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ON THE REASONING EXPRESSIONS AND THE NATURE OF DEDUCTION RULES

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Abstract

Reasoning expressions are those which express the reaasoning procedure by means of only deduction rules and the initial formulas (axioms or assumptions) without the help of any intermediate results. They express the procedure systematically, completely and concisely. The deduction rules are mappings from formulas (premises) to formula (conclusion). The elementary rules are certain propositional connectives (but not necessarily truth functions) while the higher rules are certain quantifiers. Besides, the detachment rule is an inverse of the connective implication, and is itself the kernel of deduction method; while another inverse of implication (i. e. the suggestion rule) is the kernel of induction method.

§1. Reasoning Expressions

We need a long time to understand the reasoning procedure. When we begin studying geometry in middle school we are told to write the reason for each step. However it requires only to write out the theorems (formulas) used without mentioning the deduction rule applied. Evidently the deduction rules applied are neglected and the reasoning procedure is expressed imcompletely. Later on, when representing the reasoning procedure we generally give a list of formulas (i. e. intermediate results) connected by such words as "hence", "therefore", "together with so and so we have…", etc. Sometimes we use also words "Beside", "On the other hand", etc. to hint that we restart the reasoning. Some persons, however, would not bother about such hints at all, they use solely "hence" and "therefore", or solely the symbol "$\Rightarrow$", so that the readers could hardly follow their reasoning. In the papers published in periodicals the authors, as a rule, omit a great deal of (obvious) intermediate results which should be supplied by readers when reading the papers. Since such supply is difficult to be done it is the main cause that such papers are very difficult to read.

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At the early stage of mathematical logic, B. Russell and A. N. Whitehead[5] emphasized the distinction between principles (i. e. deduction rules) and (provable) propositions([5] p. 98, especially the note, p. 106, etc.). Yet they did not correctly know the nature of deduction rules (for example, they listed the detachment rule as one of primitive propositions), hence their words about the distinction are not to the point. It is from later logicians that we know correctly the nature of deduction rules and can distinguish them from provable formulas. To compare with production in a factory, deduction rules correspond to machines, the (provable) formulas correspond to raw materials, applying rules corresponds to driving a machine. In order to express completely the reasoning procedure, therefore, it is necessary to write out fully in every step the applied rules together with corresponding provable formulas. This is the requirement proposed by mathematical logicians. Such expressions are very cumbersome and unmanageable indeed. Except very few instances at the very beginning, no one would use such expressions henceforth. Logicians develop various concise abbreviated formats instead. All of them, abbreviated or not, depend heavily upon immediate results, without which we could not understand the expressions at all. In other words, we have to list successively concrete intermediate results. Reasoning procedure would be expressed by a list of concrete formulas (perhaps attached with applied rules).

In contrast to this, we may express the calculating procedure without the help of any intermediate results. For example, we have the following reasoning procedure (in the Gentzen's sequent calculus):

$$\text{(A)}\qquad \dfrac{\dfrac{(4)\ H\to A\supset B \qquad \dfrac{(1)\ A\to A \quad (2)\ B\to B}{(3)\ A\supset B,\ A\to B}\ \supset\to}{(5)\ H,\ A\to B}\ \text{Cut} \qquad (6)\ H,\ B\to D}{(7)\ H,\ A\to D}\ \text{Cut}$$

In calculation, this corresponds to the following:

$$\text{(B)}\qquad \dfrac{\dfrac{(4)\ 4 \qquad \dfrac{(1)\ 3 \quad (2)\ 5}{(3)\ 8}\ +}{(5)\ 32}\ \times \qquad (6)\ 3}{(7)\ 96}\ \times$$

But we have a much better expression to take place of (B), that is

(C) $(4*(3+5))*3$ or

$((4)*((1)+(2))*(6)$.

(C) is superior to (B) not only in its conciseness but also in that it expresses the whole calculation procedure without the help of any intermediate results. Furthermore, the initial datas may be arbitrary (e. g. variables). Thus we get the concepts of "algebraic forms", "equations", etc., with which we promote greatly mathematical science. Besides, because we make use of variables and discard the help of concrete intermediate results, we should think about such deep properties as the commutative and associative laws which would be neglected if we were calculating for ever the concrete numbers (as do pupils in primary schools). It is evident that the expression (A) corresponds to (B), hence we may say that the expression we now use for reasoning procedure remain in the stage of primary school. It is urgent that we should have a better expression (corresponding to algebraic expression) for reasoning procedure. Such expressions would be called reasoning expressions.

We would ask: "could we (imitating (C)) write (A) as follows?"

(D) Cut Cut (4)$\supset\to$(1)(2)(6)

According to the present usage, "$\supset\to$(1)(2)" denotes only the application of the rule $\supset\to$ to (1) (2), and not the formula (3) (i. e. the result of the application); hence "Cut (4)$\subset\to$(1)(2)" is not well-formed and so neither is (D), that means (D) is not available. It is, however, very natural and convenient to let "$\supset\to$ (1)(2)" denote the formula (3), i.e. the result of the application of the rule $\supset\to$to (1)(2) (and not only the application itself). If so then (D) is available.

(D) is called reasoning expression which we may define recursively as follows.

Definition (1) (Initial) Formulas are reasoning expressions;

(2) If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are reasoning expressions, and R is n-placed rule (i. e. deriving a conclusion from n premises), then $R\alpha_1\alpha_2\cdots\alpha_n$ is reasoning expression;

(3) Reasoning expressions are those obtained by (1)(2).

We would note that there are two kinds of rules, elementary and higher rules, the formulation of item (2) should be modified correspondingly for these two kinds of rules (see below).

The reasoning expression is strange to us, and it is difficult to write and read at present. But from the experience about calculating expressions and about programs in computers, it is evident that with only very few training we could write and read reasoning expressions quite easily just as we do for algebraic expressions. We propose that reasoning expressions should be taught and employed as early as possible, preferably at the same time when we learn algebraic expressions; instead of teaching set theory in middle school it is better teaching reasoning expressions. Thus we could write and read the concise reasoning expression without having to check and to supply the long list of intermediate results.

§2. The Nature of Deduction Rules

Since we regard "$R\alpha_1\alpha_2\cdots\alpha_n$" as the result of applying R to the premises $\alpha_1, \alpha_2, \cdots, \alpha_n$, what is commonly written as

$$R;\ \alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n \vdash \beta$$

should be written as $R\alpha_1\alpha_2\cdots\alpha_n=\beta$. Deduction rules are regarded as mappings from formulas to formula, i.e. as propositional functions.

Rules are divided into two kinds, elementary and higher. The former does not contain bound variables and corresponds to propositional connectives; the latter contains bound variable and corresponds to quantifiers.

The most important elementary rule is the detachment rule, it reads (in ordinary formation):

D: $$\alpha\to\beta,\ \alpha\vdash\beta,$$

and, as mentioned above, we would write

$$D(\alpha\to\beta)\alpha=\beta.$$

The higher rules may be characterized as follows. At least one of its premises contains at least one variable which could not be substituted by constants or complex expressions when being applied. Such variables are called rule variables (or hard variables). They are genuine bound variables.

There are two important higher rules, the substitution rule and generalization rule.

The substitution rules are still divided into two, that is, substitution for individual variables and substitution for propositional variables; they read:

(S_x^ξ): $\alpha(x)\overset{x}{\vdash}\alpha(\xi)$;

(S_p^β): $\alpha(p)\overset{p}{\vdash}\alpha(\beta)$.

When the rule being applied, the x or p in it should be variable proper and could not be substituted by constants or complex expressions. For example, the following deductions are illegal:

$$\alpha(3)\vdash\alpha(16),$$

or

$$\alpha(x^2+2)\vdash\alpha(16).$$

It should be emphasized that the symbolism "$\alpha(x)$" and "$\alpha(\xi)$"need many explanations to clarify them; hence it would be better to use the substitution operators $\binom{\xi}{x}$, $\binom{\beta}{p}$ instead. And then the substitution rules would read thus:

(S_x^ξ): $\alpha\overset{x}{\vdash}\binom{\xi}{x}\alpha$;

(S_p^β): $\alpha\overset{p}{\vdash}\binom{\beta}{p}\alpha$.

The operrtor $\binom{\xi}{x}$ may be defined recursively as follows $\left(\text{similarly for } \binom{\beta}{p}\right)$:

(1) $\binom{\xi}{x}\alpha=f(\cdots,\ \xi,\ \cdots)$ when α is the atomic formula $f(\cdots,\ x,\ \cdots)$;

(2) $\binom{\xi}{x}\alpha=\neg\binom{\xi}{x}\beta$ when α is $\neg\beta$;

(3) $\binom{\xi}{x}\alpha=\binom{\xi}{x}\beta\circ\binom{\xi}{x}\gamma$ when α is $\beta\circ\gamma$ and $\circ$ is a two-placed connective;

(4) $\binom{\xi}{x}\alpha=\binom{Q}{x}\beta$ when α is $\binom{Q}{x}\beta$ and Q is one of $\forall$ or $\exists$;

(5) $\binom{\xi}{x}\alpha=\binom{Q}{y}\binom{\xi}{x}\beta$ when α is $\binom{Q}{y}\beta$, $y\neq x$ and y is not free in ξ;

(6) $\binom{\xi}{x}\alpha=\binom{Q}{z}\binom{\xi}{x}\binom{z}{y}\beta$ when α is $\binom{Q}{y}\beta$, $y\neq x$ and y occurs free in ξ, here z is the first variable which occur free neither in α nor in ξ.

Another important higher rule is the generalization rule, which reads (in ordinary formation):

$\binom{G}{x}$: $\alpha \mid\!\underline{x}\, \binom{\forall}{x}\alpha$;

$\binom{G_1}{x}$: $\beta\rightarrow\alpha \mid\!\underline{x}\, \beta\rightarrow\binom{\forall}{x}\alpha$, where x is not free in β.

As mentioned above, we would write

$$\binom{S\xi}{x}\alpha=\binom{\xi}{x}\alpha;\quad \binom{S\beta}{p}\alpha=\binom{\beta}{p}\alpha;$$

$$\binom{G}{x}\alpha=\binom{\forall}{x}\alpha;$$

$$\binom{G_1}{x}(\beta\rightarrow\alpha)=\beta\rightarrow\binom{\forall}{x}\alpha\quad (x \text{ is not free in } \beta).$$

It is obvious that, for example, the substitution rule is sharply distinguished from the operator of substitution in the ordinary formation; the former is used in reasoning procedure while the latter is used in the formula-formation procedure. But since we regard the deduction rule as mappings from formulas to formula, is there any difference between them?

Now, we have $\binom{S\xi}{x}\alpha=\binom{\xi}{x}\alpha$; the only difference between rules and operators seems to be: the deduction rules are applied only to provable formulas while the operators could be applied to arbitrary formulas, not necessarily provable ones. However, in the deduction from assumptions we have applied the deduction rules to unprovable formulas (i. e. assumptions) already; hence it seems that we have no reason to keep this distinction any longer. In other words, we should identify the substitution rule and the substitution operator, denoted by $\binom{\xi}{x}$, $\binom{\beta}{p}$; and should identify the generalization rule with the universal quantifier, denoted by $\binom{\forall}{x}$. These rules would read thus:

$$\binom{\xi}{x}\alpha=\binom{\xi}{x}\alpha;\quad \binom{\beta}{p}\alpha=\binom{\beta}{p}\alpha;$$

$$\binom{\forall}{x}\alpha=\binom{\forall}{x}\alpha,\quad \left(\text{and } \binom{\forall}{x}(\beta\to\alpha)=\beta\to\binom{\forall}{x}\alpha\right).$$

The first three are tautologies; i.e. special cases of the laws $x=x$. The proper contents of them should be formulated thus:

If α is provable, then so is $\binom{\xi}{x}\alpha$ and $\binom{\beta}{p}\alpha$;

If α is provable, then so is $\binom{\forall}{x}\alpha$.

Or we may write these two sentences as two items in the recursive definition of "provable formulas (or theorems)".

What about the detachment rule D? We know that the connective $\to$ transforms α and β into $\alpha\to\beta$, while the detachment rule transforms $\alpha\to\beta$ and α into β. Remembering that the addition $+$ transforms a and b into $a+b$, while its inverse transforms $a+b$ and a into b, we see that, in a certain sense, the detachment rule D is an inverse of $\to$. Since substraction $-$ may be characterized as:

$b-a$ denotes the c such that $a+c=b$;

in the same way, the rule D may be characterized as:

Dpq denotes the r such that $p\to r=q$.

It is evident that there is no truth function which is inverse of $\to$, so the rule D is not a truth function.

By the way we note that the inverse of $\neg$ is $\neg$ itself, since from $q=\neg p$ we have $p=\neg q$. There is no inverses of $\wedge$ and $\vee$, even if we allow deduction rules to be their inverses.

Since the connective $\to$ is not commutative, it would have two inverses. The first, from $\alpha\to\beta$ and α to get β (the necessary condition of α). This is the detachment rule D, which is the kernal of the deduction method. The second, from $\alpha\to\beta$ and β to get α (the sufficient condition of β). We call it the suggestion rule, denoted by S, which is the kernal of the induction method. It is well known that if $\alpha\to\beta$ and α are universal valid then so is β; therefore, the detachment rule (and the deduction method) always gives valid conclusion from valid premises. In contrast, even if $\alpha\to\beta$ and β are universal valid, it is not necessary that α should be valid; hence it is quite possible that from valid premises we may suggest invalid conclusion, i. e. we may obtain invalid conclusion from induction method. Owing to this fact logicians generally blame induction method for its containing errors, or justify induction with unnatural and untenable arguments (e. g. with probability interpretation, etc.). They do not grasp the true nature of the induction method. Either from daily life experience or from the development of science we know that it is very important to derive necessary conditions or to suggest sufficient conditions for given premises. Of course, the necessary and sufficient conditions should be appropriate to the current circumstance. If the sufficient condition suggested is not valid in the current circumstances, it is not appropriate, and we should suggest new ones instead, until we find the most appropriate conclusion. Similarly, although the necessary conditions are always valid, yet it may be unappropriate to us on account of being too trivial or being irrelevant to our purpose. In this case we have to derive again new necessary conditions as well. In a word, to get an appropriate conclusion we need to go along by trial and error method whether in induction or in deduction (cf. Moh [2]).

By introducing new connectives D and S (the two inverses of $\rightarrow$) the propositional calculus would have a lot of new subjects to study, and many old problems would have new formulations. For example, we generally say that α and β are mutually deducible with help of $\alpha_1, \alpha_2, \cdots, \alpha_h$ under the rules $R_1, R_2, \cdots, R_k$. Formerly we verify this fact by means of deduction: to derive α from β, and to derive β from α. Now we may say that if there are two reasoning expressions $F(x)$ and $G(x)$, which are built up from x and $\alpha_1, \alpha_2, \cdots, \alpha_h$ by means of $R_1, R_2, \cdots, R_k$, such that we have

$$\alpha=F(\beta) \quad \text{and} \quad \beta=G(\alpha),$$

then α and β are mutually deducible. Sometimes it is easier to find out the expressions $F(x)$ and $G(x)$.

§ 3. The Deletion of Two Higher Rules—Substitution Rule and Generalization Rule

As well known, we have:

(1) $\begin{pmatrix}\forall\\ x\end{pmatrix}\alpha=\begin{pmatrix}\forall\\ y\end{pmatrix}\begin{pmatrix}y\\ x\end{pmatrix}\alpha$ (y not occurs in α) (relettering of bound variable);

(2) When y does not occur free in ξ and $x\neq y$, then

$$\begin{pmatrix}\xi\\ x\end{pmatrix}\begin{pmatrix}\forall\\ y\end{pmatrix}\alpha=\begin{pmatrix}\forall\\ y\end{pmatrix}\begin{pmatrix}\xi\\ x\end{pmatrix}\alpha;$$

(3) $\begin{pmatrix}\forall\\ x\end{pmatrix}(\alpha\rightarrow\beta)\rightarrow\cdot\begin{pmatrix}\forall\\ x\end{pmatrix}\alpha\rightarrow\begin{pmatrix}\forall\\ x\end{pmatrix}\beta;$

Now, for the new connective D, we have

(4) $\binom{\xi}{x}D\alpha\beta=D\binom{\xi}{x}\alpha\binom{\xi}{x}\beta$;

(5) $\binom{\forall}{x}D\alpha\beta=DD(3)\binom{\forall}{x}\alpha\binom{\forall}{x}\beta$.

The formula (4) may be regarded as one item in the recursive definition of operator $\binom{\xi}{x}$, or we may prove it as follows. Let α be $\beta\rightarrow\gamma$; then $D\alpha\beta=\gamma$, and $\binom{\xi}{x}\alpha=\binom{\xi}{x}\beta\rightarrow\binom{\xi}{x}\gamma$; hence both sides be $\binom{\xi}{x}\gamma$, and we get (4).

If the main connective in (3) were "=", then by the same method we would get (5) $\binom{\forall}{x}D\alpha\beta=D\binom{\forall}{x}\alpha\binom{\forall}{x}\beta$. Since the main connective in (3) is $\rightarrow$, we should prove (5) as follows. Let α be $\beta\rightarrow\gamma$, then $D(3)\binom{\forall}{x}\alpha$ is $\binom{\forall}{x}\beta\rightarrow\binom{\forall}{x}\gamma$, and then we get (5) easily.

Suppose the bound variables has been suitably relettered. From (2) we may either transfer the operator $\binom{\xi}{x}$ to the right of $\binom{\forall}{x}$, or conversely. From (4) we may transfer the operator $\binom{\xi}{x}$ to the right of D; although (5) is much more complicate than (4), yet we could still transfer the operator $\binom{\forall}{x}$ to the right of D.

By induction, the substitution operator $\binom{\xi}{x}$ may be transferred to the right, and eventually to the front of axioms. If, as remarked by J. von Neumann, we regard the result of substitution for axioms as new axioms (axiom schemes), then the substitution rule is unnecessary and may be deleted.

By induction again, the universal quantifier may be also transferred to the right, and also eventually to the front of axioms. If, following W. V. Quine, we regard the universal closure of axioms as new axioms, then the generalization rule may be derived as a metatheorem, and may be deleted from primitive rules.

It should be emphasized that the assertion that we may delete these two rules in our axiom system means no more than that we may transfer the two operators to the right until they come to the front of axioms. As for the operators $\binom{\xi}{x}$ and $\binom{\forall}{x}$, they are indispensable indeed. We need universal quantifier to form negation and implication such as $\neg\binom{\forall}{x}\alpha$ and $\binom{\forall}{x}\alpha\cdot\rightarrow\beta$; we need substitution operator to write out the axiom $\binom{\forall}{x}\alpha \to \binom{\xi}{x}\alpha$. Some writer would write it as $\binom{\forall}{x}\alpha(x) \to \alpha(\xi)$; however, the very meaning of $\alpha(x)$, $\alpha(\xi)$ should be explained with the help of $\binom{\xi}{x}$.

(Added in proof). It should be emphasized that the reasoning expressions wonld play important roles in mechanical proof (e. g. the proof of the correctness of programs). We wonld discuss this more in detail in other papers. Besides, it is desirable to give eorresponding reasoning expressions in every mateematical papers so that, firstly, it would help readers so understand the deduction itself, secondly, when we are accustomed to the reasoning expressians it may take the place of the long list of inter mediate formnlas (which is used up to now).

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