&\leqslant\frac{|\alpha|}{2\pi}\left(1+\frac{|\alpha|}{\operatorname{Im}\alpha}\right)\sum_{j=1}^{\infty}\int\left|\frac{f(\tau)}{(\tau-z-j\alpha)^2}\right|d\tau\\&\leqslant\frac{M|\alpha|}{2\pi}\left(1+\frac{|\alpha|}{\operatorname{Im}\alpha}\right)\sum_{j=1}^{\infty}\frac{1}{[\operatorname{Im}(z+j\alpha)]^{1+u}}\\&\leqslant\frac{M(|\alpha|+|\alpha|^2)}{2\pi(\operatorname{Im}\alpha)^{2+u}}\sum_{j=1}^{\infty}\frac{1}{j^{1+u}}\quad\text{(by Lemma 1)}.\end{aligned}\tag{4.4}$$

Noting (4.2)—(4.4), we see the series in (4.1) converges uniformly on $y\geqslant 0$ by Abel's transformation and

$$\Phi^+(z)=\frac{1}{a+b}(S^+f)(z)+\frac{\alpha}{a+b}\sum_{j=1}^{\infty}a_j\int\frac{f(\tau)d\tau}{[\tau-z-(j+1)\alpha][\tau-z-j\alpha]}.\tag{4.5}$$

To verify condition (B), we need only verify the series in $(4.1)\in\hat{H}$ on $y\geqslant 0$ since (A) is already fulfiled. For any z_1, z_2 ($\operatorname{Im}z_1\geqslant 0$, $\operatorname{Im}z_2\geqslant 0$), by Lemma 1,

$$\begin{aligned}&\left|\sum_{j=1}^{\infty}e^{ij\theta}(T_\alpha^jS^+f)(z_2)-\sum_{j=1}^{\infty}e^{ij\theta}(T_\alpha^jS^+f)(z_1)\right|\\&\leqslant\frac{1}{2\pi}\sum_{j=1}^{\infty}\left|\int f(\tau)\left(\frac{1}{\tau-z_1-j\alpha}-\frac{1}{\tau-z_2-j\alpha}\right)d\tau\right|\\&\leqslant\frac{|z_1-z_2|}{2\pi}\sum_{j=1}^{\infty}\left[\int\left|\frac{f(\tau)}{(\tau-z_2-j\alpha)^2}\right|d\tau\int\left|\frac{f(\tau)}{(\tau-z_1-j\alpha)^2}\right|d\tau\right]^{1/2}\end{aligned}$$

$$\leqslant M|z_1-z_2|\sum_{j=1}^{\infty}[\mathrm{Im}(z_2+j\alpha)\mathrm{Im}(z_1+j\alpha)]^{-(1+u)/2}$$
$$\leqslant\frac{M|z_1-z_2|}{(\mathrm{Im}\,\alpha)^{1+u}}\sum_{j=1}^{\infty}\frac{1}{j^{1+u}}. \tag{4.6}$$

To verify H condition at ∞ for $\Phi^+(z)$, we need only verify it for its boundary values; by noting (4.5), it is sufficient to verify it for function

$$\Phi(x)=\sum_{j=1}^{\infty}a_j\int\frac{f(\tau)d\tau}{[\tau-x-(j+1)\alpha][\tau-x-j\alpha]},\quad -\infty<x<+\infty.$$

Note that $f\in\hat{H}_0^u$ and we may assume $u<1/2$. Take $0<v<u$. Then,

$$|\Phi(x_1)-\Phi(x_2)|$$
$$\leqslant\frac{2A|x_1-x_2|^u}{1-\cos\theta}\sum_{j=1}^{\infty}\int\frac{d\tau}{|\tau+x_1|^u|\tau+x_2|^u|\tau-(j+1)\alpha||\tau-j\alpha|}$$
$$\leqslant\frac{2A|x_1-x_2|^u}{1-\cos\theta}\left(1+\frac{|\alpha|}{\mathrm{Im}\,\alpha}\right)\sum_{j=1}^{\infty}\int\frac{d\tau}{|\tau+x_1|^u|\tau+x_2|^u|\tau-j\alpha|^2}$$
$$\leqslant\frac{4A|\alpha||x_1-x_2|^v}{(1-\cos\alpha)\mathrm{Im}\,\alpha}\sum_{j=1}^{\infty}\left\{\int\frac{d\tau}{|\tau+x_1|^v|\tau+x_2|^u|\tau-j\alpha|^2}\right.$$
$$\left.+\int\frac{d\tau}{|\tau+x_1|^u|\tau+x_2|^v|\tau-j\alpha|^2}\right\}$$
$$\leqslant\frac{4A|\alpha||x_1-x_2|^v}{(1-\cos\theta)\mathrm{Im}\,\alpha}\sum_{j=1}^{\infty}\left\{\left(\int\frac{d\tau}{|\tau|^{2v}|\tau-x_1-j\alpha|^2}\int\frac{d\tau}{|\tau|^{2u}|\tau-x_2-j\alpha|^2}\right)^{1/2}\right.$$
$$\left.+\left(\int\frac{d\tau}{|\tau|^{2u}|\tau-x_1-j\alpha|^2}\int\frac{d\tau}{|\tau|^{2v}|\tau-x_2-j\alpha|^2}\right)^{1/2}\right\}$$
$$\leqslant\frac{4AM|\alpha||x_1-x_2|^v}{(1-\cos\theta)(\mathrm{Im}\,\alpha)^2}\sum_{j=1}^{\infty}\left(\frac{1}{|x_1+j\alpha|^v|x_2+j\alpha|^u}+\frac{1}{|x_1+j\alpha|^u|x_2+j\alpha|^v}\right)\frac{1}{j}$$
$$\leqslant\frac{8AM|\alpha|}{(1-\cos\theta)(\mathrm{Im}\,\alpha)^{2+u-v}}\left|\frac{1}{x_1}-\frac{1}{x_2}\right|^v\sum_{j=1}^{\infty}\left|\frac{x_1}{x_1+j\alpha}\right|^v\left|\frac{x_2}{x_2+j\alpha}\right|^v\frac{1}{j^{1+u-v}}$$
$$\leqslant\frac{16AM|\alpha|^2}{(1-\cos\theta)(\mathrm{Im}\,\alpha)^{2+u-v}}\left|\frac{1}{x_1}-\frac{1}{x_2}\right|^v\sum_{j=1}^{\infty}\frac{1}{j^{1+u-v}}.$$

It may be similarly discussed for $\mu=e^{i\theta}$, $0<\theta<2\pi$.

Thus, we obtain the following theorem.

Theorem 4. (i) *When $\lambda=e^{i\theta}$, $0<\theta<2\pi$, problem* (2.4) *in $A^+\hat{H}_0$ has unique solution* (4.1); *when $\lambda=1$, its condition of solvability is*

$$\frac{1}{a+b}\sum_{j=0}^{\infty}(T_\alpha^jS^+f)(z)\in A^+\hat{H}_0 \tag{4.7}$$

and the left-hand member is its unique solution when this condition is fulfiled. (ii) *When $\mu=e^{i\theta}$, $0<\theta<2\pi$, problem* (2.5) *in $A^-\hat{H}_0$ has unique solution*

$$\Phi^-(z)=\frac{1}{a-b}\sum_{j=0}^{\infty}e^{ij\theta}(T_\beta^jS^-f)(z),\quad y\leqslant 0; \tag{4.8}$$

when $\mu=1$, its condition of solvability is

$$\frac{1}{a-b}\sum_{j=0}^{\infty}(T_\beta^jS^-f)(z)\in A^-\hat{H}_0, \tag{4.9}$$

and the left-hand member is its unique solution if this condition is fulfiled.

Remark 1. By (4.6), we find that, if series (4.7) converges at certain point on $y \geqslant 0$, then it also converges for any $y \geqslant 0$.

Remark 2. By Privalov's theorem[5], we know that, to verify series in (4.7) $\in A^+\hat{H}_0$, we need only verify it takes value 0 at ∞ and its boundary values $\in \hat{H}$.

§ 5. The Case $|\lambda|>1$ or $|\mu|>1$

Now we discuss the case $|\lambda|>1$. We have

Lemma 2. *If $f(t)\in\hat{H}_0$ and σ is real, then $e^{i\sigma t}f(t)\in\hat{H}_0$.*

Proof It is sufficient to prove $e^{i\sigma t}f(t)\in H$ in the neighborhood of ∞. For any t_1, t_2 $(-\infty<t_1,\ t_2<+\infty,\ |t_1|\geqslant|t_2|)$,

$$\begin{aligned}\Delta &= |e^{i\sigma t_2}f(t_2)-e^{i\sigma t_1}f(t_1)|\\ &\leqslant |f(t_2)-f(t_1)|+|f(t_2)|\,|e^{i\sigma t_2}-e^{i\sigma t_1}|.\end{aligned}$$

When $|t_2-t_1|\geqslant 1$, then

$$\begin{aligned}\Delta &\leqslant |f(t_2)-f(t_1)|+A|t_2|^{-u}\\ &\leqslant |f(t_2)-f(t_1)|+A|t_1t_2|^{-u/2}|t_2-t_1|^{u/2}\\ &= |f(t_2)-f(t_1)|+A\left|\frac{1}{t_2}-\frac{1}{t_1}\right|^{u/2};\end{aligned}$$

when $|t_2-t_1|<1$, then

$$\begin{aligned}\Delta &\leqslant |f(t_2)-f(t_1)|+A|\sigma|\,|t_1t_2|^{-u/2}|t_2-t_1|\\ &\leqslant |f(t_2)-f(t_1)|+A|\sigma|\left|\frac{1}{t_2}-\frac{1}{t_1}\right|^{u/2}.\end{aligned}$$

The lemma is proved.

For $f(t)\in\hat{H}_0$, we introduce the following operators (σ is a real parameter):

$$(L_\sigma f)(z)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int\frac{e^{-i\sigma\tau}f(\tau)}{\tau-z}\,d\tau, & y\neq 0,\\[2ex] \dfrac{1}{\pi i}\displaystyle\int\frac{e^{-i\sigma\tau}f(\tau)}{\tau-z}\,d\tau, & y=0;\end{cases}\tag{5.1}$$

$$(L_\sigma^{\pm}f)(z)=\begin{cases}(L_\sigma f)(z), & y\neq 0,\\ \pm\dfrac{1}{2}e^{-i\sigma z}f(z)+\dfrac{1}{2}(L_\sigma f)(z), & y=0.\end{cases}\tag{5.2}$$

The following simple fact would be used frequently.

Lemma 3. (i) *If $F^+(z)\in A^+\hat{H}_0$, then $(L_\sigma^-F^+)(z)$ may be extended to an entire function with*

$$\lim_{\substack{z\to 0\\ y\leqslant 0}}(L_\sigma^-F^+)(z)=0,\tag{5.3}$$

$$\lim_{\substack{z\to 0\\ y\geqslant 0}}[(L_\sigma^-F^+)(z)+e^{-i\sigma z}F^+(z)]=0.\tag{5.4}$$

(ii) *If $F^-(z)\in A^-\hat{H}_0$, then $(L_\sigma^+F^-)(z)$ may be extended to an entire function with*

$$\lim_{\substack{z\to 0\\ y\geqslant 0}}(L_\sigma^+F^-)(z)=0,\tag{5.5}$$

$$\lim_{\substack{z\to 0\\ y\leq 0}}[(L_\sigma^+F^-)(z)-e^{-i\sigma z}F^-(z)]=0. \tag{5.6}$$

Proof (i) By Plemelj's formulas

$$(L_\sigma^+F^+)(t)-(L_\sigma^-F^+)(t)=e^{-i\sigma t}F^+(t).$$

By this, on putting

$$\Psi(z)=\begin{cases}(L_\sigma^+F^+)(z)-e^{-i\sigma z}F^+(z), & y\geq 0, \quad (5.7)\\ (L_\sigma^-F^+)(z), & y\leq 0, \quad (5.8)\end{cases}$$

we know that $\Psi(z)$ is an entire function. Obviously, we have (5.3) and (5.4).

(ii) may be similarly proved.

For simplicity, symbol $(L^-F^+)(z)$ in (i) will denote the mentioned entire function, but it should be reminded that its expression is given by (5.2) when $y\geq 0$ and by (5.7) when $y\leq 0$. Symbol $(L^+F^-)(z)$ in (ii) is similar.

Lemma 4. *If $f\in\hat{H}_0$, then*

(i) *when $\sigma\leq 0$,*

$$(L_\sigma^-S^+f)(z)=0 \quad (\textit{for any } z), \tag{5.9}$$

$$(L_\sigma^+S^+f)(z)=e^{-i\sigma z}(S^+f)(z), \quad y\geq 0, \tag{5.10}$$

$$(L_\sigma^+S^-f)(z)=e^{-i\sigma z}(S^+f)(z)-(L_\sigma^+f)(z), \quad y\geq 0, \tag{5.11}$$

$$(L_\sigma^-S^-f)(z)=-(L_\sigma^-f)(z), \quad y\leq 0; \tag{5.12}$$

(ii) *when $\sigma\geq 0$,*

$$(L_\sigma^+S^-f)(z)=0 \quad (\textit{for any } z), \tag{5.13}$$

$$(L_\sigma^-S^-f)(z)=-e^{-i\sigma z}(S^-f)(z), \quad y\leq 0, \tag{5.14}$$

$$(L_\sigma^-S^+f)(z)=(L_\sigma^-f)(z)-e^{-i\sigma z}(S^-f)(z), \quad y\leq 0, \tag{5.15}$$

$$(L_\sigma^+S^+f)(z)=(L_\sigma^+f)(z), \quad y\geq 0. \tag{5.16}$$

Proof $(L_\sigma^-S^+f)(z)$ is entire by Lemma 3 and

$$\lim_{z\to\infty}(L_\sigma^-S^+f)(z)=0, \quad \sigma\leq 0,$$

by (5.3) and (5.4). Hence (5.9) is valid.

Noting the expression (5.7) for the entire function $(L_\sigma^-S^+f)(z)$ on $y\geq 0$, we obtain (5.10) by (5.9). Using them, we have (5.11) and (5.12) respectively.

(ii) may be proved analogously.

Lemma 5. (i) *If $F^+(z)\in A^+\hat{H}_0$ and* $\mathrm{Im}\,\alpha>0$, *then*

$$(L_\sigma^-T_\alpha F^+)(z)=e^{i\sigma\alpha}(T_\alpha L_\sigma^-F^+)(z) \quad (\textit{for any } z), \tag{5.17}$$

$$(L_\sigma^+T_\alpha F^+)(z)=e^{i\sigma\alpha}(T_\alpha L_\sigma^+F^+)(z), \quad y\geq 0. \tag{5.18}$$

(ii) *If $F^-(z)\in A^-\hat{H}_0$ and* $\mathrm{Im}\,\beta<0$, *then*

$$(L_\sigma^+T_\beta F^-)(z)=e^{i\sigma\beta}(T_\beta L_\sigma^+F^-)(z) \quad (\textit{for any } z), \tag{5.19}$$

$$(L_\sigma^-T_\beta F^-)(z)=e^{i\sigma\beta}(T_\beta L_\sigma^-F^-)(z), \quad y\leq 0. \tag{5.20}$$

Proof Noting the expression (5.7) of $L_\sigma^-F^+(z)$ on $y\geq 0$, we get

$$(T_\alpha L_\sigma^-F^+)(z)=(T_\alpha L_\sigma^+F^+)(z)-e^{-i\sigma(z+\alpha)}(T_\alpha F^+)(z), \quad y\geq 0,$$

and noting the expression of $(L_\sigma^-T_\alpha F^+)(z)$ on $y\geq 0$, we get

$$(L_\sigma^+ T_\alpha F^+)(z) - (L_\sigma^- T_\alpha F^+)(z) = e^{-i\sigma z}(T_\alpha F^+)(z),\ y \geqslant 0.$$

Substituting the latter into the former, we obtain

$$(T_\alpha L_\sigma^- F^+)(z) - e^{-i\sigma\alpha}(L_\sigma^- T_\alpha F^+)(z)$$
$$= (T_\alpha L_\sigma^+ F^+)(z) - e^{-i\sigma\alpha}(L_\sigma^+ T_\alpha F^+)(z),\ \ y \geqslant 0.$$

Noting that the left-hand member is an entire function and considering the properties of both sides at ∞, we know (5.17) and (5.18) are valid.

(ii) may be similarly proved.

Let

$$\xi = \ln|\lambda|/\operatorname{Im}\alpha \tag{5.21}$$

and denote $\lambda e^{i\xi\alpha} = e^{i\theta}(0 \leqslant \theta < 2\pi)$. Applying operators L_ξ^+ and L_ξ^- to (2.7), we get, by Lemma 5

$$(L_\xi^+ \Phi^+)(z) = e^{i\theta}(T_\alpha L_\xi^+ \Phi^+)(z) + \frac{1}{a+b}(L_\xi^+ S^+ f)(z), \quad y \geqslant 0, \tag{5.22}$$

$$(L_\xi^- \Phi^+)(z) = e^{i\theta}(T_\alpha L_\xi^- \Phi^+)(z) + \frac{1}{a+b}(L_\xi^- S^+ f)(z), \quad \text{for any } z. \tag{5.23}$$

Denote $w^+(z) = (L_\xi^+ \Phi^+)(z)$, then (5.22) becomes the following problem in $A^+\hat{H}_0$:

$$w^+(z) = e^{i\theta}(T_\alpha w^+)(z) + \frac{1}{a+b}(L_\xi^+ S^+ f)(z), \quad y \geqslant 0. \tag{5.24}$$

$(L_\xi^- \Phi^+)(z)$ is an entire function by Lemma 3 and $\in A^-\hat{H}_0$, $e^{i\xi z}(L_\xi^- \Phi^+)(z) \in A^+\hat{H}_0$. Therefore, on putting $w^-(z) = (L_\xi^- \Phi^+)(z)$, (5.23) becomes the following problem of entire function:

$$\begin{cases} w^-(z) = e^{i\theta}(T_\alpha w^-)(z) + \dfrac{1}{a+b}(L_\xi^- S^+ f)(z) \quad (\text{for any } z), & (5.25) \\ w^-(z) \in A^-\hat{H}_0, & (5.26) \\ e^{i\xi z} w^-(z) \in A^+\hat{H}_0, & (5.27) \end{cases}$$

which will be called a problem in $A_*\hat{H}_0$.

The following lemma is obvious.

Lemma 6. *If $\Phi^+(z)$ is the solution of problem (2.7) in $A^+\hat{H}_0$, then $(L_\xi^+\Phi^+)(z)$ is the solution of problem (5.24) in $A^+{}_0\hat{H}$ and $(L_\xi^-\Phi^+)(z)$ is the solution of problem (5.25) in $A_*\hat{H}_0$; conversely, if $w^+(z)$ is the solution of problem (5.24) in $A^+\hat{H}_0$ and $w^-(z)$ is the solution of prcblem (5.24) in $A_*\hat{H}_0$, then $\Phi^+(z) = e^{i\xi z}[w^+(z) - w^-(z)]$ is the solution of problem (2.7) in $A^+\hat{H}_0$.*

Problem (5.24) in $A^+\hat{H}_0$ has been already solved in § 4. But we should note that problem (5.25) is not a problem in $A^+\hat{H}_0$ in general, because the given function $(L_\xi^- S^+ f)(z)$ and solution $w^-(z) \overline{\in} A^+\hat{H}_0$ in general (otherwise, they are identical to zero). However, by noting (5.26) and applying operator $T_{-\alpha}$ to (5.25), the problem is transferred to the following problem in $A^-\hat{H}_0$:

$$w^-(z) = e^{-i\theta}(T_{-\alpha} w^-)(z) - \frac{e^{-i\theta}}{a+b}(T_{-\alpha} L_\xi^- S^+ f)(z), \quad y \leqslant 0. \tag{5.28}$$

No doubt, the solution of (5.25) in $A_*\hat{H}_0$ is necessarily the solution of (5.28) in

$A^-\hat{H}_0$. We prove that the converse is also true.

First, if $w^-(z)$ is the solution of (5.28) in $A^-\hat{H}_0$, then (5.26) is valid evidently.

Secondly, by noting that the function $(T_{-\alpha}w^-)(z)$ in the right-hand member of (5.28) is analytic in $y<\operatorname{Im}\alpha$ and continuous on $y\leqslant\operatorname{Im}\alpha$ ($\in\hat{H}_0$ in fact) while $(T_{-\alpha}L_\xi^- S^+f)(z)$ is an entire function, it is easy to extend $w^-(z)$ to an entire function by (5.28) (retaining the symbol w^-), then (5.28) is valid in the whole plane, i. e., (5.25) is fulfiled.

We call such a function $w^-(z)$ satisfying (5.25) and (5.26) an upper pseudo-solution of (5.25) in $A_*\hat{H}_0$. Evidently, it possesses the following property:

$$\lim_{z\to\infty} w^-(z)=0,\quad 0\leqslant y\leqslant \operatorname{Im}\alpha. \tag{5.29}$$

Thus, we have

Lemma 7. *The solution $w^-(z)$ of (5.27) in $A^-\hat{H}_0$ (after extension) is an upper pseudo-solution of (5.25) in $A_*\hat{H}_0$.*

Note that the upper pseudo-solution could not be thought as the solution of (5.25) in $A_*\hat{H}_0$ yet since we have abandoned the condition (5.27).

Analogously, we call an entire function satisfying (5.25) and (5.27) a lower pseudo-solution of problem (5.25) in A_*a_0.

It is very evident that an entire function being both an upper and a lower pseudo-solution of (5.25) in $A_*\hat{H}_0$ is the solution in A_*H_0.

Lemma 8. *If $w^-(z)$ is the solution of problem (5.28) in $A^-\hat{H}_0$, then $w^*(z)=e^{-i\xi z}(L^\pm_{-\xi}w^-)(z)$ is a lower pseudo-solution of problem (5.25) in $A_*\hat{H}_0$.*

Proof By Lemma 3, $w^*(z)$ is an entire function and evidently

$$e^{i\xi z}w^*(z)=(L^\pm_{-\xi}w^-)(z)\in A^+\hat{H}_0,$$

i. e., $w^*(z)$ satisfies (5.27).

We may prove

$$\begin{aligned}(T_\alpha L^\pm_{-\xi}w^-)(z)&=e^{-i\theta}e^{i\xi\alpha}(L^\pm_{-\xi}w^-)(z)\\ &\quad-\frac{e^{-i\theta}}{a+b}e^{-i\xi(z+\alpha)}(L_\xi^- S^+f)(z)\quad(\text{for any } z).\end{aligned}\tag{5.30}$$

For this purpose, note that application of $T_\alpha L^\pm_{-\xi}$ to the right-hand terms of (5.28) would give respectively

$$\begin{aligned}(T_\alpha L^\pm_{-\xi}T_{-\alpha}w^-)(z)&=e^{i\xi\alpha}(L^\pm_{-\xi}w^-)(z)\quad(\text{for any } z),\\ (T_\alpha L^\pm_{-\xi}T_{-\alpha}L_\xi^- S^+f)(z)&=e^{i\xi\alpha}(L^\pm_{-\xi}L_\xi^- S^+f)(z)\quad(\text{for any } z).\end{aligned}\tag{5.31}$$

In particular, by (5.11) and (5.7), for $y\geqslant0$,

$$\begin{aligned}(T_\alpha L^\pm_{-\xi}T_{-\alpha}L_\xi^- S^+f)(z)&=e^{i\xi\alpha}[e^{i\xi z}(L_\xi^+S^+f)(z)-(S^+f)(z)]\\ &=e^{i\xi(z+\alpha)}(L_\xi^- S^+f)(z);\end{aligned}\tag{5.32}$$

but the functions in both sides are entire and so it is valid in the whole plane. Combining (5.31) and (5.32), we get (5.30), by which we have

$$w^*(z)=e^{-i\theta}(T_{-\alpha}w^*)(z)-\frac{e^{-i\theta}}{a+b}(T_{-\alpha}L_\xi^- S^+f)(z) \quad (\text{for any } z). \tag{5.33}$$

This means $w^*(z)$ satisfies (5.25) and the lemma is proved.

By the way, we mention that

$$\lim_{z\to\infty} w^*(z)=\lim_{z\to 0} e^{-i\xi z}(L_{-\xi}^{+}w^-)(z)=0, \quad 0\leqslant y\leqslant \operatorname{Im}\alpha. \tag{5.34}$$

Lemma 9. *The solution $w^-(z)$ of (5.28) in $A^-\hat{H}_0$ is identical to $w^*(z)$.*

Proof Denote $\delta(z)=w^-(z)-w^*(z)$, which is entire as proved before. We have

$$\delta(z)=e^{i\theta}(T_\alpha\delta)(z) \quad (\text{for any } z), \tag{5.35}$$

$$\lim_{z\to\infty}\delta(z)=0, \quad 0\leqslant y\leqslant \operatorname{Im}\alpha. \tag{5.36}$$

They follow from (5.28) (valid in the whole plane!), (5.33) and (5.29), (5.34) respectively. Hence $\delta(z)\equiv 0$.

By Lemmas 7, 8, 9, we know that the solution of (5.28) in $A^-\hat{H}_0$ is also the solution of (5.25) in $A_*\hat{H}_0$. Therefore, solving these two problems are equivalent. Thus, Lemma 6 may be restated as follows.

Lemma 10. *If $\Phi^+(z)$ is the solution of (2.7) in $A^+\hat{H}_0$, then $(L_\xi^+\Phi^+)(z)$ is the solution of (5.24) in $A^+\hat{H}_0$ and $(L_\xi^-\Phi^+)(z)$ is that of (5.28) in $A^-\hat{H}_0$; conversely, if $w^+(z)$ is the solution of (5.24) in $A^+\hat{H}_0$ and $w^-(z)$ is that of (5.28) in $A^-\hat{H}_0$, then*

$$\Phi(z)=e^{i\xi z}[w^+(z)-w^-(z)] \tag{5.37}$$

is the solution of (2.7) in $A^+\hat{H}_0$.

Remark 3. By (5.10) we know that $(L_{-\xi}^{+}w^+)(z)=e^{i\xi z}w^+(z)$. As we have shown $(L_{-\xi}^{+}w^-)(z)=e^{i\xi z}w^-(z)$, we may write (5.37) in another form:

$$\Phi(z)=(L_{-\xi}^{+}w^+)(z)-(L_{-\xi}^{+}w^-)(z). \tag{5.38}$$

Since problems (5.24) in $A^+\hat{H}_0$ and (5.28) in $A^-\hat{H}_0$ have been solved in § 4, thereby we have

Theorem 5. *If $|\lambda|>1$ and denote $\lambda=e^{-i\xi\alpha}e^{i\theta}(0\leqslant\theta<2\pi)$ where ξ is given by (5.21), then*

(i) *when $\theta\neq 0$, problem (2.7) has a unique solution in $A^+\hat{H}_0$ with boundary value*

$$\begin{aligned}\Phi^+(t)=&\frac{f(t)}{2(a+b)}+\frac{e^{i\xi t}}{a+b}\sum_{j=0}^{\infty}\frac{e^{ij\theta}}{2\pi i}\int\frac{e^{-i\xi\tau}f(\tau)}{\tau-t-j\alpha}d\tau\\&+\frac{e^{i\xi t}}{a+b}\sum_{j=1}^{\infty}\frac{e^{-ij\theta}}{2\pi i}\int\frac{e^{-i\xi\tau}f(\tau)}{\tau-t+j\alpha}d\tau\\&-\frac{1}{a-b}\sum_{j=1}^{\infty}\frac{1}{\lambda^j 2\pi i}\int\frac{f(\tau)d\tau}{\tau-t+j\alpha};\end{aligned} \tag{5.39}$$

(ii) *when $\theta=0$, then the conditions of solvability of the problem (2.7) in $A^+\hat{H}_0$ are*

$$\sum_{j=1}^{\infty}\frac{1}{2\pi i}\int\frac{e^{-i\xi\tau}f(\tau)}{\tau-z-j\alpha}d\tau\in A^+\hat{H}_0, \tag{5.40}$$

$$\sum_{j=1}^{\infty}\frac{1}{2\pi i}\int\frac{e^{-i\xi\tau}f(\tau)}{\tau-z+j\alpha}d\tau\in A^-\hat{H}_0, \tag{5.41}$$

and it has a unique solution with boundary value (5.39) *if they are satisfied.*

Proof (i) When $\theta\neq 0$, we need only calculate $w^{\pm}(z)$ by Lemma 10. Since $w^{+}(z)$ is the solution of (5.24) in $A^{+}\hat{H}_0$, using the results in § 4, we have

$$w^{+}(z)=\frac{1}{a+b}\sum_{j=0}^{\infty}e^{ij\theta}(T_{\alpha}^{j}L_{\xi}^{+}f)(z)$$

$$=\frac{1}{a+b}(L_{\xi}^{+}f)(z)+\frac{1}{a+b}\sum_{j=0}^{\infty}\frac{e^{ij\theta}}{2\pi i}\int\frac{e^{-i\xi\tau}f(\tau)}{\tau-z-j\alpha}d\tau,\ y\geqslant 0. \tag{5.42}$$

Similarly

$$w^{-}(z)=-\frac{1}{a+b}\sum_{j=1}^{\infty}e^{-ij\theta}(T_{-\alpha}^{j}L_{\xi}^{-}S^{+}f)(z)$$

$$=-\frac{1}{a+b}\sum_{j=1}^{\infty}\frac{e^{-ij\theta}}{2\pi i}\int\frac{e^{-i\xi\tau}f(\tau)}{\tau-z+j\alpha}d\tau$$

$$+\frac{e^{-i\xi z}}{a+b}\sum_{j=1}^{\infty}\frac{1}{\lambda^{j}2\pi i}\int\frac{f(\tau)d\tau}{\tau-z+j\alpha},\ y\leqslant 0. \tag{5.43}$$

By (5.42) and (5.43), we obtain (5.39).

(ii) When $\theta=0$, by the results in § 4, the condition of solvability of (5.24) in $A^{+}\hat{H}_0$ is that the series of the last term in (5.42) $\in A^{+}\hat{H}_0$, i. e., the condition (5.40). Similarly, that of (5.28) in $A^{-}\hat{H}_0$ is that the series of the last term in (5.43) $\in A^{-}\hat{H}_0$, which is actually true by results in § 3. Hence, for requiring $w^{-}(z)$ in (5.43) $\in A^{-}\hat{H}_0$, we need only require (5.41).

As an illustration, we take $f(t)=e^{i\xi t}/(t+i)$. In this case, (5.40) is not fulfiled while (5.41) is fulfiled. So (5.40) and (5.41) is independent to each other.

For problem (2.8) in $A^{-}\hat{H}_0$, we have analogous results:

Theorem 6. *If* $|\mu|>1$ *and denote* $\mu=e^{-in\beta}e^{i\theta}(0\leqslant\theta<2\pi)$, *where*

$$\eta=\ln|\mu|/\operatorname{Im}\beta, \tag{5.44}$$

then,

(i) *when* $\theta\neq 0$, (2.8) *has unique solution* (2.6) *with boundary value*

$$\Phi^{-}(t)=-\frac{f(t)}{2(a-b)}+\frac{e^{i\eta t}}{a-b}\sum_{j=1}^{\infty}\frac{e^{ij\theta}}{2\pi i}\int\frac{e^{-i\eta\tau}f(\tau)}{\tau-t-j\beta}d\tau$$

$$+\frac{e^{i\eta t}}{a-b}\sum_{j=1}^{\infty}\frac{e^{-ij\theta}}{2\pi i}\int\frac{e^{-i\eta\tau}f(\tau)}{\tau-t+j\beta}d\tau$$

$$-\frac{1}{a-b}\sum_{j=1}^{\infty}\frac{1}{\mu^{j}2\pi i}\int\frac{f(\tau)d\tau}{\tau-t+j\beta}; \tag{5.45}$$

(ii) *when* $\theta=0$, *the conditions of solvability for* (2.8) *are*

$$\sum_{j=1}^{\infty}\frac{1}{2\pi i}\int\frac{e^{-i\eta\tau}f(\tau)}{\tau-z-j\beta}d\tau\in A^{-}\hat{H}_0, \tag{5.46}$$

$$\sum_{j=1}^{\infty}\frac{1}{2\pi i}\int\frac{e^{-i\eta\tau}f(\tau)}{\tau-z+j\beta}d\tau\in A^{+}\hat{H}_0, \tag{5.47}$$

and it has a unique solution still with boundary value (5.45) if they are fulfiled.

§6. The Solution of Equation (1.9)

We call the set

$$S=\{(\lambda,\ \mu)\,|\,|\lambda|\geqslant 1,\ |\mu|\geqslant 1,\ \lambda e^{i\xi\alpha}=1 \text{ or } \mu e^{i\eta\beta}=1\},$$

where ξ, η are given by (5.21) and (5.44) respectively, the singular set of equation (1.9). By the previous discussions, we obtain the main result of this paper as follows.

Theorem 7. *If $f(t)\in\hat{H}_0$, then*

(i) *when $(\lambda,\ \mu)\overline{\in}S$, equation (1.9) in $\hat{H}_0$ has unique solution*

$$\varphi(t)=\Phi^+(t)-\Phi^-(t), \tag{6.1}$$

where Φ^+ and Φ^- are given in Theorems 3—6 respectively according to the corresponding cases;

(ii) *when $(\lambda,\ \mu)\in S$, the conditions of solvability for (1.9) in $\hat{H}_0$ are given in Theorems 4—6 respectively, and it also has unique solution (6.1) if they are fulfiled.*

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