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ON ESTIMATIONS OF TRIGONOMETRIC SUMS OVER PRIMES IN SHORT INTERVALS (III)***

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(*Dedicated to the Tenth Anniversary of CAM*)

Abstract

In this paper the following result is proved: There is an absolute positive integer c such that for every large odd integer N the Diophantine equation with prime variables $N=p_1+p_2+p_3$, $N/3-U<p_j\leqslant N/3+U$, $j=1, 2, 3$, is solvable for $U=N^{2/3}\log^c N$. Moreover, an asymptotic formula for the number of the solutions is given.

§1. Statement of Results

Through this paper, $c, c_1, c_2, \cdots$ stand for positive constants, $p, p_1, p_2, \cdots$ primes, $e(\theta)=e^{2\pi i\theta}$, and $l=\log x$. Let α be a real number, $\Lambda(n)$ the Mangoldt function: $\Lambda(n)=\log p$, if $n=p^k$, $k\geqslant 1$; $=0$, otherwise, and $x\geqslant A\geqslant 2$,

$$S(\alpha;\, x,\, A)=\sum_{x-A<n\leqslant x}\Lambda(n)e(n\alpha).$$

In [1] and [2] we proved the following two theorems by some purely analytic methods.

Theorem 1.[2, Theorem 2] *Let ε be an arbitrary positive constant,*

$$x^{91/96+\varepsilon}\leqslant A\leqslant x.$$

Then for any given positive c_1 there exist c_2 and c_3 such that for any α satisfying

$$\alpha=a/q+\lambda,\qquad (a,\, q)=1 \tag{1}$$

and
$$1\leqslant q\leqslant l^{c_2},\quad A^{-1}l^{c_3}<|\lambda|\leqslant(ql^{c_2})^{-1},$$

we have
$$S(\alpha;\, x,\, A)\ll Al^{-c_1}.$$

Theorem 2.[2, Theorem 3] *Let N be a large odd integer. The Diophantine equation with prime variables*

$$\begin{cases} N=p_1+p_2+p_3, \\ N/3-U<p_j<N/3+U,\ j=1,\,2,\,3 \end{cases} \tag{2}$$

is solvable for

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$$U = N^{91/96+\varepsilon},$$

where ε is an arbitrary positive constant. Moreover, the number of the solutions

$$T(N,\ U) = 3\mathfrak{S}(N)U^2(\log N)^{-3} + O(U^2(\log N)^{-4}) \tag{3}$$

where

$$\mathfrak{S}(N) = \prod_{p|N}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p\nmid N}\left(1+\frac{1}{(p-1)^3}\right) > \frac{1}{2},\ 2\nmid N. \tag{4}$$

In the present paper we shall improve these two theorems by developing the methods in [1] and [2] and using a new result of Zhan[4] for the zero density estimate of Dirichlet L-functions in short intervals. In fact, we shall prove

Theorem 3. *For any given c_4, there exist c_5, c_6 and c_7 such that for*

$$x^{2/3}l^{c_5} \leqslant A \leqslant x, \tag{5}$$

and α satisfying (1) *and*

$$1 \leqslant q \ll l^{c_6},\ A^{-1}l^{c_7} \ll |\lambda| \leqslant (qx^{1/6})^{-1}, \tag{6}$$

or

$$l^{c_6} \ll q \leqslant x^{1/6},\ |\lambda| \leqslant (qx^{1/6})^{-1}, \tag{7}$$

we have

$$S(\alpha;\ x,\ A) \ll Al^{-c_4}, \tag{8}$$

Theorem 4. *There is an absolute positive integer c such that the Diophantine equation* (2) *with prime variables is solvable for*

$$U = N^{2/3}\log^c N,$$

and the asymptotic formula (3) *is true.*

§2. Some Lemmas

To prove Theorem 3 and Theorem 4 we shall need the following lemmas.

Lemma 1.[3, Theorem 18.1.5] *Let $2 \leqslant T \ll x$, χ a character mod$q$, and $\rho = \rho_\chi = \beta_\chi + i\gamma_\chi = \beta + i\gamma$ the non-trivial zero of the Dirichlet L-function. Then we have*

$$\psi(x,\ \chi) = \sum_{n\leqslant x}\chi(n)\Lambda(n) = E_0x - \sum_{|\gamma|\leqslant T}\frac{x^\rho - 1}{\rho} + R_1(x,\ \chi,\ T),$$

$$R_1(x,\ \chi,\ T) \ll xT^{-1}(\log qx)^2,$$

where $E_0 = 1$ if χ is principal; $=0$, otherwise.

Lemma 2.[3, Lemmas 21.1.4, 21.2.2] *Let $F(u)$ be a real function, $G(u)$ a monotonic function satisfying $|G(u)| \leqslant M$ for $a \leqslant u \leqslant b$. Then,* (i) *if $F'(u)$ is monotonic and $|F'(u)| \geqslant m > 0$ for $a \leqslant u \leqslant b$, we have*

$$\int_a^b G(u)e(F(u))du \ll m^{-1}M;$$

(ii) *if $|F''(u)| \geqslant r > 0$ for $a \leqslant u \leqslant b$, we have*

$$\int_a^b G(u)e(F(u))du \ll r^{-1/2}M.$$

Lemma 3.[4, Theorem 3] *Let* $N(\theta, T_1, H_1, q)$ *denote the number of zero of* $\prod_{\chi \bmod q} L(s, \chi)$ *in the region*

$$1/2 \leqslant \theta \leqslant \operatorname{Re} s \leqslant 1, \quad 0 \leqslant T_1 \leqslant \operatorname{Im} s \leqslant T_1 + H_1.$$

Then for $H_1 \geqslant T_1^{1/3}$ *we have*

$$N(\theta, T_1, H_1, q) \ll \begin{cases} (qH_1)^{4(1-\theta)/(3-2\theta)} (\log qH_1)^9, & 1/2 \leqslant \theta \leqslant 3/4, \\ t(qH_1)^{8(1-\theta)/3} (\log qH_1)^{216}, & 3/4 \leqslant \theta \leqslant 1. \end{cases}$$

Lemma 4.[3, Theorem 17.3.2] *For any given* $\varepsilon > 0$ *there exists* $c_8 = c_8(\varepsilon)$ *such that for any real character* $\chi \bmod q$ *and*

$$\sigma \geqslant 1 - c_8 q^{-\varepsilon},$$

we have $L(\sigma, \chi) \neq 0$.

Lemma 5.[3, Theorem 17.4.2] *Let* $q \geqslant 1$, $s = \sigma + it$. *Then there is* c_9 *such that* $\prod_{\chi \bmod q} L(s,\chi)$ *has no zeros in the region*

$$\sigma \geqslant 1 - c_9 (\log q + (\log(|t| + 2))^{4/5})^{-1},$$

exept for the possible exceptional zero mod$q$.

Lemma 6.[3, Corollary 30.3.2] *Using the notations of Lemma 3, we have*

$$N(\theta, 0, H_1, q) \leqslant \min(qH_1 \log(qH_1), (qH_1)^{5(1-\theta)/2} (\log qH_1)^{13}).$$

§3 Proof of Theorem 3

It is well-known that

$$S(\alpha; x, A) = \frac{1}{\phi(q)} \sum_{\chi \bmod p} \Big(\sum_{h=1}^{q} \bar{\chi}(h) e\Big(\frac{ah}{q}\Big)\Big) S(\lambda, \chi) + O(l^2)$$
$$\ll q^{1/2} \phi^{-1}(q) \sum_{\chi \bmod q} |S(\lambda, \chi)| + l^2,$$

where $\phi(n)$ is Euler function and

$$S(\lambda, \chi) = \sum_{x-A<n\leqslant x} \chi(n) \Lambda(n) e(n\lambda).$$

From Lemma 1 it is derived that for $T \ll x$,

$$S(\alpha; x, A) \ll q^{1/2} \phi^{-1}(q) \sum_{\chi \bmod q} \Big| \int_{x-A}^{x} \Big(\sum_{|\gamma|\leqslant T} y^{\rho-1} \Big) e(\lambda y)\, dy \Big|$$
$$+ q^{1/2} \phi^{-1}(q) \min(A, |\lambda|^{-1}) + q^{1/2} (1 + |\lambda| A) x T^{-1} l^2.$$

In what follows we take

$$T = A^{-1} (1 + |\lambda| A) x q^{1/2} l^{c_{10}+2}, \tag{9}$$

and then, if the condition $T \ll x$ is true, we have

$$S(\alpha; x, A) \ll I(\lambda, q) l + q^{-1/2} l \min(A, |\lambda|^{-1}) + A l^{-c_{10}}, \tag{10}$$

where

$$I(\lambda, q) = q^{-1/2} \sum_{\chi \bmod q} \sum_{|\gamma|\leqslant T} \Big| \int_{x-A}^{x} y^{\rho-1} e(\lambda y)\, dy \Big|. \tag{11}$$

Thus, the proof of Theorem 3 is reduced to the estimation of $I(\lambda, q)$. First we are going to prove the following lemma.

Lemma 6. *Let x be a sufficiently large positive number,*

$$2\leqslant A\leqslant x/100. \tag{12}$$

Then, (i) *if*

$$0\leqslant|\lambda|\leqslant xA^{-2}/10, \tag{13}$$

we have

$$I(\lambda, q)\ll q^{-1/2}Al^2 \max_{\substack{1/2\leqslant\beta\leqslant1\\ 0\leqslant T_1\leqslant 2T}} x^{\beta-1}N(\beta, T_1, xA^{-1}, q); \tag{14}$$

(ii) *if*

$$|\lambda|>xA^{-2}/10, \tag{15}$$

we have

$$I(\lambda, q)\ll q^{-1/2}Al^2\sqrt{\frac{x}{|\lambda|A^2}} \max_{\substack{1/2\leqslant\beta\leqslant1\\ 0\leqslant T_1\leqslant 2T}} x^{\beta-1}N(\beta, T_1, 10|\lambda|A, q), \tag{16}$$

where $N(\theta, T_1, H_1, q)$ is defined in Lemma 3, and T is given by (9).

Proof Obviously, we may assume $\lambda\geqslant0$. By Lemma 2 we have

$$\int_{x-A}^{x} y^{\rho-1}e(\lambda y)\,dy\ll x^{\beta-1}\min\left(A, \frac{x}{\sqrt{|\gamma|}}, x\big(\min_{x-A\leqslant y\leqslant x}(\gamma+2\pi\lambda y)\big)^{-1}\right). \tag{17}$$

Setting $y=x+v$, for $x-A\leqslant y\leqslant x$ we have $-A\leqslant v\leqslant 0$; hence

$$|2\pi\lambda v|\leqslant 2\pi\lambda A. \tag{18}$$

Let $H\geqslant1$ be a parameter satisfying

$$T>H\geqslant 10\lambda A. \tag{19}$$

Noticing (18) and (19), it follows from (11) and (17) that

$$I(\lambda, q)\ll q^{-1/2}\sum_{\chi \bmod q}\sum_{\substack{|\gamma|\leqslant T\\ |\gamma+2\pi\lambda x|<H}} x^{\beta-1}\min\left(A, \frac{x}{\sqrt{|\gamma|}}\right)$$
$$+q^{-1/2}\sum_{\chi \bmod q}\sum_{k=2}^{K}\sum_{\substack{|\gamma|\leqslant T\\ (k-1)H\leqslant|\gamma+2\pi\lambda x|<kH}} x^{\beta-1}\frac{x}{(k-1)H}, \tag{20}$$

where $K=[T/H]+1$. If (13) holds we tak $H=xA^{-1}$, and then from (20) and (9) we obtain

$$I(\lambda, q)\ll q^{-1/2}Al\max_{|T_1|\leqslant 2T}\sum_{\chi \bmod q}\sum_{T_1\leqslant\gamma\leqslant T_1+xA^{-1}} x^{\beta-1},$$

from which (14) is derived at once in a standard way. If (15) holds, we take $H=10\lambda A$. Now, by (12) we have

$$-7\lambda x\leqslant\gamma\leqslant-6\lambda x, \text{ for } |\gamma+2\pi\lambda x|\leqslant H.$$

Hence it follows that, for $|\gamma+2\pi\lambda x|\leqslant H$,

$$\min(A, x/\sqrt{|\gamma|})\ll\min(A, \sqrt{x/\lambda})\ll\sqrt{x/\lambda}.$$

By use of this and $x/H\ll\sqrt{x/\lambda}$, it is derived from (20) that

$$I(\lambda, q)\ll q^{-1/2}Al\sqrt{\frac{x}{\lambda A^2}}\max_{|T_1|\leqslant 2T}\sum_{\chi \bmod q}\sum_{T_1\leqslant\gamma\leqslant T_1-10\lambda A} x^{\beta-1},$$

and then (16) follows immediately.

On having Lemma 6 we can apply Lemm3 to the estimation of $I(\lambda, q)$. Now, we take

$$c_5=657+3c_4, \quad c_{10}=c_4, \tag{21}$$

and assume that

$$x^{2/3}l^{c_5}\leqslant A\leqslant 2x^{2/3}l^{c_5}. \tag{22}$$

Lemma 7. *Under the conditions* (21) *and* (22), *if* (13) *holds and* $q\leqslant x^{1/6}$, *then we have*

$$I(\lambda,\ q)\ll Al^{-c_4-1}.$$

Proof Not losing generality we can assume $\lambda\geqslant 0$. It is easy to see that under the conditions (13), (21), (22) and $q\leqslant x^{1/6}$, we have

$$T\ll x^2A^{-2}q^{1/2}l^{c_4+2}\ll x^{3/4},$$

where T is given by (9), and then $xA^{-1}\gg T^{1/3}$. Thus, using Lemma 3 we find from (14) that

$$I(\lambda,\ q)\ll q^{-1/2}Al^2\Big\{\max_{\substack{1/2\leqslant\beta\leqslant 1\\ |\gamma|\leqslant 3T}} x^{\beta-1}(qxA^{-1})^{4(1-\beta)/(3-2\beta)}l^9 + \max_{\substack{3/4\leqslant\beta\leqslant 1\\ |\gamma|\leqslant 3T}} x^{\beta-1}(qxA^{-1})^{8(1-\beta)/3}l^{216}\Big\}. \tag{24}$$

Now we are going to deal with the first term in the bracket. Let

$$g(\beta)=\log\{x^{\beta-1}(qxA^{-1})^{4(1-\beta)/(3-2\beta)}\}.$$

We have

$$g'(\beta)=\log x-4(3-2\beta)^{-2}\log(qxA^{-1}).$$

Under the conditions of the lemma, $qxA^{-1}\leqslant x^{1/2}$. Therefore, for $1/2\leqslant\beta\leqslant 3/4$,

$$g'(\beta)>0.$$

Thus, we get

$$q^{-1/2}\max_{1/2\leqslant\beta\leqslant 3/4} x^{\beta-1}(qxA^{-1})^{4(1-\beta)/(3-2\beta)}\ll q^{-1/2}x^{-1/4}(qxA^{-1})^{2/3} \ll q^{1/6}x^{5/12}A^{-2/3}\ll l^{-438-2c_4}. \tag{25}$$

The second term in the bracket of (24) can be estimated as follows. Take $c_{11}=218+c_4$. If $q\leqslant l^{2c_{11}}$, by Lemmas 4 and 5 we can conclude that there is no zero of $L(s,\ \chi)$ ($\chi\bmod q$) in the region:

$$|\mathrm{Im}\, s|\ll x, \quad \mathrm{Re}\, s\geqslant 1-c_{12}l^{-4/5}.$$

From this fact and $T\ll x$, it follows that under the conditions of the lemma we have

$$q^{-1/2}\max_{\substack{3/4\leqslant\beta\leqslant 1/2\\ |\gamma|\leqslant 3T}} x^{\beta-1}(qxA^{-1})^{8(1-\beta)/3} \ll q^{1/6}x^{5/12}A^{-2/3}+(A^{8/3}x^{-5/3})^{-c_{12}l^{-4/5}}\ll l^{-438-2c_4}. \tag{26}$$

If $l^{2c_{11}}\leqslant q\leqslant x^{1/6}$, it is trivial that

$$q^{-1/2}\max_{3/4\leqslant\beta\leqslant 1/2} x^{\beta-1}(qxA^{-1})^{8(1-\beta)/3}\ll q^{1/6}x^{5/12}A^{-2/3}+q^{-1/2}\ll l^{-219-c_4}. \tag{27}$$

From (24), (25), (26) and (27), the lemma follows.

Lemma 8. *Under the conditions* (15), (21) *and* (22), *if* $q \leqslant x^{1/6}$ *and*

$$|\lambda| \leqslant (qx^{1/6})^{-1} \tag{28}$$

hold, then we have

$$I(\lambda, q) \ll Al^{-c_4-1}.$$

Proof The argument is similar to that of Lemma 7. Assume that $\lambda \geqslant 0$. Under the conditions of this lemma, the parameter T given by (9) satisfies

$$T \ll \lambda x q^{1/2} l^{c_4+2} \ll x^{5/6} l^{c_4+2},$$

and hence $\lambda A \gg xA^{-1} \gg x^{1/3} l^{-c_5} \gg T^{1/3}$. Now, we can applyLemma 3 to (16), and obtain

$$I(\lambda, q) \ll q^{-1/2} Al^2 \sqrt{\frac{x}{\lambda A^2}} \Big\{ \max_{1/2 \leqslant \beta \leqslant 3/4} x^{\beta-1} (\lambda q A)^{4(1-\beta)/(3-2\beta)} l^9$$
$$+ \max_{\substack{1/2 \leqslant \beta \leqslant 3/4 \\ |\gamma| \leqslant 3T}} x^{\beta-1} (\lambda q A)^{8(1-\beta)/3} l^{217} \Big\}. \tag{29}$$

The first term in the bracket can be estimated in a similar way as we estimate that in (24). Noticing that

$$-1/2 + 4(1-\beta)/(3-2\beta) > 0, \quad 1/2 \leqslant \beta \leqslant 3/4,$$

we find from (28) that

$$q^{-1/2} \sqrt{\frac{x}{\lambda A^2}} \max_{1/2 \leqslant \beta \leqslant 3/4} x^{\beta-1} (\lambda q A)^{4(1-\beta)/(3-2\beta)}$$
$$\ll x^{1/12} x^{1/2} A^{-1} \max_{1/2 \leqslant \beta \leqslant 3/4} x^{\beta-1} (xA^{-1/6})^{4(1-\beta)/(3-2\beta)}$$
$$\ll (x^{1/6})^{-1/6} x^{1/4} A^{-1/3} \ll l^{-219-c_4}. \tag{30}$$

Similarly to proving (26) and (27), we can get under the conditions of the lemma: (i) if $q \leqslant l^{2c_{11}}$, it follows that

$$q^{-1/2} \sqrt{\frac{x}{\lambda A^2}} \max_{\substack{3/4 \leqslant \beta \leqslant 1 \\ |\gamma| \leqslant 3T}} x^{\beta-1} (\lambda q A)^{8(1-\beta)/3}$$
$$\ll x^{1/4} (\lambda q)^{1/6} A^{-1/2} + q^{-1/2} (\lambda A^2 x^{-1})^{-1/2} ((\lambda q A)^{8/3} x^{-1})^{c_{14} l^{-4/5}}$$
$$\ll x^{1/4} (\lambda q)^{1/6} A^{-1/3} + (A^{8/3} x^{-5/3})^{-c_{14} l^{-4/5}} \ll l^{-219-c_4}, \tag{31}$$

by Lemmas 4 and 5; (ii) if $q > l^{2c_{11}}$, it is easy to see that

$$q^{-1/2} \sqrt{\frac{x}{\lambda A^2}} \max_{\substack{3/4 \leqslant \beta \leqslant 1 \\ |\gamma| \leqslant 3T}} x^{\beta-1} (\lambda q A)^{8(1-\beta)/3}$$
$$\ll x^{1/4} (\lambda q)^{1/6} A^{-1/3} + q^{-1/2} \ll l^{219-c_4}. \tag{32}$$

Summing up (29), (30), (31) and (32), we complete the proof of the lemma.

Proof of Theorem 3. Now, we are going to prove that Theorem 3 is true for

$$c_6 = 2c_4 + 2, \; c_7 = c_4 + 1, \tag{33}$$

and c_5 given by (21). At first, from Lemmas 7 and 8 we can easily conclude that the theorem is true if (22) holds. In fact, when $q \leqslant x^{1/6}$ and $|\lambda q| \leqslant x^{-1/6}$, the parameter T given by (9) ($c_{10} = c_4$) satisfies $T \ll x$ (see the proofs of Lemmas 7 and 8). Therefore, from (10), Lemmas 7 and 8, the desired conclusion follows at once.

Secondly, we prove that the theorem holds if

$$2x^{2/3}l^{c_5}<A\leqslant x/100.$$

Let $x_1=x$, $A_1=x_1^{2/3}(\log x_1)^{c_5}$, and

$$x_{j+1}=x_j-A_j,\quad A_{j+1}=x_{j+1}^{2/4}(\log x_{j+1})^{c_5}.$$

Now, there exists a positive integer J satisfying

$$x_{J+2}<x-A\leqslant x_{J+1}.$$

Trivially, $x_1>x_2>\cdots>x_{J+1}\geqslant 99x/100$, and

$$x^{2/3}(\log x)^{c_5}\geqslant A_1>A_2>\cdots>A_{J+1}\geqslant(1/2)x^{2/3}(\log x)^{c_5}$$

for sufficient large x. Putting $B=x_J-x+A$, we have

$$S(\alpha;\,x,\,A)=\sum_{j=1}^{J-1}S(\alpha;\,x_j,\,A_j)+(S(\alpha;\,x_J,\,B).$$

Noticing the definition of A_j and $A_J\leqslant B\leqslant 2A_J$, from the above discussion we obtain

$$S(\alpha;\,x,\,A)\ll l^{-c_4}\Big(\sum_{j=1}^{J-1}A_j+B\Big)=Al^{-c_4};$$

this is what we need.

At last, we prove that the theorem is also true for $x/100<A\leqslant x$. Obviously, we only need to treat the case $A=x$.

Before giving the proof, it is necessary to give the following remark. In the above we have proved that the theorem holds for

$$x^{2/3}l^{c_5}<A\leqslant x/100. \tag{34}$$

In fact, from the proofs of Lemmas 7 and 8 we can assert that under condition (34) the theorem is still true if conditions (6) and (7) are replaced by

$$1\leqslant q\ll l^{c_6},\quad A^{-1}l^{c_7}\ll|\lambda|\leqslant(q(xf(x))^{1/6})^{-1}, \tag{35}$$

and

$$l^{c_6}\ll q\leqslant(xf(x))^{1/6},\quad |\lambda|\leqslant(q(xf(x))^{1/6})^{-1}, \tag{36}$$

respectively where $f(x)$ is a real function satisfying

$$l^{-c_{15}}\ll f(x)\ll l^{c_{16}}, \tag{37}$$

c_{15} and c_{16} being any given positive constants.

Let $x_1=x$, $A_1=x_1/100$, and

$$x_{j+1}=x_j-A_j\ A_{j+1}=x_{j+1}/100.$$

Clearly, $x_j=(99/100)^{j+1}x$, and hence there exists an integer J such that $x_{J+1}<xl^{-c_4}<x_J$. Thus we have

$$S(\alpha;\,x,\,x)=\sum_{j=1}^{J}S(\alpha;\,x_j,\,A_j)+O(xl^{-c_4}). \tag{38}$$

Now, take $f_j(x_j)x_j=x$ for $1\leqslant j\leqslant J$. Clearly, condition (37) is true for x_j and f_j, and so we have

$$S(\alpha;\,x_j,\,A_j)\ll A_j(\log x_j)^{-c_4}$$

for $1\leqslant j\leqslant J$ if condition (35) or (36) is true for x_j, A_j and f_j; that means the above estimate follows if

$$1\leqslant q\ll l^{c_6},\quad A_j^{-1}l^{c_7}\ll|\lambda|\leqslant(qx^{1/6})^{-1}. \tag{39}$$

or

$$l^{c_6} \ll q \leqslant x^{1/6}, \quad |\lambda| \leqslant (qx^{1/6})^{-1} \tag{40}$$

holds. From this, (38), and the definition of A_j, it is derived that if the condition

$$1 \leqslant q \ll l^{c_6}, \quad x^{-1} l^{c_7+c_4} \ll |\lambda| \leqslant (qx^{1/6})^{-1}$$

or

$$l^{c_6} \ll q \leqslant x^{1/6}, \quad |\lambda| \leqslant (qx^{1/6})^{-1}$$

holds, we have

$$S(\alpha;\ x,\ x) \ll x l^{-c_4}.$$

This shows that the theorem is true for $A=x$ when $c_5=657+3c_4$, $c_6=2c_4+2$ and $c_7=2c_4+1$. The proof is completed.

Remark. Theorem 3 is still true if comditions (6) and (7) are replaced by (35) and (36) respectively.

§ 4. Proof of Theorem 4

Let $N_1=N/3-U$, $N_2=N/3+U$, and take $c_4=3$, $c_6=8$, $c_7=4$, $c=c_5=666$. By Theorem 3 ($x=N_2$, $A=2U$), there is

$$S(\alpha;\ N_2,\ 2U) \ll U \log^{-3} N,$$

if (6) or (7) holds. From this and

$$S(\alpha;\ N_2,\ 2U) = \log(N/3) \sum_{N_1<p\leqslant N_2} e(p\alpha) + O(N^{-1}U^2), \tag{41}$$

it follows that if (6) or (7) holds we have

$$\sum_{N_1<p\leqslant N_2} e(p\alpha) \ll U \log^{-4} N.$$

Let $Q_1=\log^8 N_2$, $Q=N_2^{1/6}$, $\tau=U\log^{-4} N_2$, and $I(q,\ a)$ denote the interval $[a/q-\tau^{-1},\ a/q+\tau^{-1}]$,

$$E_1 = \bigcup_{1\leqslant q\leqslant Q_1} \ \bigcup_{\substack{0\leqslant a<q \\ (a,q)=1}} I(q,\ a),$$

and

$$E_2 = [-Q^{-1},\ 1-Q^{-1}] \setminus E_1.$$

By the well-known Dirichlet's lemma [3, Lemma 19.3.5], it is derived that for every $\alpha\in E_2$ (6) or (7) ($x=N_2$, $A=2U$) holds. Hence, we hvae

$$\sum_{N_1<p\leqslant N_2} e(p\alpha) \ll U \log^{-4} N, \ \alpha \in E_2$$

and then

$$\int_{E_2} \Big| \sum_{N_1\leqslant p\leqslant N} e(p\alpha) \Big|^3 d\alpha \ll U^2 \log^{-4} N.$$

Consequently,

$$\begin{aligned} T(N,\ U) &= \int_{-1/Q}^{1-1/Q} \Big(\sum_{N_1<p\leqslant N_2} e(p\alpha) \Big)^3 e(-N\alpha)\, d\alpha \\ &= \int_{E_1} + \int_{E_2} = \int_{E_1} + O(U^2 \log^{-4} N). \end{aligned} \tag{42}$$

Now we are going to calculate the integral $\int_{E_1}$. By (41) we have

$$\left(\sum_{N_1<p\leq N_2} e(p\alpha)\right)^3=(\log N/3)^{-3}S^3(\alpha;\ N_2,\ 2U)+O(N^{-1}U^4\log^{-1}N).$$

The measure of E_1 is equal to

$$\sum_{q\leq Q_1}\phi(q)\,(2U^{-1}\log^4 N_2)\ll U^{-2}Q_1^2\log^4 N\ll U^{-1}\log^{20} N. \tag{43}$$

From the last two formulas we obtain

$$T_1(N,\ U)=\int_{E_1}=(\log N/3)^{-3}\int_{E_1} S^3(\alpha;\ N_2,\ 2U)e(-N\alpha)\,d\alpha+O(N^{-1}U^3\log^{19}N). \tag{44}$$

By Lemma 1 we have

$$\sum_{N_1<n\leq N_2}\chi(n)\Lambda(n)e(n\lambda)=E_0\int_{N_1}^{N_2}e(\lambda y)\,dy-\sum_{|\gamma|\leq T}\int_{N_1}^{N_2}e(\lambda y)y^{\rho-1}\,dy$$
$$+\int_{N_1}^{N_2}e(\lambda y)\,dR_1(y,\ \chi,\ T),$$

where $T=NU^{-1}(\log N)^{c_{17}}$, c_{17} is a positive constant to be choosen later. When $\alpha=a/q+\lambda\in E_1$, there is $|\lambda|\leq U^{-1}\log^4 N$, and hence

$$\int_{N_1}^{N_2}e(\lambda y)\,dR_1(y,\chi,\ T)\ll NT^{-1}\log^2 N+|\lambda|UNT^{-1}\log^2 N\ll U(\log N)^{-c_{17}+6}.$$

From the last two formulas it follows that if $\alpha\in a/q+\lambda\in E_1$, there is

$$S\Big(\frac{a}{q}+\lambda;\ N_2,\ 2U\Big)=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\Big(\sum_{h=1}^{q}\bar\chi(h)e\Big(\frac{ah}{q}\Big)\Big)\sum_{N_1<n\leq N_2}\chi(n)\Lambda(n)e(n\lambda)$$
$$+O(\log^2 N)$$
$$=\frac{\mu(q)}{\phi(q)}\int_{N}^{N_3}e(\lambda y)\,dy+I+O(q^{1/2}U(\log N)^{-c_{17}+6}),$$

where $\mu(n)$ is Möbius function, and

$$I\ll\frac{q^{1/2}}{\phi(q)}U\sum_{\chi \bmod q}\sum_{|\gamma|\leq T}N^{\beta-1}.$$

By Lemmas 5 and 6 we have, for $q\leq Q_1$,

$$I\ll Uq^{-1/2}\log N\max_{\substack{1/2\leq\beta\leq1\\|\gamma|\leq T}}N^{\beta-1}N(\beta,\ 0,\ T,\ q)$$
$$\ll Uq^{-1/2}\log^{24}N\max_{\substack{1/2\leq\beta\leq1\\|\gamma|\leq T}}N^{\beta-1}(qT)^{5(1-\beta)/2}$$
$$\ll Uq^{-1/2}\log^{14}N\{N^{-1/2}(qT)^{5/4}+(N^{-1}(qT)^{5/2})^{c_{18}(\log N)^{-4/5}}\}.$$
$$\ll U\exp(-c_{19}(\log N)^{1/5}),$$

using $T=NU^{-1}(\log N)^{c_{17}}=T^{1/3}(\log N)^{-c_1 c_{17}}$ in the last step. Therefore, for $a/q+\lambda\in E_1$ there is

$$S\Big(\frac{a}{q}+\lambda;\ N_2,\ 2U\Big)=\frac{\mu(q)}{\phi(q)}e\Big(\frac{\lambda N}{3}\Big)\frac{\sin(2\pi\lambda U)}{\pi\lambda}+O(U(\log N)^{-c_{17}+10}),$$

and hence

$$S^3\Big(\frac{a}{q}+\lambda;\ N_2,\ 2U\Big)=\frac{\mu(q)}{\phi^3(q)}e(\lambda N)\frac{\sin^3(2\pi\lambda U)}{(\pi\lambda)^3}+O(U^3(\log N)^{-c_{17}+10}).$$

From this and (43), we deduce that

$$\int_E S^3(\alpha;\, N_2,\, 2U)e(-N\alpha)d\alpha$$

$$=\sum_{q<Q_1}\frac{\mu(q)}{\phi^3(q)}\Bigg(\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{-a}{q}N\right)\Bigg)\int_{-1/\tau}^{1/\tau}\frac{\sin^3(2\pi\lambda U)}{(\pi\lambda)^3}d\lambda+O\left(U^2(\log N)^{-c_{17}+30}\right).$$

Using

$$\int_{-1/\tau}^{1/\tau}\frac{\sin^3(2\pi\lambda U)}{(\pi\lambda)^3}d\lambda=\frac{4U^2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^3 y}{y^3}dy+O(U^2\log^{-8}N)$$

$$=3U^2+O(U^2\log^{-8}N)$$

and taking $c_{17}=31$, from the last formula and (44) we have

$$T_1(N,\, U)=\frac{3U^2}{\log^3 N}\sum_{q<Q_1}\frac{\mu(q)}{\phi^3(q)}\Bigg(\sum_{\substack{a=1\\(a\ q)=1}}^{q} e\left(\frac{-a}{q}N\right)\Bigg)+O(U^4\log^{-4}N)$$

$$=3U^2\log^{-3}N\mathfrak{S}(N)+O(U^2\log^{-4}N).$$

From this and (41) the asymptotic formula (3) is derived. Since (4) is a well-known result (see [3, (20.2.11)]), the proof is completed.

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