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HUNGARIAN SEMIGROUPS, THE ARITHMETIC OF GENERALIZED RENEWAL SEQUENCES AND SEMI-*P*-FUNCTIONS

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Abstract

In this paper the authors discuss the additional conditions under which a Hungarian semigroup is possessed of the fundamental Delphic properties. It is proved that both the positive generalized renewal sequence semigroup and the tame semi-p-function semigroup are Hungarian semigroups and possessed of the fundamental Delphic properties. Then the arithmetic properties of these two classes of the special semigroups are studied respectively.

§1. Hungarian Semigroups and the Fundamental Delphic Properties

In this section we adopt the concepts and notations of [9]. Let S be a commutative first countable Hausdorff topological semigroup with unity. We denote the semigroup operation by "multiplication".

Definition 1. 1. *Let $u, v \in S$, v is called a factor of u, denoted by $v|u$, if there exists $v' \in S$ such that $u = vv'$. u and v are called mutually associate" and denoted by $u \sim v$, if we have simultaneously $v|u$ and $u|v$. If in S $u \sim v$ always implies $u = v$, S is called "non-associate".*

In the following we always denote the unity by 1, its set of associates by $A(1)$ and the set of associates of each $u \in S$ by $A(u)$.

Definition 1. 2. *Call $u \in S$ "prime" (not effectively decomposable, without effective factor) if $u \notin A(1)$ and $v|u$ always implies $v \in A(u) \cup A(1)$. Call $u \in S$ "antiprime" if u has no prime factor. The elements which are neither prime nor antiprime are called "composed".*

Evidently, all $u \in A(1)$ are antiprime.

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Definition 1.3. *Call $u \in S$ "infinitely divisible" (denoted by i. d. in short), if for any positive integer n there always exists the n-th root $u^{\frac{1}{n}} \in S$, such that*

$$u = \underbrace{u^{\frac{1}{n}} u^{\frac{1}{n}} \cdots u^{\frac{1}{n}}}_{n \text{ in number}} = (u^{\frac{1}{n}})^n.$$

Definition 1.4.*) *Call $u \in S$ "infinitesimally divisible" (denoted by i. t. d. in short), if u can be decomposed as an infinitesimal triangular array $\{u_{kj} \in S: k=1, 2, \cdots; j=1, 2, \cdots, n_k\}$ such that*

$$u = \prod_{j=1}^{n_k} u_{kj} \quad (\text{all } k=1, 2, \cdots)$$

and satisfies

$$\lim_{k\to\infty} u_{kj} = 1. \quad (\text{uniformly for } j).$$

"Infinitely divisible" is an algebraical concept, "infinitesimally divisible" is a topological concept, they are different from each other. If in a topological semi-group S, all i. t. d. elements are i. d. elements at the same time, we say that S has "weak central limit property".

D. G. Kendall, after summarizing the arithmetic properties of probability distribution convolution semigroup, of renewal sequence semigroup and p-function semigroup, promulgated a class of abstract topological semigroup which he named Delphic, by the place where he accomplished his article[4]. This class of semigroups possesses the above mentioned fundamental properties, namely:

Theorem 1.1. *If $u \in S$ is i. d., it is also i. t. d.*

Theorem 1.2. *Antiprime elements are all i. d.*

Theorem 1.3. *All composed element $u \in S$ can always be represented (not neccessary uniquely) as*

$$u = w \prod_{j \in I} v_j,$$

where w is antiprime, all v_j are prime, and I is a non-empty subset of positive integers.

Kendall studied in detail the arithmetic properties of Delphic semigroups which is one of his brilliant contributions to mathematics. His works had been carried much further by R. Davidson[5,6].

But in some semigroups (such as the probability distribution convolution semigroup and nonperiodic renewal sequence semigroup) the Delphic homomorphism does not exist, and in others it may be very difficult to find, especially in some abstract topological semigroups. So one needs to define a kind of semigroups without introducing the existence condition of the Delphic homomorphism but requiring still the most fundamental properties that positive

*) This concept was first introduced by Rùzsa and Székely (See [9], but in different form). It is easily seen that they are equivalent if S is first countable.

integer multiplication semigroup has, that is, all composite numbers can be decomposed as product of primes. By this idea, Rùzsa and Szèkely put forward the concept of Hungarian semigroup[9].

Definition 1. 5. *Call S a Hungarian semigroup if the following conditions are satisfied:*

A) *For each $u \in S$ the set of associates $A(u)$ is closed, so one may construct the factor semigroup $S/\sim$;*

B) *The set of factors of each element in $S/\sim$ is compact subset of $S/\sim$;*

C) *For each $u \in S$ the set of associates $A(u) = uA(1)$.*

Theorem 1. 4. *If S is a Hungarian semigroup, then Theorem. 1.3 holds.*

In the following we shall continue to find the conditions under which the other Delphic properties hold.

Theorem 1.5. *If S is a Hungarian semigroup without idempotent element other than unity, then all antiprime elements are i. t. d.. Therefore, if S also has weak central limit property (that is,* i. t. d.$\Rightarrow$i. d.), *then Theorem* 1.2 *holds*[9].

Theorem 1.6. *If S is a Hungarian semigroup without idempotent element other than unity, and if in the factor semigroup* $S/\sim$, *the existence of n-th root implies uniqueness and also all elements in $A(1)$ are i. d. and i. t. d., then Theorem* 1.1 *holds.*

Proof Since S does not involve idempotent elements other than unity, neither does $S/\sim$. Suppose $\tilde{u} \in S/\sim$ is i. d., denote its n-th root by $\tilde{u}^{\frac{1}{n}}$ for all n, so $\tilde{u}^{\frac{1}{n}} | \tilde{u}$. Since the set of factors of $\tilde{u}$ in $S/\sim$ is compact, there in $\{\tilde{u}^{\frac{1}{n}}\}$ must be sub-sequence $\{\tilde{u}^{\frac{1}{n_k}}\}$ that converges to a certain $\tilde{v}$, so $\tilde{u}^{\frac{1}{n}} \to \tilde{v}$ by the uniqueness of n-th root. Therefore $\tilde{u}^{\frac{1}{n}} = (\tilde{u}^{\frac{1}{2n}})^2 \to \tilde{v}^2$, $\tilde{v} = \tilde{v}^2$ and $\tilde{v} = 1$ which implies $\tilde{u}$ is i. t. d. That means for all $u \in S$, i. d$\Rightarrow$i. t. d.

§2. The Arithmetic of Positive Generalized Renewal Sequences

Definition 2.1. *Call real sequeuce $\mathbf{u} = \{u_n\}$ generalized renewal sequence ($u_0 = 1$ or undefined), if there exists a nonnegative sequence $\mathbf{f} = \{f_n\}$ ($f_0 = 0$ or undefined), such that*

$$u_n = f_n + \sum_{r=1}^{n-1} f_r u_{n-r} \quad (all\ n \geqslant 1). \tag{2.1}$$

If moreover

$$\sum_{n=1}^{\infty} f_n \leqslant 1, \tag{2.2}$$

then $\mathbf{u}$ is called a renewal sequence. They are called positive (renewal or generalized *renewal sequences*), *if* $u_1>0$.

As usual the class of renewal sequences and generalized renewal sequences are denoted by $\mathscr{R}$ and $\widetilde{\mathscr{R}}$ respectively, and by $\mathscr{R}^+$ and $\widetilde{\mathscr{R}}^+$ their respective positive subclass.

The concept of generalized renewal sequences are the natural generalization of the probabilistic model——renewal sequences, to the pure mathematical domain, proposed first by J. F. C. Kingman[7], who established the important relations between them

Lemma 2.1. $\mathbf{u}\in\mathscr{R}\Leftrightarrow\mathbf{u}\in\widetilde{\mathscr{R}}$ *and* $u_n\leqslant 1$, $(\forall n)$.

Lemma 2.2. $\mathbf{u}\in\widetilde{\mathscr{R}}\Leftrightarrow$ *for each* $k\geqslant 1$, *there exists* $\mathbf{u}(k)\in\mathscr{R}$ *and* $c_k>0$ *such that*

$$u_n=c_k^n u_n(k)\ (\text{for } n=1,\ 2,\ \cdots,\ k).$$

The fundamental properties of $\widetilde{\mathscr{R}}$ were systematically studied by Liang Zhishun and Huang Zhirui[10]. From there one can see easily the following theorem.

Theorem 2.1. *By pointwise multiplication and pointwise convergence topology* $\widetilde{\mathscr{R}}$ *is a commutative first countable topological semigroup with unity, and is the closed sub-semigroup of the real sequences semigroup* $R^{z+}\cdot\widetilde{\mathscr{R}}^+$ *is the hereditary sub-semigroup of* $\widetilde{\mathscr{R}}$, *it is also a commutative first countable topological semigroup with unity. In this section we use* $\mathbf{1}=(1,\ 1,\ \cdots\cdots)$ *to denote the unity of* $\widetilde{\mathscr{R}}$.

Lemma 2.3. *The semigroup* $\widetilde{\mathscr{R}}^+$ *is associate, the set of associates of unity* $\mathscr{A}(\mathbf{1})$ *isall the integral power sequences* $\mathbf{v}(\infty,\ c)=\{c^n\}$ *for* $c>0$. *Meanwhile each element of* $\mathscr{A}(\mathbf{1})$ *is i. d. and i. t. d..*

Proof We need only to prove $\mathscr{A}(\mathbf{1})=\{\mathbf{v}(\infty,\ c):\ c>0\}$. Since the rest facts are evident.

The fact $\mathbf{v}(\infty,\ c)\mathbf{v}(\infty,\ c^{-1})=\mathbf{1}$ (all $c>0$) implies $\{\mathbf{v}(\infty,\ c):\ c>0\}\subset\mathscr{A}(\mathbf{1})$. Conversely, let $\mathbf{u}\in\mathscr{A}(\mathbf{1})$, that means, there exists $\mathbf{v}\in\widetilde{\mathscr{R}}^+$ such that $\mathbf{uv}=\mathbf{1}$, then by the facts that $u_n\in u_{n-1}u_1$, $v_n\geqslant v_{n-1}v_1$, $u_n=v_n^{-1}$ (all n), one can deduce that $u_n=u_1^n$ $(\forall n)$ by mathematical induction, so $\mathbf{u}\in\{\mathbf{v}(\infty,\ c):\ c>0\}$.

Lemma 2.4. *For* $\mathbf{u}\in\widetilde{\mathscr{R}}^+$ *its set of associates* $\mathscr{A}(\mathbf{u})=\mathbf{u}\mathscr{A}(\mathbf{1})$ *and is closed in* $\widetilde{\mathscr{R}}^+$.

Proof Evidently $\mathbf{uv}(\infty,\ c)\sim\mathbf{u}\ (\forall c>0)$. Conversely, let $\mathbf{v}\in\mathscr{A}(\mathbf{u})$, then there exist $\mathbf{w}$, $\mathbf{s}\in\widetilde{\mathscr{R}}^+$ such that $\mathbf{v}=\mathbf{uw}$ and $\mathbf{u}=\mathbf{sv}$, so $\mathbf{u}=(\mathbf{sw})\mathbf{u}$. Since evidently the semigroup $\widetilde{\mathscr{R}}^+$ is cancellative, therefore $\mathbf{sw}=\mathbf{1}$, i. e., $\mathbf{w}\in\mathscr{A}(\mathbf{1})$. Hence, $\mathbf{v}\in\mathbf{u}\mathscr{A}(\mathbf{1})$. Evidently $\mathscr{A}(\mathbf{1})$ is closed, since if $\mathbf{v}(\infty,\ c_n)\to\mathbf{v}\in\widetilde{\mathscr{R}}^+$, then neccessarily $c_n\to c>0$ and $\mathbf{v}=\mathbf{v}(\infty,\ c)\in\mathscr{A}(\mathbf{1})$, which shows that all $\mathscr{A}(\mathbf{u})=\mathbf{u}\mathscr{A}(\mathbf{1})$ are closed.

From Lemma 2.4 and [9], one can easily deduce

Lemma 2.5. *By the relation of asscciates,* $\widetilde{\mathscr{R}}^+$ *can form the factor semigroup* $\widetilde{\mathscr{R}}^+/\sim$, *which is first countable.*

For further studies of the properties of the factor semigroup $\widetilde{\mathscr{R}}^{+}/\sim$, we mention that in each $\mathscr{A}(\mathbf{u})$, there exists one and only one $\tilde{\mathbf{u}}\in\mathscr{A}(\mathbf{u})$, such that $\tilde{u}_1=1$. Consider the sub-semigroup $\widetilde{\mathscr{R}}^1=\{\mathbf{u}\in\widetilde{\mathscr{R}}^{+}: u_1=1\}$ of $\widetilde{\mathscr{R}}^{+}$. If we establish 1-1 correspondence between $\mathbf{u}\in\widetilde{\mathscr{R}}'$ and $\mathscr{A}(\mathbf{u})\in\widetilde{\mathscr{R}}^{+}/\sim$, then we can prove the topological isomorphism between $\widetilde{\mathscr{R}}^1$ and $\widetilde{\mathscr{R}}^{+}/\sim$ (algebraic isomorphism and topological homeomorphism). So we can use the simpler in construction semigroup $\widetilde{\mathscr{R}}^1$ instead of $\widetilde{\mathscr{R}}^{+}/\sim$.

Lemma 2. 6. *$\widetilde{\mathscr{R}}^1$ is a first countable topological semi-group and for each $\mathbf{u}\in\widetilde{\mathscr{R}}^1$, the set of factors in $\widetilde{\mathscr{R}}^1$ $\mathscr{F}(\mathbf{u})=\{\mathbf{v}\in\widetilde{\mathscr{R}}^1: \mathbf{v}|\mathbf{u}\}$ is compact.*

Proof Evidently $\mathscr{F}(\mathbf{u})\subset\{1\}\times[1, u_2]\times[1, u_3]\times\cdots$ is compact, so $\mathscr{F}(\mathbf{u})$ is also compact since it is closed.

By the above lemmas we obtain

Theorem 2.2. *$\widetilde{\mathscr{R}}^{+}$ is a Hungarian semigroup.*

Corollary. *For $\widetilde{\mathscr{R}}^{+}$, Theorem 1.3, holds.*

We can further deduce that for $\widetilde{\mathscr{R}}^{+}$, Theorem 1.1, Theorem 1.2 also hold.

Theorem 2.3. *For $\widetilde{\mathscr{R}}^{+}$, Theorem 1.1 holds.*

Proof We need to prove that for $\widetilde{\mathscr{R}}^1$, Theorem 1.1 holds. In fact if $\mathbf{u}\in\widetilde{\mathscr{R}}^1$ is i. d. it can be decomposed as triangular array $\{\mathbf{u}(k, j)\in\widetilde{\mathscr{R}}^1: k=1, 2, \cdots; j=1, 2, \cdots, k\}$ in which

$$\mathbf{u}(k, j)=\mathbf{u}^{\frac{1}{k}}\to\mathbf{1}\quad(k\to\infty \text{ uniformly for } j).$$

Therefore $\mathbf{u}$ is i. t. d. So for each $\mathbf{u}\in\widetilde{\mathscr{R}}^{+}$ which is i. d., since each element of $\mathscr{A}(\mathbf{1})$ is i. d., $\mathbf{u}\mathbf{v}(\infty, u_1^{-1})\in\widetilde{\mathscr{R}}^1$ is also i. d. and since $\mathbf{v}(\infty, u_1)$ is i. t. d., so do $\mathbf{u}=(\mathbf{u}\mathbf{v}(\infty, u_1^{-1}))\cdot\mathbf{v}(\infty, u_1)$.

In order to prove Theorem 1.2, by Theorem 1.5, we need only to prove that $\widetilde{\mathscr{R}}^{+}$ has weak central limit property. He Yuanjiang [14] afforded a very well criterior for the central limit property by introducing the so called "countable homomorphism" and succeeded in solving the complicative problem. In abstract semigroups, to find sufficient conditions for the criterior of central limit property without introducing the homomorphism seems not easy. Here we shall follow the "direct method" of D. G. Kendall [3].

Lemma 2. 7. *$\mathbf{u}\in\widetilde{\mathscr{R}}^{+}$ is i. d. iff $\mathbf{u}$ is a Kaluza sequence*[12].

Theorem 2.4. *If $\mathbf{u}\in\widetilde{\mathscr{R}}^1$ is i. t. d., then $\mathbf{u}$ is i. d..*

Proof Since $\mathbf{u}\in\widetilde{\mathscr{R}}^1$ is i. t. d., it can be decomposed as triangular array $\{\mathbf{u}(k, j)\in\widetilde{\mathscr{R}}^1: k=1, 2,\cdots; j=1, 2, \cdots, n_k\}$ and satisfies

$$\lim_{k\to\infty}\mathbf{u}(k, j)=\mathbf{1}\quad(\text{uniformly in j}).\tag{2.3}$$

Denote the f-sequences of $\mathbf{u}(k, j)$ by $\mathbf{f}(k, j)$ with $f_1(k, j)=1$ and $\lim_{k\to\infty} f_n(k, j)=0$ (uniformly in j) for $n\geqslant 2$. We have

$$\frac{u_n u_{n-2}}{u_{n-1}^2}=\prod_{j=1}^{n_k}\left\{\frac{1+A(k,\ j)+f_n(k,\ j)/u_{n-1}(k,\ j)}{1+B(k,\ j)}\right\},$$

in which

$$\begin{aligned}A(k,\ j)&=\sum_{r=2}^{n-1}f_r(k,\ j)u_{n-r}(k,\ j)/u_{n-1}(k,\ j),\\ B(k,\ j)&=\sum_{r=2}^{n-1}f_r(k,j)u_{n-r-1}(k,\ j)/u_{n-2}(k,\ j).\end{aligned}\qquad (n\geqslant 3)$$

From (2.3) we know that

$$\lim_{k\to\infty}u_m(k,\ j)=1\quad(\text{uniformly for } 1\leqslant m\leqslant n \text{ and } 1\leqslant j\leqslant n_k),$$

$$\lim_{k\to\infty}f_r(k,\ j)=0\quad(\text{uniformly for } 2\leqslant r\leqslant n \text{ and } 1\leqslant j\leqslant n_k).$$

So

$$\lim_{k\to\infty}A(k,\ j)=\lim_{k\to\infty}B(k,\ j)=0\quad(\text{uniformly for } 1\leqslant j\leqslant n_k).$$

Using the inequality

$$z-z^2\leqslant\log(1+z)\leqslant z\quad(0\leqslant z<1/2)$$

$$\begin{aligned}\log\frac{u_nu_{n-2}}{u_{n-1}^2}&\geqslant\sum_{j=1}^{n_k}\left\{\frac{f_n(k,\ j)}{u_{n-1}(k,\ j)}+(A(k,\ j)-B(k,\ j))-\left(A(k,\ j)+\frac{f_n(k,j)}{u_{n-1}(k,\ j)}\right)^2\right\}\\ &\geqslant\sum_{j=1}^{n_k}\left\{\frac{f_n(k,j)}{u_{n-1}(k,j)}+\left(\frac{1}{u_{n-1}(k\ \ j)}-\frac{1}{u_{n-2}(k,\ j)}\right)(u_n(k,\ j)-u_{n-1}(k,\ j))\right.\\ &\qquad\left.-\left(\frac{u_n(k,\ j)}{u_{n-1}(k,\ j)}-1\right)^2\right\}\\ &=\sum_{j=1}^{n_k}\left\{\frac{f_n(k,\ j)}{u_{n-1}(k,\ j)}-\left(\frac{u_n(k,\ j)}{u_{n-1}(k,\ j)}-1\right)\left(\frac{u_n(k,\ j)}{u_{n-1}(k,\ j)}+\frac{u_{n-1}(k,\ j)}{u_{n-2}(k,\ j)}-2\right)\right\}\\ &\geqslant\sum_{j=1}^{n_k}\left\{\frac{f_n(k,\ j)}{u_{n-1}(k,\ j)}-2(u_n(k,\ j)-1)^2\right\}.\end{aligned}$$

Since $e^{x-1}\leqslant x^2$ (for $1\leqslant x\leqslant 2$), we have

$$\sum_{j=1}^{n_k}(u_n(k,j)-1)\leqslant 2\log\prod_{j=1}^{n_k}u_n(k,\ j)=2\log u_n<\infty,$$

and

$$\sum_{j=1}^{n_k}(u_n(k,\ j)-1)^2\to 0,\quad(k\to\infty).$$

$$\log\frac{u_nu_{n-2}}{u_{n-1}^2}\geqslant 0,\quad\text{i. e.}\quad\frac{u_nu_{n-2}}{u_{n-1}^2}\geqslant 1.$$

So by Lemma 2.7 **u** is i.d.

Corollary 1. $\mathbf{u}\in\widetilde{\mathscr{R}}^+$ *i. t. d.* $\Rightarrow$ *i. d.*

Corollary 2. *For* $\widetilde{\mathscr{R}}^+$, *Theorem* 1.2 *holds.*

By the above Theorems 2.2, 2.3 and 2.4, we know that $\widetilde{\mathscr{R}}^+$ posseses the fundamental properties of Delphic semigroup. In the following we continue to discuss the further arithmetic properties of $\widetilde{\mathscr{R}}^+$. First we have

Theorem 2.5. $I_0(\widetilde{\mathscr{R}}^+)=\mathscr{A}(\mathbf{1})$.

This result was first obtained by Chen Zaifu [15], He Yuanjiang [14], by using the result of renewal sequences in [5], gave a simpler proof of this fact. In the following we turn to the criterior of the prime class. In [10] Liang Zhishun and Huang Zhirui obtained

Lemma 2.8. *If* $\mathbf{u}\in\widetilde{\mathscr{R}}^{+}$, *then there exists limit (finite or infinite)*

$$c=\lim_{n\to\infty} u_n^{\frac{1}{n}}, \tag{2.4}$$

and

$$u_n\leqslant c^n \quad (\forall n).$$

Definition 2.2. *For* $\mathbf{u}\in\widetilde{\mathscr{R}}^{+}$, *when the limit* $c<\infty$ *in* (2.4) *we call it "tame" otherwise we call it "wild".*

Lemma 2.9. *if* $\mathbf{u}\in\widetilde{\mathscr{R}}^{+}$ *is "tame", then* $\mathbf{uv}(\infty, c^{-1})\in\mathscr{R}^{+}$, *where* c *is the limit in* (2.4).

It can be proved by Lemma 2.1.

Let us denote the class of tame positive generalized renewal sequence by $T((\widetilde{\mathscr{R}}^{+})$ and write

$$\mathscr{R}^{1}=\{\mathbf{u}\in\widetilde{\mathscr{R}}^{+}: \lim_{n\to\infty} u_n^{\frac{1}{n}}=1\}.$$

Lemma 2. 10. $T(\widetilde{\mathscr{R}}^{+})$ *is the hereditary subsemigroup of* $\widetilde{\mathscr{R}}^{+}$ *and* $\mathscr{R}^{1}$ *is that of* $\mathscr{R}^{+}$, *and moreover, the factor semigroup* $T(\widetilde{\mathscr{R}}^{+})/\sim$ *is topologically isomorphic to* $\mathscr{R}^{1}$.

Theorem 2.6. *Let* $\mathbf{u}\in T(\widetilde{\mathscr{R}}^{+})$ *and* $\lim_{n\to\infty} u_n^{\frac{1}{n}}=c$, *thin* $\tilde{\mathbf{u}}=\mathbf{uv}(\infty, c^{-1})\in\mathscr{R}^{1}$ *and* $\mathbf{u}$ *is prime in* $\widetilde{\mathscr{R}}^{+}$ *iff* $\tilde{\mathbf{u}}$ *is prime in* $\mathscr{R}^{+}$.

By this theorem, the criterior for the prime in renewal sequence [5,16] can be applied to the tame generalized renewal sequences. As for the wild sequences, we can give out many examples of i. d., but it isn't known yet whether there are prime.

§3. The Arithmetic of Standard Semi-p-Functions

Definition 3.1. *Each real function* $\mathbf{p}=\{p(t)\in R: t>0\}$ *on* $(0, \infty)$ *corresponds a family of functions*

$$F_n(t_1,\cdots, t_n; p)=\sum_{s=1}^{n}(-1)^{s-1}\sum_{0=k_0<k_1<\cdots<k_s=n}\sum_{r=1}^{s}p\Big(\sum_{i=k_{r-1}+1}^{k_r} t_i\Big), \tag{3.1}$$

$(n=1, 2, \cdots, t_1, t_2, \cdots, t_n)>0)$. *It is called the family of F-functions of* $\mathbf{p}$ [12].

Definition 3.2. *Let* $\mathbf{p}$ *be a real function defined on* $(0, \infty)$. *If it satisfies the inequalities*

$$F_n(t_1, \cdots, t_n; \mathbf{p})\geqslant 0 \quad (\forall n\geqslant 1;\ t_1, \cdots t_n>0) \tag{3.2}$$

it is called semi-p-function; if it satisfies further the inequalities

$$\sum_{k=1}^{n} F_k(t_1, \cdots, t_k; \mathbf{p})\leqslant 1 \quad (\forall n\geqslant 1;\ t_1, \cdots, t_n>0) \tag{3.3}$$

it is called p-function. When

$$\lim_{t\to 0} p(t)=1,$$

we call them standard in both cases.

In this paper we consider only the standard p-functions and semi-p-functions. As usual we denote by $\mathscr{P}$ and $\widetilde{\mathscr{P}}$ the class of p-functions and semi-p-functions respectively.

The concept of semi-p-function was first introduced by Kingman[6], and the following fundamental properties were obtained.

Lemma 3.1. $\mathbf{p}\in\widetilde{\mathscr{P}}\Leftrightarrow \mathbf{p}\in\mathscr{P}$ *and* $p(t)\leqslant 1,\ (\forall t>0)$.

Lemma 3.2. $\mathbf{p}\in\widetilde{\mathscr{P}}\Leftrightarrow$ *for each* $\tau>0$, *there are* $\mathbf{p}_\tau\in\mathscr{P}$, $\lambda_\tau\in R$, *such that*

$$p(t)=p_\tau(t)e^{\lambda_\tau t}\,(\forall 0<t\leqslant\tau). \tag{3.4}$$

Lemma 3.3. *If* $\mathbf{p}\in\widetilde{\mathscr{P}}$, *then* $p(t)>0(\forall t>0)$ *and exists limit*

$$-\infty<\lambda=\lim_{t\to\infty}\frac{1}{t}\log p(t)\leqslant\infty \tag{3.5}$$

and

$$p(t)\leqslant e^{\lambda t}\quad(\forall t>0). \tag{3.6}$$

Lemma 3.4. $\mathbf{p}\in\widetilde{\mathscr{P}}\Leftrightarrow \mathbf{p}$ *is continuous on* $(0,\infty)$, $\lim_{t\to 0} p(t)=1$, *and*

$$\mathbf{u}(p,h)=\{p(nh): n\geqslant 1\}\in\widetilde{\mathscr{R}}\quad(\forall h>0).$$

Definition 3.3. $\mathbf{p}\in\widetilde{\mathscr{P}}$ *is called "tame" if the limit* $\lambda<\infty$ *in* (3.5), *otherwise it is called "wild", and denote the two classes by* $\mathrm{T}(\widetilde{\mathscr{P}})$ *and* $\mathrm{W}(\widetilde{\mathscr{P}})$ *respectively.*

In [13] and [11] we have studied some of the properties of semi-p-functions, there we obtained:

Theorem 3.1. *By point-wise multiplication and point-wise convergence topology,* $\widetilde{\mathscr{P}}$ *forms a commutative topological semigroup with unity* $\mathbf{1}=\{p(t)=1: t>0\}$. *It is associative, the set of associates of unity* $\mathscr{A}(\mathbf{1})=\{\mathbf{p}:\ p(t)=e^{\lambda t},\ -\infty<\lambda<\infty\}$ *is closed, each element there is i. d. and i. t. d.. For each* $\mathbf{p}\in\widetilde{\mathscr{P}}$, *its set of associates is* $\mathscr{A}(\mathbf{p})=\mathbf{p}\mathscr{A}(\mathbf{1})$ *and is closed (in* $\widetilde{\mathscr{P}}$*).*

Proof The first conclusion can be seen in [13, 11]. Let us go to prove the associate properties. Evidently each $\mathbf{p}=\{e^{\lambda t};\ t>0\}\in\mathscr{A}(\mathbf{1})$, since $\mathbf{p}^{-1}=\{e^{-\lambda t}:\ t>0\}$ and $\mathbf{p}\mathbf{p}^{-1}=\mathbf{1}$. Conversely, let $\mathbf{p}\in\mathscr{A}(\mathbf{1})$, there must exists $\mathbf{q}\in\widetilde{\mathscr{P}}$, such that $\mathbf{pq}=\mathbf{1}$. Therefore $\mathbf{q}=\mathbf{p}^{-1}$ and simultaneously we have $p(t+s)\geqslant p(t)p(s)$, $p^{-1}(t+s)\geqslant p^{-1}(t)p^{-1}(s)$. So $\mathbf{p}$ satisfies the functional equation $p(s+t)=p(s)p(t)\quad(\forall s,\ t>0)$. So $\mathbf{p}$ is continuous and positive. From [11] we know that it must be an exponential function, that is, there exists real number λ, such that $\mathbf{p}=\{e^{\lambda t}:\ t>0\}$. Evidently $\mathscr{A}(\mathbf{1})$ is closed, with elements all being i. d. and i. t. d. By a similar method one easily proves that $\mathscr{A}(\mathbf{p})=\mathbf{p}\mathscr{A}(\mathbf{1})$ and $\mathscr{A}(\mathbf{p})$ is closed.

Theorem 3.2. $\mathrm{T}(\widetilde{\mathscr{P}})$ *is the hereditary sub-semigroup of* $\widetilde{\mathscr{P}}$, *and is an open subset of it. Therefore* $\mathrm{T}(\widetilde{\mathscr{P}})$ *is a commutative topological semigroup with unity, the set of associates of its elements is the same as that in* $\widetilde{\mathscr{P}}$ *and is closed (in* $\mathrm{T}(\widetilde{\mathscr{P}})$*).*

Proof Evidently $T(\widetilde{\mathscr{P}})$ is the sub-semigroup of $\widetilde{\mathscr{P}}$, and has unity, $\mathscr{A}(\mathbf{1})\subset T(\widetilde{\mathscr{P}})$. So for each $\mathbf{p}\in T(\widetilde{\mathscr{P}})$ its set of associates in $T(\widetilde{\mathscr{P}})$ is the same as that in $\widetilde{\mathscr{P}}$, namely $\mathbf{t}\mathscr{A}(\mathbf{1})$. Let us prove the hereditary property. Let $\mathbf{p}\in T(\widetilde{\mathscr{P}})$, $\mathbf{p}=\mathbf{qr}$, $\mathbf{q}$, $\mathbf{r}\in\widetilde{\mathscr{P}}$. Denote their limits in (3.5) by λ_p, λ_q, λ_r respectively, we have the relation

$$\lambda_q+\lambda_r=\lambda_p<\infty\Rightarrow\lambda_q<\infty \quad\text{and}\quad \lambda_r<\infty,$$

which means that $\mathbf{q}$, $\mathbf{r}\in T(\widetilde{\mathscr{P}})$. Let us prove that $T(\widetilde{\mathscr{P}})$ is open in $\widetilde{\mathscr{P}}$, that is, $W(\widetilde{\mathscr{P}})$ is closed in $\widetilde{\mathscr{P}}$. We take $\{\mathbf{p}_\alpha\}$ as a net of $W(\widetilde{\mathscr{P}})$ that converges in $\widetilde{\mathscr{P}}$. Let $\mathbf{p}=\lim_\alpha \mathbf{p}_\alpha\in\widetilde{\mathscr{P}}$, we must have $\mathbf{p}\in W(\widetilde{\mathscr{P}})$. Since if $\mathbf{p}\in T(\widetilde{\mathscr{P}})$, and its limit in (3.5) is $\lambda_p<\infty$, then for a certain $\varepsilon>0$, there must exists a certain α, such that $\lambda_{p_\alpha}<\lambda_p+\varepsilon$, which would show that $\mathbf{p}_\alpha\in T(\widetilde{\mathscr{P}})$ contradicts the fact $\mathbf{p}_\alpha\in W(\widetilde{\mathscr{P}})$. Since $T(\widetilde{\mathscr{P}})$ is open in $\widetilde{\mathscr{P}}$, and for each $\mathbf{p}\in T(\widetilde{\mathscr{P}})$, $\mathscr{A}(\mathbf{p})$ is closed in $\widetilde{\mathscr{P}}$, $\mathscr{A}(\mathbf{p})\subset T(\widetilde{\mathscr{P}})$ and must be closed in $T(\widetilde{\mathscr{P}})$.

Denote

$$\mathscr{P}^1=\left\{\mathbf{p}\in\widetilde{\mathscr{P}}:\ \lambda_p=\lim_{t\to\infty}\frac{1}{t}\log p(t)=0\right\},$$

and call $\mathbf{p}\in\mathscr{P}^1$ fundamental p-function. From Lemma 3.3 and Lemma 3.1 one can easily see that $\mathscr{P}^1\subset\mathscr{P}$, that is, all fundamental p-functions are standard.

Lemma 3.5. *$\mathscr{P}^1$ is the hereditary sub-semigroup of $\mathscr{P}$, and by the pointwise convergence topology it is first countable.*

Proof The former conclusion is evident. Use the results of Davidson in [5], we know that the pointwise convergence topology of $\mathscr{P}$ is equivalent to the topology of uniform convergence in finite interval, and is first countable and so does $\mathscr{P}^1$.

Corollary 1. *By the pointwise convergence topology, $T(\widetilde{\mathscr{P}})$ is first countable.*

Proof Map $\mathscr{P}^1\times R$ to $T(\widetilde{\mathscr{P}})$ by $\varphi(p,\lambda)=\{p(t)e^{\lambda t}\}$, it is a homeomorphic mapping, the conclusion follows from the fact that $\mathscr{P}^1$ and R are first countable.

Corollary 2. *$T(\widetilde{\mathscr{P}})/\sim$ is first countable and topologically isomorphic to $\mathscr{P}^1$ (algebraically isomorphic and topologically homeomorphic).*

Corollary 3. *For each $\mathbf{p}\in\widetilde{\mathscr{P}}^1$, its set of factors in $\mathscr{P}^1$ is compact.*

This can be easily seen by the hereditary property of $\mathscr{P}^1$ in $\mathscr{P}$.

From Theorem 3.2 and Lemma 3.5 we deduce immediately

Theorem 3.3. *$T(\widetilde{\mathscr{P}})$ is a Hungarian semigroup (by pointwise multiplication and pointwise convergence topology).*

Corollary 1. *For the semigroup $T(\widetilde{\mathscr{P}})$, Theorem 1.3 holds.*

Corollary 2. *$\mathbf{p}\in T(\widetilde{\mathscr{P}})$ is prime, compositeor antiprime if and only if $\tilde{\mathbf{p}}\sim\mathbf{p}$ and $\tilde{\mathbf{p}}\in\mathscr{P}^1$ is prime, compositeor antiprime respectively in $\widetilde{\mathscr{P}}^1$.*

Corollary 3. *If $\mathbf{p}\in I(\widetilde{\mathscr{P}})$ is i. d., then $\tilde{\mathbf{p}}\sim\mathbf{p}$ and $\tilde{\mathbf{p}}\in\mathscr{P}^1$ is i. d. in $\mathscr{P}^1$.*

Theorem 3. 4. *For $T(\widetilde{\mathscr{P}})$, Theorem 1.2 holds.*

Proof From the hereditary properties of $\mathscr{P}^1$ in $\mathscr{P}$, we know that $\mathbf{p}\in\mathscr{P}^1$ is antiprime in $\mathscr{P}^1$ if and only if it is antiprime in $\mathscr{P}$. Since $\mathscr{P}$ is a Delphic semigroup and, satisfies Theorem 1.2, so does $\mathscr{P}^1$. From Theorem 3.1, each element of $\mathscr{A}(\mathbf{1})$ is i. d. so for $T(\widetilde{\mathscr{P}})$ Theorem 1.2 holds.

Theorem 3. 5. *For $T(\widetilde{\mathscr{P}})$ Theorem 1.1 holds.*

Proof From the hereditary property of $\mathscr{P}^1$ in $\mathscr{P}$, and $\mathscr{P}$ satisfies Theorem 1.1 so does $\mathscr{P}^1$. Since each $\mathbf{p}\in\mathscr{A}(\mathbf{1})$ is i. t. d. so for $\mathrm{T}(\widetilde{\mathscr{P}})$ Theorem 1.1 holds.

From Theorem 3.3, Theorem 3.4 and Theorem 3.5 we have deduce that semigroup $\mathrm{T}(\widetilde{\mathscr{P}})$ possesses the three fundamental properties of Delphic semigroups. So its class of antiprime is equivalent to the class I_0 (that is the class of i. d. elements which have no prime factors).

Theorem 3. 6. *The class I_0 of $\mathrm{T}(\widetilde{\mathscr{P}})$ is in fact $\mathscr{A}(\mathbf{1})$, that is the class of all exponential functons* $\{\mathbf{p}$: $p(t)=e^{\lambda t}$, $(t>0)$, $-\infty>\lambda>+\infty\}$.

Proof From the result of Улановский[8], the class I_0 of $\mathscr{P}$ is the class of all bounded exponential functions $\{\mathbf{p}$: $p(t)=e^{-\lambda t}$, $(t>0)$, $\lambda\geqslant 0\}$. By the hereditary property of $\mathscr{P}^1$ in $\mathscr{P}$ we obtain $I_0(\mathscr{P}^1)=\{\mathbf{1}\}$, therefore the class I_0 is really $\mathscr{A}(\mathbf{1})$.

Theorem 3. 7. $\mathbf{p}$ *is prime in $T(\widetilde{\mathscr{P}})$* $\Leftrightarrow\tilde{\mathbf{p}}\sim\mathbf{p}$ *and* $\tilde{\mathbf{p}}\in\mathscr{P}^1$ *is prime in* $\mathscr{P}$.

Proof The necessity is evident. To prove the sufficiency, suppose $\tilde{\mathbf{p}}\in\mathscr{P}^1$ is prime in $\mathscr{P}^1$ From the hereditary property, $\tilde{\mathbf{p}}$ is prime in $\mathscr{P}^1$. Therefore all $\mathbf{p}\sim\tilde{\mathbf{p}}$ are prime in $\mathrm{T}(\widetilde{\mathscr{P}})$.

In Theorem 3.6 we obtained the full description of the class I_0 of $\mathrm{T}(\widetilde{\mathscr{P}})$. Theorem 3.7 shows that the key of criterior of prime elements in $\mathrm{T}(\widetilde{\mathscr{P}})$ is the criterion of prime elements in $\mathscr{P}$. From this, by using the results of Davidson[5], Qian Shixian[17] and He Qimei[18] in recognition of prime elements in $\mathscr{P}$ and the following lemma we deduce that it need only to recognize the prime elements in $\mathscr{P}^1$.

Lemma 3.6. $\mathbf{p}\in\mathscr{P}$ *and is prime in* $\mathscr{P}\Leftrightarrow\mathbf{p}\in\mathscr{P}^1$ *and is prime in* $\mathscr{P}^1$.

Proof Necessity. Since $\mathbf{p}$ is prime in $\mathscr{P}$, we must have $\mathbf{p}\in\mathscr{P}^1$. If not so, the corres ponding limit λ in (3.5) must be less than zero, $\mathbf{p}$ must have in $\mathscr{P}$ a factory $\mathbf{q}=\{e^{\lambda t}: t>0\}$ and so it is not prime in $\mathscr{P}$. Moreover since it has no effective factor in $\mathscr{P}$, so does in $\mathscr{P}^1$.

Sufficiency. Since $\mathscr{P}^1$ is hereditary in $\mathscr{P}$, that $\mathbf{p}$ has no effective factor in $\mathscr{P}^1$ implies that it has no effective factor in $\mathscr{P}^1$ too.

In order to study completely the arithmetic properties of the semigroup $\widetilde{\mathscr{P}}$, it remains only to make clear the construction of $W(\widetilde{\mathscr{P}})$.

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