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ON THE ACCURACY OF THE QUASI-CONFORMING AND GENERALIZED CONFORMING FINITE ELEMENTS

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Abstract

It is observed in practice that the numerical accuracy of the two unconventional plate elements, i. e., the nine parameter quasi-conforming and generalized conforming elements, is better than that of the usual Zienkiewicz in compatible cubic element and of a new element proposed recently by Specht, although all these elements have the same asymptotical rate of convergence $O(h)$ in the energy norm. In the paper a careful error analysis for the quasi-conforming and generalized conforming elements is given. It is shown that the consistency error due to nonconformity of the two unconventional elements is of order $O(h^2)$, one order high than that of other conventional nonconforming elements with nine parameters.

§1. Introduction

There are two kinds of unconventional plate bending elements introduced by Chinese scholars during last years, namely, the quasi-conforming[1] and the generalized conforming[2] elements. Numerical calculations show good results of these elements. By use of the so-called two set parameter method a theoretical analysis for both elements mentioned above is given in [3], [4]. It is proved the two elements satisfy the F-E-M-Test[5], which ensures convergence of the elements. An error estimate is also derived for the nine parameter generalized conforming element which indicates that the difference between the true solution and its finite element approximation is of order $O(h)$ in the energy norm, like Zienkiewicz's incompatible cubic element. However, numerical experiments[6] give evidence of a better convergence performance for the nine parameter quasi-conforming and generalized conforming elements than that for Zienkiewicz's element and Specht's element[7]. How to explain the discrepancy between the theoretical analysis and numerical computations of these two unconventional

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elements? The present paper intends to find an answer to this issue.

Through a careful analysis we have found that the both elements under consideration possess a very specific convergence property, i. e. the consistency error due to nonconformity of the elements has an order $O(h^2)$, one order higher than that of Zienkiewicz's element and Specht's element. As we know, according to the second Strang lemma [8], the error of every nonconforming element consists of two parts, one arises from the interpolation error and the other is a consistency error due to nonconformity of the element. In most cases the order of the consistency error is lower than or equal to that of the interpolation error. Since every element with nine parameters may contain only a full quadratic polynomial, but not a full cubic one, its interpolation error then permits at most of order $O(h)$. Now the consistency error of the two unconventional elements is of order $O(h^2)$, higher than its interpolation error $O(h)$. This particular property seems to be never seen beforefor usual nonconforming elements. It may be a good reason to explain nice convergence behaviours of these two nine parameter elements.

§ 2. Quasi-Comforming Element

Let u^* be the solution of the clamped plate bending problem and $u_h \in V_h$ be a finite element approximation of u^*. According to the well known Strang lemma, the following error estimate in the energy norm for nonconforming approximations holds:

$$|u^*-u_h|_{2,h} \leqslant C\big(\inf_{v_h\in V_h} |u^*-v_h|_{2,h} + \sup_{w_h\in V_h} |E_h(u^*, w_h)|/|w_h|_{2,h}\big), \tag{1}$$

where $h=\max_K h_K$, h_K is the diameter of element K, the semi-norm $|v_h|_{2,h}$ is defined by $|v_h|^2_{2,h}=\sum_K |v_h|^2_{2,K}$, and the consistency error functional is

$$E_h(u^*, w) = E_1(u^*, w) + E_2(u^*, w) + E_3(u^*, w) \tag{2}$$

with

$$E_1(u^*, w) = \sum_K \sum_{F\subset\partial K} \int_F \Big[\Delta u^* - (1-\sigma)\frac{\partial^2 u^*}{\partial s^2}\Big]\frac{\partial w}{\partial n}\,ds, \tag{3}$$

$$E_2(u^*, w) = \sum_K \sum_{F\subset\partial K} \int_F (1-\sigma)\frac{\partial^2 u^*}{\partial n\partial s}\frac{\partial w}{\partial s}\,ds, \tag{4}$$

$$E_3(u^*, w) = \sum_K \sum_{F\subset\partial K} \int_F \frac{\partial(\Delta u^*)}{\partial n}\,ds, \tag{5}$$

$0<\sigma<1/2$ is the Poisson ratio.

Now we analyse carefully the consistency error functional $E_h(u^*, w_h)$ related to the nine parameter quasi-conforming element. Given a triangular element K with vertice $P_i(x_i, y_i)$, $i=1, 2, 3$, we denote by λ_i the area coordinates for the triangle K and put

$$\xi_1=x_2-x_3,\ \xi_2=x_3-x_1,\ \xi_3=x_1-x_2,$$
$$\eta_1=y_2-y_3,\ \eta_2=y_3-y_1,\ \eta_3=y_1-y_2,$$
$$l_{12}^2=\xi_3^2+\eta_3^2,\ l_{23}^2=\xi_1^2+\eta_1^2,\ l_{31}^2=\xi_2^2+\eta_2^2,$$
$$\Delta=\text{the area of } K.$$

As described in [3], the nine parameter quasi-conforming element is identical with a special nonconforming element. The shape function u on each triangle K of this new element is a nine dimensional subspace of a full cubic polynomial space (see [3]). It takes

$$d_1=u_1,\ d_2=u_2,\ d_3=u_3,\ d_4=\frac{1}{l_{12}}\int_{p_1}^{p_2} u\,ds,$$
$$d_5=\frac{1}{l_{23}}\int_{p_2}^{p_3} u\,ds,\ d_6=\frac{1}{l_{31}}\int_{p_3}^{p_1} u\,ds,\ d_7=\int_{p_1}^{p_2}\frac{\partial u}{\partial n}ds,$$
$$d_8=\int_{p_2}^{p_3}\frac{\partial u}{\partial n}ds,\ d_9=\int_{p_3}^{p_1}\frac{\partial u}{\partial n}ds,\ d_{10}=\int_{p_1}^{p_2}\lambda_1\frac{\partial u}{\partial n}ds$$

as ten degrees of freedom. The nine nodal parameters $q_1, q_2, \cdots, q_9$ are then defined through the degrees of freedom $d_1, \cdots, d_{10}$ as follows:

$$d_1=u_1=q_1,\ d_2=u_2=q_4,\ d_3=u_3=q_7,$$
$$d_4=\frac{1}{l_{12}}\int_{p_1}^{p_2} u\,ds=\frac{1}{2}(q_1+q_4)-\frac{\xi_3}{12}(q_2-q_5)-\frac{\eta_3}{12}(q_3-q_6),$$
$$d_5=\frac{1}{l_{23}}\int_{p_2}^{p_3} u\,ds=\frac{1}{2}(q_4+q_7)-\frac{\xi_1}{12}(q_5-q_8)-\frac{\eta_1}{12}(q_6-q_9),$$
$$d_6=\frac{1}{l_{31}}\int_{p_3}^{p_1} u\,ds=\frac{1}{2}(q_7+q_1)-\frac{\xi_2}{12}(q_8-q_2)-\frac{\eta_2}{12}(q_9-q_3),$$
$$d_7=\int_{p_1}^{p_2}\frac{\partial u}{\partial n}ds=-\frac{\eta_3}{2}(q_2+q_5)+\frac{\xi_3}{2}(q_3+q_6),$$
$$d_8=\int_{p_2}^{p_3}\frac{\partial u}{\partial n}ds=-\frac{\eta_1}{2}(q_5+q_8)+\frac{\xi_1}{2}(q_6+q_9),$$
$$d_9=\int_{p_3}^{p_1}\frac{\partial u}{\partial n}ds=-\frac{\eta_2}{2}(q_8+q_2)+\frac{\xi_2}{2}(q_9+q_3),$$
$$d_{10}=\int_{p_1}^{p_2}\lambda_1\frac{\partial u}{\partial n}ds=-\frac{\eta_3}{6}(2q_2+q_5)+\frac{\xi_3}{6}(2q_3+q_6). \tag{6}$$

The components q_1, q_2, q_3 are prescribed to the nodal point p_1, q_4, q_5, q_6 to the nodal point p_2 and q_7, q_8, q_9 to the point p_3.

It has been shown in [3] that for every cubic polynomial $u\in P_3(K)$ after a rather complicated manipulation we have

$$q_1=u_1,\ q_4=u_2,\ q_7=u_3,$$
$$q_2=\frac{1}{2}u_{1x}+\frac{1}{4\Delta}\Big\{2\Big(\xi_3\int_{p_3}^{p_1}\frac{\partial u}{\partial n}ds-\xi_2\int_{p_1}^{p_2}\frac{\partial u}{\partial n}ds\Big)+\xi_3(\eta_2u_{3x}-\xi_2u_{3y})$$
$$-\xi_2(\eta_3u_{2x}-\xi_3u_{2y})\Big\}$$
$$=u_{1x}+O(h^2|u|_{3,\infty,K}),$$

$$
\begin{aligned}
q_3 = &\frac{1}{2}\, u_{1y} + \frac{1}{4\Delta}\Big\{2\Big(\eta_3 \int_{p_3}^{p_1} \frac{\partial u}{\partial n}\, ds - \eta_2 \int_{p_1}^{p_2} \frac{\partial u}{\partial n}\, ds\Big) + \eta_3(\eta_2 u_{3x} - \xi_2 u_{3y}) \\
&- \eta_2(\eta_3 u_{2x} - \xi_3 u_{2y})\Big\} \\
= &u_{1y} + O(h^2 |u|_{3,\infty,K}), \\
q_5 = &\frac{1}{2}\, u_{2x} + \frac{1}{4\Delta}\Big\{2\Big(\xi_1 \int_{p_1}^{p_2} \frac{\partial u}{\partial n}\, ds - \xi_3 \int_{p_2}^{p_3} \frac{\partial u}{\partial n}\, ds\Big) + \xi_1(\eta_3 u_{1x} - \xi_3 u_{1y}) \\
&- \xi_3(\eta_1 u_{3x} - \xi_1 u_{3y})\Big\} \\
= &u_{2x} + O(h^2 |u|_{3,\infty,K}), \\
q_6 = &\frac{1}{2}\, u_{2y} + \frac{1}{4\Delta}\Big\{2\Big(\eta_1 \int_{p_1}^{p_2} \frac{\partial u}{\partial n}\, ds - \eta_3 \int_{p_2}^{p_3} \frac{\partial u}{\partial n}\, ds\Big) + \eta_1(\eta_3 u_{1x} - \xi_3 u_{1y}) \\
&- \eta_3(\eta_1 u_{3x} - \xi_1 u_{3y})\Big\} \\
= &u_{2y} + O(h^2 |u|_{3,\infty,K}), \\
q_8 = &\frac{1}{2}\, u_{3x} + \frac{1}{4\Delta}\Big\{2\Big(\xi_2 \int_{p_2}^{p_3} \frac{\partial u}{\partial n}\, ds - \xi_1 \int_{p_3}^{p_1} \frac{\partial u}{\partial n}\, ds\Big) + \xi_2(\eta_1 u_{2x} - \xi_1 u_{2y}) \\
&- \xi_1(\eta_2 u_{1x} - \xi_2 u_{1y})\Big\} \\
= &u_{3x} + O(h^2 |u|_{3,\infty,K}), \\
q_9 = &\frac{1}{2}\, u_{3y} + \frac{1}{4\Delta}\Big\{2\Big(\eta_2 \int_{p_2}^{p_3} \frac{\partial u}{\partial n}\, ds - \eta_1 \int_{p_3}^{p_1} \frac{\partial u}{\partial n}\, ds\Big) + \eta_2(\eta_1 u_{2x} - \xi_1 u_{2y}) \\
&- \eta_1(\eta_2 u_{1x} - \xi_2 u_{1y})\Big\} \\
= &u_{3y} + O(h^2 |u|_{3,\infty,K}),
\end{aligned}
$$

where u_i, $i=1, 2, 3$, are the function values and u_{ix}, u_{iy}, $i=1, 2, 3$, the two first derivatives of the shape function u at the vertices p_i of K. It is evident that if u is a quadratic polynomial, the nodal parameters q_2, q_3, q_5, q_6, q_8, q_9 coincide with the usual nodal values u_{1x}, u_{1y}, u_{2x}, u_{2y}, u_{3x}, u_{3y}, respectively. In general, q_2, q_3, $\cdots$, q_8, q_9 represent some perturbations of the values u_{1x}, u_{1y}, $\cdots$, u_{3x}, u_{3y}.

It is easily verified that for every cubic polynomial $u \in P_3(K)$, the equality

$$
\int_{p_2}^{p_3} \lambda_2 \frac{\partial u}{\partial n}\, ds = \frac{1}{2} \int_{p_2}^{p_3} \frac{\partial u}{\partial n}\, ds + \frac{1}{6}\Big[\Big(\frac{\partial u}{\partial n}\Big)_2 - \Big(\frac{\partial u}{\partial n}\Big)_3\Big] \tag{7}
$$

holds. By using the formulas of d_8 and q_i, it follows that

$$
\int_{p_2}^{p_3} \lambda_2 \frac{\partial u}{\partial n}\, ds = -\frac{\eta_1}{6}\,(2q_5 + q_8) + \frac{\xi_1}{6}(2q_6 + q_9). \tag{8}
$$

Similarly,

$$
\int_{p_3}^{p_1} \lambda_3 \frac{\partial u}{\partial n}\, ds = -\frac{\eta_2}{6}\,(2q_8 + q_2) + \frac{\xi_2}{6}(2q_9 + q_3). \tag{9}
$$

From (6), (8) and (9) it is seen that the integrals

$$
\int_{p_1}^{p_2} \lambda_1 \frac{\partial u}{\partial n}\, ds, \quad \int_{p_2}^{p_3} \lambda_2 \frac{\partial u}{\partial n}\, ds, \quad \int_{p_3}^{p_1} \lambda_3 \frac{\partial u}{\partial n}\, ds
$$

are dependent only upon the nodal parameters q_i, related to the side p_1p_2, p_2p_3, p_3p_1, respectively. In view of the formulas of d_7, d_8, d_9 and the vanishing of nodal parameters q_i related to boundary points of the problem, we may conclude that for every linear polynomials f on F the integral $\int_F f(s)\frac{\partial u}{\partial s}ds$ is continuous at interelement sides F and turns to zero when F are free sides of elements.

On the other hand, we have

$$\int_{p_1}^{p_2}\lambda_1\frac{\partial u}{\partial s}ds=-q_1+\frac{1}{l_{12}}\int_{p_1}^{p_2}u\,ds$$

$$=-q_1+d_4=\frac{1}{2}(-q_1+q_4)-\frac{\xi_3}{12}(q_2-q_5)-\frac{\eta_3}{12}(q_3-q_6),$$

$$\int_{p_1}^{p_2}\lambda_2\frac{\partial u}{\partial s}ds=q_4-d_4=\frac{1}{2}(-q_1+q_4)+\frac{\xi_3}{12}(q_2-q_5)+\frac{\eta_3}{12}(q_3-q_6),$$

$$\int_{p_2}^{p_3}\lambda_2\frac{\partial u}{\partial s}ds=-q_4+\frac{1}{l_{23}}\int_{p_2}^{p_3}u\,ds$$

$$=-q_4+d_5=\frac{1}{2}(-q_4+q_7)-\frac{\xi_1}{12}(q_5-q_8)-\frac{\eta_1}{12}(q_6-q_9),$$

$$\int_{p_2}^{p_3}\lambda_3\frac{\partial u}{\partial s}ds=q_7-d_5=\frac{1}{2}(-q_4+q_7)+\frac{\xi_1}{12}(q_5-q_8)+\frac{\eta_1}{12}(q_6-q_9),$$

$$\int_{p_3}^{p_1}\lambda_3\frac{\partial u}{\partial s}ds=-q_7+\frac{1}{l_{31}}\int_{p_3}^{p_1}u\,ds$$

$$=-q_7+d_6=\frac{1}{2}(-q_7+q_1)-\frac{\xi_2}{12}(q_8-q_2)-\frac{\eta_2}{12}(q_9-q_3),$$

$$\int_{p_3}^{p_1}\lambda_1\frac{\partial u}{\partial s}ds=q_1-d_6=\frac{1}{2}(-q_7+q_1)+\frac{\xi_2}{12}(q_8-q_2)+\frac{\eta_2}{12}q_9-q_3).$$

Therefore, for every linear polynomial f on F, the integral $\int_F f(s)\frac{\partial u}{\partial s}ds$ is also continuous at interelement sides F and turns to zero when F are free sides of elements.

Now we consider the first two terms $E_1(u^*, w_h)$ and $E_2(u^*, w_h)$ of the consistency error functional $E_h(u^*, w_h)$. For every $\varphi\in L^2(F)$ let

$$P_0^F\varphi=\frac{1}{|F|}\int_F\varphi\,ds,\quad |F|=\int_F 1\,ds$$

be the mean value of φ over F. The remainder term is

$$R_0^F\varphi=\varphi-P_0^F\varphi.$$

Further, let $P_1\varphi$ be the piecewise linear interpolation at the vertices of triangles. The remainder term then is

$$R_1\varphi=\varphi-P_1\varphi.$$

Then, application of interpolation theory yields

$$\int_F(R_0^F\varphi)^2ds\leqslant Ch_K|\varphi|_{1,K}^2,\quad\int_{\partial b}(R_1\varphi)^2d\varphi\leqslant Ch_K^3|\varphi|_{2,K}^2. \tag{10}$$

Set

$$\Delta u^* - (1-\sigma)\frac{\partial^2 u^*}{\partial s^2} = \varphi_1(s), \quad (1-\sigma)\frac{\partial^2 u^*}{\partial n\,\partial s} = \varphi_2(s),$$

the piecewise linear interpolations $P_1\varphi$ and $P_1\varphi_2$ are continuous functions over all elements K. As mentioned above, for every linear polynomial f on F the integrals

$$\int_F f(s)\,\frac{\partial u}{\partial n}\,ds, \quad \int_F f(s)\,\frac{\partial u_h}{\partial s}\,ds$$

are continuous at interelement sides F and vinish when F are free sides of elements. Therefore,

$$\sum_K \sum_{F\subset\partial K} \int_F P_1\varphi\,\frac{\partial w_h}{\partial n}\,ds = 0, \quad \sum_K \sum_{F\subset\partial K} \int_F P_1\varphi\,\frac{\partial w}{\partial s}\,ds = 0,$$

$$\sum_K \sum_{F\subset\partial K} \int_F \varphi_1 P_0^F\left(\frac{\partial w_h}{\partial n}\right)ds = 0, \quad \sum_K \sum_{F\subset\partial K} \int_F \varphi_2 P_0^F\left(\frac{\partial w_h}{\partial s}\right)ds = 0,$$

and so

$$E_1(u^*, w_h) = \sum_K \sum_{F\subset 2K} \int_F \varphi_1\,\frac{\partial w_h}{\partial n}\,ds = \sum_K \sum_{F\subset\partial K} \int_F R_1\varphi_1 R_0^F\left(\frac{\partial w_h}{\partial n}\right)ds,$$

$$E_2(u^*, w_h) = \sum_K \sum_{F\subset\partial K} \int_F \varphi_2\,\frac{\partial w_h}{\partial s}\,ds = \sum_K \sum_{F\subset\partial K} \int_F R_1\varphi_2 R_0^F\left(\frac{\partial w_h}{\partial s}\right)ds.$$

Applying Schwarz's inequality and the inequalities of (10) gives

$$|E_1(u^*, w_h)| = \sum_K \sum_{F\subset\partial K}\left(\int_F (R_1\varphi_1)^2\,ds\right)^{\frac{1}{2}}\left(\int_F R_0^F\left(\frac{\partial w_h}{\partial n}\right)\right)^2 ds\Big)^{\frac{1}{2}}$$
$$\leqslant Ch^2|\varphi_1|_2|w_h|_{2,h} \leqslant Ch^2|u^*|_4|w_h|_{2,h}, \tag{11}$$

$$|E_2(u^*, w_h)| \leqslant \sum_K \sum_{F\subset\partial K}\left(\int_F (R_1\varphi_2)^2\,ds\right)^{\frac{1}{2}}\left(\int_F\left(R_0^F\left(\frac{\partial w_h}{\partial s}\right)\right)^2 ds\right)^{\frac{1}{2}}$$
$$\leqslant Ch^2|\varphi_2|_2|w_h|_{2,h} \leqslant Ch^2|u^*|_4|w_h|_{2,h}. \tag{12}$$

As for the third term $E_3(u^*, w_h)$ of $E_h(u^*, w_h)$, since the shape function w_h is continuous at the vertices of triangles, the piecewise linear interpolation $P_1 w_h$ is continuous function over all elements and then

$$\sum_K \sum_{F\subset\partial K} \int_F \varphi_3 P_1 w_h\,ds = 0$$

with $\varphi_3 = -\frac{\partial(\Delta u^*)}{\partial n}$. Moreover, the continuity of $\int_F w_h ds$ on interelement sides gives

$$\sum_K \sum_{F\subset\partial K} \int_F (P_0\varphi_3)\,w_h\,ds = 0$$

and so

$$E_3(u^*, w_h) = \sum_K \sum_{F\subset\partial K} \int_F \varphi_3 w_h\,ds = \sum_K \sum_{F\subset\partial K} \int_F (R_0\varphi_3)(P_1 w_h)\,ds.$$

Using the inequalities of (10) we have

$$|E_3(u^*, w_h)| \leqslant \sum_K \sum_{F\subset\partial K}\left(\int_F (R_0\varphi_3)^2\,ds\right)^{\frac{1}{2}}\left(\int_F (P_1 w_h)^2\,ds\right)^{\frac{1}{2}}$$
$$\leqslant Ch^2|\varphi_3|_1|w_h|_{2,h} \leqslant Ch^2|u^*|_4|w_h|_{2,h}, \tag{13}$$

which in conjunction with (11) and (12) yields that

$$|E_h(u^*, w_h)| \leqslant Ch^2 |u^*|_4 |w_h|_{2,h}.$$

Therefore

$$\sup_{w_h \in V_h} \frac{|E_h(u^*, w_h)|}{|w_h|_{2,h}} \leqslant Ch^2 |u^*|_4.$$

It means that the consistency error of the nine parameter quasi-conforming element is of order $O(h^2)$, one order higher than that of other nine parameter conventional elements, being of order $O(h)$.

§ 3. Generalized Comforming Element

According to [4], the nine parameter generalized conforming element may also be identical with a nonconforming cubic one, having a special set of nine degrees of freedom as follows:

$$d_1 = \int_{p_1}^{p_2} \frac{\partial u}{\partial s} ds, \qquad d_2 = \int_{p_2}^{p_3} \frac{\partial u}{\partial s} ds,$$

$$d_3 = 60\left\{\frac{1}{l_{12}} \int_{p_1}^{p_2} u\left(\lambda_2 - \frac{1}{2}\right) ds + \frac{1}{l_{23}} \int_{p_2}^{p_3} u\left(\lambda_3 - \frac{1}{2}\right) ds + \frac{1}{l_{31}} \int_{p_3}^{p_1} u\left(\lambda_1 - \frac{1}{2}\right) ds\right\},$$

$d_4, d_5, \cdots, d_9$ are the same as those for the quasi-conforming element. The nine parameters, denoted again by q_i, $i=1, 2, \cdots, 9$, are defined through d_i, $i=1, 2, \cdots, 9$, in the following way

$$d_1 = \int_{p_1}^{p_2} \frac{\partial u}{\partial s} ds = q_4 - q_1, \qquad d_2 = \int_{p_2}^{p_3} \frac{\partial u}{\partial s} ds = q_7 - q_4,$$

$$d_3 = -\frac{1}{2}(\xi_1 q_2 + \eta_1 q_3 + \xi_2 q_5 + \eta_2 q_6 + \xi_3 q_8 + \eta_3 q_9),$$

the formulations of $d_4, \cdots, d_9$ by $q_1, q_2, \cdots, q_9$ are the same as those for the quasi-conforming element.

It is seen that the definition of the first three degrees of freedom d_1, d_2 and d_3 in the quasi-conforming and in the generalized conforming elements is quite different and there is an additional degree of freedom d_{10} in the quasi-conforming element. In spite of all these differences, howerer, it is proved in [4] that the relationship between the nodal parameters q_i, $i=1, 2, \cdots, 9$, of the generalized conforming element and its shape function u is exactly the same as that for the quasi-conforming element (see section 2).

Suppose u is the shape function of the generalized conforming element, being an incomplete cubic polynomial with nine terms on the triangle K. Let us consider the integral $\int_{p_1}^{p_2} \lambda_1 \frac{\partial u}{\partial n} ds$, corresponding to the tenth degree of freedom of the quasi-conforming element. As mentioned above, for every cubic polynomial $u \in P_3(K)$ the equality

$$\int_{p_1}^{p_2} \lambda_1 \frac{\partial u}{\partial n} ds = \frac{1}{2} \int_{p_1}^{p_2} \frac{\partial u}{\partial n} ds + \frac{1}{6} \left[\left(\frac{\partial u}{\partial n} \right)_1 - \left(\frac{\partial u}{\partial n} \right)_2 \right]$$

hold. Substituting $d_7 = \int_{p_1}^{p_2} \frac{\partial u}{\partial n} ds$ and q_i as formulated in Section 2 into the right-hand side, after some calculations we find that

$$\int_{p_1}^{p_2} \lambda_1 \frac{\partial u}{\partial n} ds = -\frac{\eta_3}{6} (2q_2 + q_5) + \frac{\xi_3}{6} (2q_3 + q_6),$$

which has exactly the same form as the tenth degree of freedom d_{10} of the quasi-conforming element, discretized by the nodal parameters q_i.

Similarly, the integrals $\int_{p_2}^{p_3} \lambda_2 \frac{\partial u}{\partial n} ds$, $\int_{p_3}^{p_1} \lambda_3 \frac{\partial u}{\partial n} ds$ as well as the integrals

$$\int_{p_1}^{p_2} \lambda_1 \frac{\partial u}{\partial s} ds, \quad \int_{p_2}^{p_3} \lambda_2 \frac{\partial u}{\partial s} ds, \quad \int_{p_3}^{p_1} \lambda_3 \frac{\partial u}{\partial s} ds$$

may also be formulated through the nodal parameters q_i in the same fashion as those for the quasi-conforming element.

In view of the discussion in Section 2, the consistency error functional of the generalized conforming element can be estimated as follows

$$|E_h(u^*, w_h)| \leqslant Ch^2 |u^*|_4 |w_h|_{2,h}.$$

Therefore, the consistency errors of both the generalized conforming and the quasi-conforming elements have the same order $O(h^2)$.

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