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UNIQUENESS OF VISCOSITY SOLUTIONS OF FULLY NONLINEAR SECOND ORDER PARABOLIC PDE'S

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Abstract

This paper investigates the maximum principle for viscosity solutions of fully nonlinear, second order parabolic, possibly degenerate partial differential equations. Using this maximum principle the author prove that viscosity solutions of initial and bo unoary value problem for parabolic equations are unique.

§ 1. Introduction

In this paper, we are concerned with viscosity solutions of the following parabolic type, fully nonlinear equations

$$u_t - F(x, t, u, Du, D^2u) = 0 \quad \text{in } Q, \tag{1.1}$$

where $Q = \{(x, t) \mid x \in \Omega_t, t \in (0, T]\}$ for some $T > 0$, Ω_t are open sets in R^n. F is a real valued continuous function on $\Gamma = \bar{Q} \times R \times R^n \times S^n$ and S^n denotes the space of $n \times n$ real symmetric matrices. Du and D^2u denote respectively the gradient of u and the Hessian of u with respect to the x coordinates.

Let $\partial_p Q$ denote set $(\Omega_0 \times \{t=0\}) \cup \{(x,t) \mid x \in \partial\Omega_t, t \in [0, T]\}$. We shall treat in this paper viscosity solutions of the following initial and boundary value problem

$$u(x, t) = \psi(x, t), \quad (x, t) \in \partial_p Q, \tag{1.2}$$

for parabolic equation (1.1).

Throughout this paper, we suppose that (1.1) is degenerate parabolic, i. e. for all $(x, t, z, p, X) \in \Gamma$, $Y \in S^n$, the following inequality holds

$$F(x, t, z, p, x) \leqslant F(x, t, z, p, X+Y) \quad \text{if} \quad Y \geqslant 0, \tag{1.3}$$

where we endow S^n with the usual partial ordering. Notice that this assumption covers in particular the completely degenerate case when F does not depend on X, i. e. the particular case of first order H-J eqations of the following form

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$$u_t - F(x, t, u, Du) = 0. \tag{1.4}$$

Because of this possible degeneracy and the highly nonlinear form of the equations considered here, classical solutions cannot be expected to exist.

Uniqueness of the elliptic equations, based on the maximum principle, was studied in [1, 2, 3]. In [3] parabolic equations was studied as a special case of elliptic equations, but his idea did not reflects the essential feature of parabolic type equations and the usual result cannot be expected to be obtained.

Our goal here are to develop maximum principle for viscosity solutions of parabolic equation (1.1), and to prove uniqueness of viscosity solutions of initial and boundary value problem (1.1)—(1.2).

§ 2. Viscosity Solutions

We first recall the definition of viscosity solutions of equation (1.1).

Definition 2. 1. *Let u be an upper-semicontinuous (resp. lower semicontinuous) function on Q. u is said to be a viscosity subsolution of* (1.1) *(resp. supersolution) if for all $\varphi \in C^{2,1}(Q)$ the following inequality holds at each local maximum (resp. minimum) point $(x_0, t_0) \in Q$ of $u - \varphi$*

$$\varphi_t(x_0, t_0) - F(x_0, t_0, u(x_0, t_0), D\varphi(x_0, t_0), D^2\varphi(x_0, t_0)) \leqslant 0 \tag{2.1}$$

(resp.

$$\varphi_t(x_0, t_0) - F(x_0, t_0, u(x_0\ t_0), D\varphi(x_0, t_0), D^2\varphi(x_0, t_0)) \geqslant 0). \tag{2.2}$$

Then $u \in C(Q)$ is said to be a viscosity solution of (1.1) *if u is a viscosity subsolution and supersolution of* (1.1).

Remark. *It is possible to replace local by global, or local strict or global strict.*

In order to proceed we need to define the superdifferential and subdifferential to a function u.

Definition2. 2. *Let u be a real valued function on Q. The superdifferential, $D^+u(x, t)$, is defined as the set*

$$D^+u(x, t) = \Big\{(p, q, M) \in R^n \times R \times S^n \,\Big|\, \limsup_{z\to 0}\sup_{s\to 0} \frac{1}{|z|^2 + |s|}\Big[u(x+z, t+s) - u(x, t) - p\cdot z - q\cdot s - \Big(\frac{M}{2}z, z\Big)\Big] \leqslant 0\Big\}. \tag{2.3}$$

The subdifferential, $D^-u(x, t)$, is defined as the set

$$Du^-(x, t) = \Big\{\{(p, q, M) \in \mathbf{R}^n \times \mathbf{R} \times S^n \,\Big|\, \liminf_{z\to 0}\inf_{s\to 0} \frac{1}{|z|^2 + |s|}\Big[u(x+z, t+s) - u(x, t) - p\cdot z - q\cdot s - \Big(\frac{M}{2}z, z\Big)\Big] \geqslant 0\Big\}. \tag{2.4}$$

The following similar to Lemma 2.15 of [1] can be proved by an analogous argument.

Proposition 2. 3. *Let $u \in C(Q)$. The following are equivalent.*

(i) *u is a viscosity subsolution (resp. supersolution) of* (1.1).

(ii) *$q-F(x, t, u(x, t), p, M) \leqslant 0$ (resp. $q-F(x, t, u(x, t), p, M) \geqslant 0$) for all $(x, t) \in Q$ and all $(p, q, M) \in D^+u(x, t)$ (resp. $(p, q, M) \in Du^-(x, t)$).*

In order to obtain the maximum principle we will try to compare an u. s. c., left continuous for t, bounded viscosity subsolution u of (1.1) and an l. s. c., left continuous for t, bounded viscosity supersolution v of (1.1). At first let us construct the appropriate regularizations of u and v. We set for $\varepsilon \in (0, 1]$:

$$u^\varepsilon(x, t) = \sup_{(y,\tau)\in Q_t} \left\{ u(y, \tau) - \frac{1}{\varepsilon}(|x-y|^2 + t - \tau) \right\}, \quad (x, t) \in Q^\varepsilon, \tag{2.5}$$

$$v_\varepsilon(x, t) = \inf_{(y,\tau)\in Q_t} \left\{ v(y, \tau) + \frac{1}{\varepsilon}(|x-y|^2 + t - \tau) \right\}, \quad (x, t) \in Q^\varepsilon, \tag{2.6}$$

where $Q_t = Q \cap \{\tau \leqslant t\}$ and $Q^\varepsilon = \{(x, t) \in Q \mid d_p((x, t), \partial_p Q) > \varepsilon\}$.

Lemma 2. 4. *The appropriate regularizations u^ε, v_ε satisfy the following properties*

(a) *The (y, τ)'s in (2.5)—(2.6) may be restricted by $Q_{c_0\sqrt{\varepsilon}}(x, t)$ where* $Q_\delta(x, t) = \{(y, \tau) \in Q_t \mid |x-y|^2 + t - \tau \leqslant \delta^2\}$ *for $\delta > 0$ and*

$$c_0 = [\max(2\sup|u|, 2\sup|v|)]^{1/2}.$$

(b) $\sup|u^\varepsilon| \leqslant \sup|u|$, $\sup|v_\varepsilon| \leqslant \sup|v|$.

(c) *u^ε and v_ε are Lipschitz continuous with respect to x.*

(d) *$u^\varepsilon + \frac{1}{\varepsilon}(|x|^2 + t)$ is convex in x and monotone increasing in t; $v_\varepsilon - \frac{1}{\varepsilon}(|x|^2 + t)$ is concave in x and monotone decreasing in t.*

(e) $\lim\limits_{\varepsilon\to 0} u^\varepsilon(x, t) = u(x, t)$ $\lim\limits_{\varepsilon\to 0} v_\varepsilon(x, t) = v(x, t)$, $\forall (x, t) \in Q$.

(f) *u^ε and v_ε are continuous with respect to t.*

Proof The properties (a), (b), (c) and (d) can be proved easily.

By the definition of regularization u^ε we have $u(x, t) \leqslant u^\varepsilon(x, t) \leqslant \sup\limits_{(y,\tau)\in Q_{c_0\sqrt{\varepsilon}}(x,t)} u(y, \tau)$, so $u(x, t) \leqslant \liminf\limits_{\varepsilon\to 0} u^\varepsilon(x, t) \leqslant \limsup\limits_{\varepsilon\to 0} u^\varepsilon(x, t) \leqslant \lim\limits_{\varepsilon\to 0} \sup\limits_{Q_{c_0\sqrt{\varepsilon}}(x,t)} u(y, \tau) = \limsup\limits_{(y,\tau)\to(x,t)} u(x, \tau) \leqslant u(x, t)$. Property (e) is established by these inequalities.

In order to prove property (f) set $f(x, t) = u^\varepsilon(x, t) + \varepsilon^{-1}t$ and $g(x, y, \tau) = u(y, \tau) + \varepsilon^{-1}\tau - \varepsilon^{-1}|y-x|^2$; then $f(x, t) = \sup\limits_{(y,\tau)\in Q_t} g(x, y, \tau)$. We prove that $f(x, t)$ is a continuous function of t. Using continuity from the left of g we conclude that for each fixed $(x\ t) \in Q^\varepsilon$ and for all $\delta > 0$, there exists $(y_\delta, \tau_\delta) \in Q_t$, $\tau_\delta < t$, such that $f(x, t) \leqslant g(x, y_\delta, \tau_\delta) + \delta$. Applying the monotonicity of funtion $f(x, t)$ we obtain

$$\lim_{\tau\to t^-} f(x, \tau) \leqslant f(x, t) \leqslant f(x, \tau_\delta) + \delta \leqslant \lim_{\tau\to t^-} f(x, \tau) + \delta.$$

Letting $\delta \to 0$ we obtain the continuity from the left for $f(x, \cdot)$. In an analogous manner we conclude that $f(x, \cdot)$ is right continuous.

The results for v_ε can be proved in an analogous manner.

We are now ready to relate the regularizations u^ε and v_ε with viscosity subsolutions and supersolu tions, respectively.

Lemma 2. 5. *Assume that F is degenerate parabolic and satisfies the following condition*

$$F(x, t, u, p, A)\leq F(x, t, v, p, A), \ \forall\ (x, t, u, p, A)\in\Gamma,\ v\leq u. \tag{2.7}$$

Then u^ε is a viscosity subsolution of

$$(u^\varepsilon)_t - F_\varepsilon(x, t, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon)=0 \quad in\ Q^\varepsilon, \tag{2.8}$$

where $F_\varepsilon(x, t, z, p, A)=sup\{F(y, \tau, z, p, A)\,|\,(y, \tau)\in Q_{c_0\sqrt{\varepsilon}}(x, t)\}$.

Similarly, v_ε is a viscosity supersolution of

$$(v_\varepsilon)_t - F^\varepsilon(x, t, v_\varepsilon, Dv_\varepsilon, D^2v_\varepsilon)=0 \quad in\ Q^\varepsilon, \tag{2.9}$$

where $F^\varepsilon(x, t, z, p, A)=\inf\{F(y, \tau, z, p, A)\,|\,(y, \tau)\in Q_{c_0\sqrt{\varepsilon}}(x, t)\}$.

Proof Let $\varphi\in C^{2,1}(Q^\varepsilon)$ and $(x_0, t_0)\in Q^\varepsilon$ such that $(u^\varepsilon-\varphi)(x_0, t_0)=\sup(u^\varepsilon-\varphi)$. We take point $(x_1, t_1)\in Q_{c_0\sqrt{\varepsilon}}(x_0, t_0)$ satisfying

$$u^\varepsilon(x_0, t_0)=u(x_1, t_1)-\frac{1}{\varepsilon}(|x_0-x_1|^2+t_0-t_1).$$

Define

$$\Phi_\varepsilon(x, t)=\varphi(x-x_1+x_0, t-t_1+t_0)+\frac{1}{\varepsilon}(|x_0-x_1|^2+t_0-t_1)\in C^{2,1}(Q).$$

Then we have

$$\begin{aligned}(u-\Phi_\varepsilon)(x, t)&\leq(u^\varepsilon-\varphi)(x-x_1+x_0, t-t_1+t_0)\\ &\leq(u^\varepsilon-\varphi)(x_0, t_0)=(u-\Phi_\varepsilon)(x_1, t_1),\end{aligned}$$

so (x_1, t_1) is a local maximum point of $u-\Phi_\varepsilon$. Since u is a viscosity subsolution of (1.1), we have

$$[(\Phi_\varepsilon)_t-F(\cdot, \cdot, u, D\Phi_\varepsilon, D^2\Phi_\varepsilon)]_{(x_1,t_1)}\leq 0.$$

Since F is nonincreasing and $(x_1, t_1)\in Q_{c_0\sqrt{\varepsilon}}(x_0, t_0)$, we conclude by the definition of Φ_ε that

$$[\varphi_t-F_\varepsilon(\cdot, \cdot, u^\varepsilon, D\varphi, D^2\varphi)]_{(x_0,t_0)}\leq 0.$$

This implies that u^ε is a viscosity subsolution of equation (2.8).

The result for v_ε can of course be proved in an analogous manner.

§ 3. The Maximum Principle and Uniqueness

We begin our development with an estimation for the size of the set of points with small derivatives for a semi-convex function having an interior maximum.

Lemma 3. 1. *Let Ω be a bounded open set in $\mathbf{R}^n$ and T is a positive constant, $Q=\Omega\times(0, T]$. Assume $u\in C(\bar{Q})$, $D_xu\in L^\infty(Q)$, $u_t\in L^1(Q)$ and*

$$D^2u\geq -KI \quad in\ Q\ (in\ the\ sense\ of\ distributions), \tag{3.1}$$

where $K>0$. If $(x_0, T)\in Q$ is a global strict maximum point of u, then there are a *positive constant* δ_0 *independent of* K *and a positive function* $\varepsilon(t)$ *with* $\lim_{t\to 0}\varepsilon(t)=0$ *such that*

$$\text{mes}\,(A_\delta)>0 \qquad \textit{for all } \delta\leqslant\delta_0 \tag{3.2}$$

where

$$A_\delta=\{(x,t)\in Q_\delta(x_0,T)\,|\,|Du(x,t)|\leqslant\varepsilon(\delta),$$
$$u(y,s)\leqslant u(x,t)+Du(x,t)(y-x) \text{ for all } (y,s)\in Q_\delta(x,t)\}.$$

Proof Let us take a positive constant δ_0 such that $Q_{2\delta_0}(x_0,T)\subset Q$. Define a new function $\tilde{u}$ in $\tilde{Q}=\Omega\times(0,T+\delta_0]$ by

$$\tilde{u}(x,t)=\begin{cases}u(x,t), & t\in(0,T],\\ u(x,2T-t), & t\in(T,T+\delta_0].\end{cases}$$

It is obvious that $\tilde{u}$ has global strict maximum at point $(x_0,T)\in\tilde{Q}$. Now set

$$\tilde{u}_\eta(x,t)=\int_{R^{n+1}}\rho(y,s)\tilde{u}(x-\eta y,t-\eta s)dyds,\ (x,t)\in\tilde{Q}_\eta, \tag{3.3}$$

where $\tilde{Q}_\eta=\overline{\Omega}_\eta\times[\eta,T+\delta_0-\eta]$, ρ is a mollifier in R^{n+1}.

Let us suppose that $\tilde{u}_\eta$ takes on its maximum at $(x_\eta,t_\eta)\in\mathring{\tilde{Q}}_\eta$. It follows that there exists a subsequence of (x_η,t_η) (denoted also by (x_η,t_η)) such that $(x_\eta,t_\eta)\to(x_0,T)$, so there exists $\eta_0>0$ such that $Q_{\delta_0}(x_\eta,t_\eta)\subset\tilde{Q}$ for all $\eta\leqslant\eta_0$.

In order to complete the proof of this lemma, we set

$$\varepsilon(\delta)=\frac{2}{\delta_0}[\tilde{u}(x_0,T)-\sup_{\partial_pQ_\delta(x_0,T)}\tilde{u}],$$

$$\varepsilon_\eta(\delta)=\frac{2}{\delta_0}[u(x_0,T)-\sup_{\partial_pQ_\delta(x_\eta,t_\eta)}\tilde{u}_\eta]\to\varepsilon(\delta)\quad(\eta\to 0),$$

$$Q_\delta^\eta=B_{\delta_0}(x_\eta)\times(t_\eta-\delta^2,t_\eta).$$

Denote upper contact set of $v_\eta=\tilde{u}_\eta-\sup_{\partial_pQ_\delta(x_\eta,t_\eta)}\tilde{u}_\eta$ on Q_λ^δ by Q_v^+ then we have (see [4, Chapter 7])

$$(\varepsilon_\eta(\delta))^{n+1}\leqslant C(n)\int_{Q_{v_\eta}^+}(\tilde{u}_\eta)_t\det(D^2\tilde{u}_\eta)\,dxdt. \tag{3.4}$$

$$\text{mes}(Q_{v_\eta}^+\setminus A_\delta^\eta)=0. \tag{3.5}$$

On the other hand, by (3.1) and since $\tilde{u}_\eta$ is concave down at every point $(x,t)\in A_\delta^\eta$ we obtain

$$|\det(D^2\tilde{u}_\eta)|\leqslant K^n \quad \text{in } A_\delta^\eta.$$

Combining this inequality with (3.4), (3.5) we conclude that

$$(\varepsilon_\eta(\delta))^{n+1}\leqslant C(n)K^n\int_{A_\delta^\eta}(\tilde{u}_\eta)_t\,\text{dxdt}, \text{ for all } \eta<\eta_0 \text{ and all } \delta<\delta_0 \tag{3.6}$$

where A_δ^η is defined analogously to A_δ. We claim that for any sequence $\eta_i\downarrow 0$

$$\text{mes}\,[(\limsup_{i\to\infty}A_\delta^{\eta_i})\setminus A_\delta]=0. \tag{3.7}$$

Indeed, for a. e. $(x,t)\in\limsup_{i\to\infty}A_\delta^{\eta_i}$ we have

$$\tilde{u}_{\eta_i}(x,t)\to\tilde{u}(x,t);\quad D\tilde{u}_{\eta_i}(x,t)\to D\tilde{u}(x,t).$$

We can assume w. l. o. g. that $(x, t)\in A_\delta^{\eta_i}$ for all i. By the definition of $A_\delta^{\eta_i}$ we know

$$\tilde{u}_{\eta_i}(y, s)\leqslant\tilde{u}_{\eta_i}(x, t)+D\tilde{u}_{\eta_i}(x, t)(y-x) \quad \text{for all } (y, s)\in Q_\delta(x, t),$$
$$|D\tilde{u}_{\eta_i}(x, t)|\leqslant\varepsilon(\delta).$$

We may also assume w. l. o. g. that $\tilde{u}_{\eta_i}(x, t)\to\tilde{u}(x, t)$ and $D_{\eta_i}(x, t)\to D\tilde{u}(x, t)$. Letting $i\to\infty$ we conclude that $(x, t)\in A_\delta$. Thus our claim is verified.

By (3.6) we have for $m=1, 2, \cdots\cdots$

$$[\varepsilon_{\eta_m}(\delta)]^{n+1}\leqslant C(n)K^n\int_{\bigcup\limits_{i=m}^{\infty}A_\delta^{\eta_i}}|(\tilde{u}_{\eta_m})_t|\,dx\,dt$$
$$\leqslant C(n)K^n\left[\int_{A_\delta}|\tilde{u}_t|+\int_{\bigcup\limits_{i=m}^{\infty}A_\delta^{\eta_i}}|\tilde{u}_t|+\int_{\tilde{Q}}|(\tilde{u}_{\eta_m})_t-\tilde{u}_t|\right]. \tag{3.8}$$

Letting $m\to\infty$, by (3.7) and since $\tilde{u}_t\in L^1(\tilde{Q})$ we obtain

$$[\varepsilon(\delta)]^{n+1}\leqslant C(n)K^n\int_{A_\delta}|u_t|\,dxdt, \text{ for all } \delta<\delta_0.$$

Clearly $\mathrm{mes}(A_\delta)>0$ for all $\delta<\delta_0$ and this completes the proof of this lemma.

Corollary 3.2. *If u has a global maximum at point $(x_0\ T)\in Q$, then A_δ is replaced by*

$$A'_\delta=\{(x, t)\in Q_\delta(x_0, T)\,|\,|Du(x, t)|\leqslant\varepsilon(\delta)+2\delta^2,\ u(y, s)$$
$$\leqslant u(x, t)+Du(x, t)(y-x)+\delta(|y-x|^2+|t-s|),\ (y, s)\in Q_\delta(x, t)\}. \tag{3.9}$$

We have $\mathrm{mes}(A'_\delta)>0$ also.

Remark. *If u has a global maximum at point $(x_0, t_0)\in Q$, $t_0<T$, then the assertion of the above corollary holds also.*

The next lemma deals with superdifferentials and subdifferentials.

Lemma 3.3. *Let Ω be a bounded open set in R^n, $Q=\Omega\times(-1, 1)$. Suppose that $u\in C(\bar{Q})$, $Du\in L^\infty(Q)$ and there exists a positive constant K such that function $\tilde{u}=u+K(|x|^2+t)$ is convex in x, monotone increasing in t. Then there exists a function $M(x, t)\in L^1(Q, S^n)$, for* a. e. *$(x_0, t_0)\in Q$*

$$u(x_0+x, t_0+t)=u(x_0, t_0)+Du(x_0, t_0)\cdot x+u_t(x_0, t_0)t$$
$$+\frac{1}{2}(M(x_0, t_0)x, x)+o(|x|^2+|t|). \tag{3.10}$$

Proof This is a famous theorem, its proof can be found in [5, Appendix 2]. We give a proof which is an extension of that in [1].

By the monotonicity of $\tilde{u}$ in t we conclude that u is differentiable in t at a. e. $(x, t)\in Q$ and $u_t\in L^1(Q)$. On the other hand we have

$$D^2u\geqslant-2KI \text{ in } Q \quad \text{(in the sense of distributions)}, \tag{3.11}$$

so there is a function $M\in L^1(Q, S^n)$ and a matrix valued measure $\Gamma\in M(Q, S^n)$ (see [5, Chapter 3]) such that

$$D^2u=M+\Gamma \quad \text{(in the sense of distributions)}, \tag{3.12}$$

$$M \geqslant -2KI \quad \text{a. e. } (x, t) \in Q, \tag{3.13}$$

where Γ is singular with respect to Lebesgue measure and according to the Radon-Nikodyn theorem we see that

$$\frac{d\Gamma}{dx\,dt}(x, t) = 0 \quad \text{a. e. } (x, t) \in Q. \tag{3.14}$$

Let $(x_0, t_0) \in Q$ be a Lebesgue point of u_t and M at which $Du(x)$ exists and (3.14) holds. We claim that (3.10) holds at (x_0, t_0). W. l. o. g. we can assume $u(x_0, t_0) = 0$, $u_t(x_0, t_0) = 0$, $Du(x_0, t_0) = 0$, $M(x_0, t_0) = 0$ and $(x_0, t_0) = 0$.

Let u_η be as defined in the proof of Lemma 3.1. For any $E \subset Q$ let

$$I_\eta = I_\eta^\varepsilon(E) = \varepsilon^{-n-4} \int_{Q(0, 2\varepsilon)} \chi_E(x, t) u_\eta(x, t)\, dx\, dt, \tag{3.15}$$

where $Q(0, \varepsilon) = \{(x, t) \mid |x| \leqslant \varepsilon,\ -\varepsilon^2 \leqslant t \leqslant \varepsilon^2\}$. We have

$$\begin{aligned} I_\eta = \varepsilon^{-n-4} \int_{Q(0, 2\varepsilon)} \chi_E \Big\{ [u_\eta(x, t) - u_\eta(x, 0)] + \int_0^{|x|} \Big[Du_\eta\Big(\sigma \frac{x}{|x|}, 0\Big) \\ - Du_\eta(0, 0) \Big] \frac{x}{|x|}\, d\sigma + u_\eta(0, 0) + Du_\eta(0, 0) \cdot x \Big\} dx\, dt \\ = I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.16}$$

It is obvious that $I_3, I_4 = o(1)$ as $\eta \to 0$. Noting $(0, 0)$ is Lebesgue point of u_t we obtain

$$\begin{aligned} \limsup_{\eta \to 0} |I_1| &\leqslant 8\varepsilon^{-n-2} \limsup_{\eta \to 0} \int_{Q(0, 2\varepsilon)} |(u_\eta)_t(x, \tau)|\, dx\, d\tau \\ &= 8\varepsilon^{-n-2} \int_{Q(0, 2\varepsilon)} |u_t(x, \tau)|\, dx\, d\tau = o(1) \quad (\varepsilon \to 0). \end{aligned}$$

We change coordinates and integrate by parts to obtain

$$\begin{aligned} I_2 &= \varepsilon^{-n-4} \int_0^{2\varepsilon} dr \int_{-4\varepsilon^2}^{4\varepsilon^2} dt \int_{|\omega|=1} d\omega \int_0^r d\sigma \int_0^\sigma \chi_E(r\omega, t) r^{n-1} \frac{d^2}{d\rho^2} u_\eta(\rho\omega, 0)\, d\rho \\ &= \varepsilon^{-n-4} \int_{Q(0, 2\varepsilon)} J_E\Big(|x|, \frac{x}{|x|}, t\Big) |x|^{-n+1} D^2 u_\eta(x, 0)\Big(\frac{x}{|x|}, \frac{x}{|x|}\Big) dx\, dt, \end{aligned}$$

where $J_E(\rho, \omega, t) = \int_\rho^{2\varepsilon} \int_\sigma^{2\varepsilon} r^{n-1} \chi_E(r\omega, t)\, dr\, d\sigma$. Using (3.14) and the definition of u_η we get

$$\begin{aligned} \limsup_{\eta \to 0} I_2 &\leqslant \varepsilon^{-n-4} \int_{Q(0, 2\varepsilon)} J_E\Big(|x|, \frac{x}{|x|}, t\Big) |x|^{-n+1} (|M|\, dx dt + d|\Gamma|) \\ &= o(1) \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

So we have $\lim_{\eta \to 0} I_\eta^\varepsilon(E) = o(1)$ as $\varepsilon \to 0$.

On the other hand,

$$\lim I_\varepsilon^\eta(E) = \varepsilon^{-n-4} \int_{Q(0, 2\varepsilon)} \chi_E(x, t) u(x, t)\, dx\, dt, \tag{3.17}$$

and so we conclude that

$$I_\varepsilon(E) = \int_{Q(0, 2\varepsilon)} \chi_E(x, t) u(x, t)\, dx\, dt = o(\varepsilon^{n+4}). \tag{3.18}$$

Now let $m_\varepsilon=\max\{0, \sup\limits_{Q(0,\varepsilon)} u\}$ and let $(x_\varepsilon, t_\varepsilon)$ be a point in $Q(0, \varepsilon)$ such that $u(x_\varepsilon, t_\varepsilon)=m_\varepsilon$. Function $\tilde{u}(x, t)$ is convex in x and nondecreasing in t. Let $p_\varepsilon\in R^n$ be a vector such that

$$\tilde{u}(x, t)\geqslant\tilde{u}(x, t_\varepsilon)\geqslant\tilde{u}(x_\varepsilon, t_\varepsilon)+p_\varepsilon(x-x_\varepsilon), \ \forall x\in\Omega,\ t\geqslant t_\varepsilon.$$

We have for all $t\geqslant 0$

$$u(x_\varepsilon+x, t_\varepsilon+t)\geqslant m_\varepsilon+(p_\varepsilon-2Kx_\varepsilon)\cdot x-K(|x|^2+t).$$

Let $H_\varepsilon=\left\{(x_\varepsilon+x, t_\varepsilon+t)\,\big|\,|x|^2+t\leqslant\min\left(\varepsilon^2, \frac{m_\varepsilon}{2K}\right), t\geqslant 0 \text{ and } (p_\varepsilon-2Kx_\varepsilon)x\geqslant 0\right\}$. We see that $u\geqslant\frac{m_\varepsilon}{2}$ in H_ε, and that

$$I_\varepsilon(H_\varepsilon)\geqslant C\min\{m_\varepsilon\varepsilon^{n+2}, m_\varepsilon^{2+\frac{n}{2}}\}.$$

Combining this inequality with (3.18) we conclude that

$$m_\varepsilon=o(\varepsilon^2).$$

This proves that

$$u(x, t)\leqslant o(|x|^2+|t|). \tag{3.19}$$

We now consider $l_\varepsilon=\min\{0, \inf\limits_{Q(0,\varepsilon)} u\}$. Let E_ε be the set

$$E_\varepsilon=\left\{(x, t)\in Q(0, 2\varepsilon)\,\big|\,u(x, t)\leqslant\frac{l_\varepsilon}{2}\right\}.$$

An argument similar to a previous one using inequality (3.18) shows

$$|l_\varepsilon|\,\mathrm{mes}\,(E_\varepsilon)=o(\varepsilon^{n+4}).$$

We now prove by contradiction that $l_\varepsilon=o(\varepsilon^2)$. Indeed, if $l_\varepsilon\neq o(\varepsilon^2)$, then there is a sequence $\varepsilon_i\to 0$ such that, for some $c_0>0$, $l_{\varepsilon_i}\leqslant -c_0\varepsilon^2$ for all $i\in Z^+$. Now this implies

$$\mathrm{mes}\,(E_{\varepsilon_i})=o(\varepsilon_i^{n+2}).$$

Let (x_i, t_i) be a point such that $u(x_i, t_i)=l_{\varepsilon_i}$ and $(x_i, t_i)\in Q(0, \varepsilon_i)$. Since $\mathrm{mes}\,(E_{\varepsilon_i})=o(\varepsilon_i^{n+2})$, it follows that there is a sequence of points $(y_i, s_i)\in Q(0, 2\varepsilon_i)$, $s_i\leqslant t_i$ such that

$$u(y_i, s_i)\geqslant\frac{l_{\varepsilon_i}}{2}, \quad |y_i-x_i|^2+|t_i-s_i|=o(\varepsilon_i^2).$$

Applying the monotonicity we have

$$u(y_i, t_i)\geqslant u(y_i, s_i)+K(s_i-t_i)\geqslant\frac{1}{2}l_{\varepsilon_i}+o(\varepsilon_i^2)\geqslant\frac{3}{4}l_{\varepsilon_i}.$$

From this it follows that there are points $z_i\in\overline{x_iy_i}$ such that

$$|Du(z_i, t_i)|\geqslant(o(\varepsilon_i))^{-1}|l_{\varepsilon_i}| \quad \text{as } i\to\infty$$

$$\text{and } |D\tilde{u}(z_i, t_i)|\geqslant|Du(z_i, t_i)|-2K|z_i|\geqslant\frac{|l_{\varepsilon_i}|}{o(\varepsilon_i)} \quad \text{as } i\to\infty.$$

Since $\tilde{u}$ is convex in x

$$\tilde{u}(x, t_i)\geqslant u(z_i, t_i)+D\tilde{u}(z_i, t_i)(x-z_i) \quad \text{for all } x\in\Omega.$$

Therefore

$$u(x, t_i)\geqslant u(z_i, t_i)+D\tilde{u}(z_i, t_i)(x-z_i)-K|x|^2.$$

Taking $\tilde{x}_i = z_i + \varepsilon_i \dfrac{D\tilde{u}(z_i, t_i)}{|D\tilde{u}(z_i, t_i)|}$ we find that $(\tilde{x}_i, t_i) \in Q(0, 2\varepsilon_i)$ and

$$u(\tilde{x}_i, t_i) \geqslant l_{\varepsilon_i} - 4K\varepsilon_i^2 + \frac{|l_{\varepsilon_i}|}{o(1)} = \frac{|l_{\varepsilon_i}|}{o(1)} \geqslant \frac{\varepsilon_i^2}{o(1)}.$$

This contradicts (3.19) and so proves

$$u(x, t) \geqslant -o(|x|^2 + |t|).$$

This and (3.19) show that (3.10) holds for all Lebesgue points of u_t and M at which Du exists and (3.14) holds. The set of such points includes a. e. points in Q and so this proves our lemma.

We are now ready to prove our main theorem.

Theorem 3. 4. *Let Ω be a bounded open set in R^n and T be a positive constant, $Q = \Omega \times (0, T]$. Let $\varphi \in C^{2,1}(\Omega \times \Omega \times (0, T])$, set $w(x, y, t) = u^\varepsilon(x, t) - v_\varepsilon(y, t)$ for (x, t), $(y, t) \in Q^\varepsilon$ and assume that $w - \varphi$ achives its maximum over $\bar{\Omega}_\varepsilon \times \bar{\Omega}_\varepsilon \times [\varepsilon, T]$ at a point $(\bar{x}, \bar{y}, \bar{t}) \in \Omega_\varepsilon \times \Omega_\varepsilon \times (\varepsilon, T]$. Then, if (1.3) and (2.7) hold, there exist matrices X, $Y \in S^n$ such that*

$$\varphi_t(\bar{x}, \bar{y}, \bar{t}) - F_\varepsilon(\bar{x}, \bar{t}, u^\varepsilon(\bar{x}, \bar{t}), D_x\varphi(\bar{x}, \bar{y}, \bar{t}), X) + F^\varepsilon(\bar{y}, \bar{t}, v_\varepsilon(\bar{y}, \bar{t}),$$
$$-D_y\varphi(\bar{x}, \bar{y}, \bar{t}), -Y) \leqslant 0, \tag{3.20}$$

$$-\frac{2}{\varepsilon} I_{2n} \leqslant \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant D^2\varphi(\bar{x}, \bar{y}, \bar{t}). \tag{3.21}$$

Proof We know from Lemma 2.4 that $u^\varepsilon \in C(\bar{Q}^\varepsilon)$, $Du^\varepsilon \in L^\infty(Q^\varepsilon)$ and $u^\varepsilon + \frac{1}{\varepsilon}(|x|^2 + t)$ is convex in x and nondecreasing in t. According to Lemma 3.3 there exists a function $M_1 \in L^1(Q^\varepsilon, S^n)$, $M_1 \geqslant -\frac{2}{\varepsilon} I$ such that for a. e. $(x, t) \in Q^\varepsilon$

$$u^\varepsilon(x + \Delta x, t + \Delta t) = u^\varepsilon(x, t) + Du^\varepsilon(x, t) \cdot \Delta x + u_t^\varepsilon(x, t) \cdot \Delta t$$
$$+ \frac{1}{2}(M_1(x, t)\Delta x, \Delta x) + o(|\Delta x|^2 + |\Delta t|). \tag{3.22}$$

Furthermore $u_t^\varepsilon \in L^1(Q^\varepsilon)$. Similarly, there is a function $M_2 \in L^1(Q^\varepsilon, S^n)$, $M_2 \geqslant -\frac{2}{\varepsilon} I$ such that for a. e. $(x, t) \in Q^\varepsilon$,

$$-v_\varepsilon(y + \Delta y, t + \Delta t) = -v_\varepsilon(y, t) - Dv_\varepsilon(y, t)\Delta y - (v_\varepsilon)_t(y, t)\Delta t$$
$$+ \frac{1}{2}(M_2\Delta y, \Delta y) + o(|\Delta y|^2 + |\Delta t|). \tag{3.23}$$

From these we conclude that for a. e. (x, y, t)

$$w(x + \Delta x, y + \Delta y, t + \Delta t) = w(x, y, t) + Dw(x, y, t)\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + w_t\Delta t$$
$$+ \frac{1}{2} M(\Delta x, y)\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + o(|\Delta x|^2 + |\Delta y|^2 + |\Delta t|), \tag{3.24}$$

where $M=\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \geqslant -\frac{2}{\varepsilon} I_{2n}$ for a. e. (x, y, t).

By (3.22), (3.23) we now obtain from Proposition 2.3 for a. e. (x, y, t)

$$u_t^\varepsilon(x, t) - F_\varepsilon(x, t, u^\varepsilon(x, t), Du^\varepsilon(x, t), M_1(x, t)) \leqslant 0,$$

$$(v_\varepsilon)_t(y, t) - F^2(y, t, v_\varepsilon(y, t), Dv_\varepsilon(y, t), -M_2(y, t)) \geqslant 0. \tag{3.25}$$

Using Lemma 3.1 for function $w-\varphi$ we can then choose a positive constant δ_0 such that mes $(S_\delta)>0$ for all $\delta<\delta_0$ where the set S_δ is defined by

$$S_\delta = \{(x, y, t) \in Q_\delta(\bar{x}, \bar{y}, \bar{t}) \mid |D(w-\varphi)| \leqslant \varepsilon(\delta) + 2\delta^2;$$
$$(w-\varphi)(x+\Delta x, y+\Delta y, t+\Delta t) \leqslant (w-\varphi)(x, y, t)$$
$$+ D(w-\varphi)\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + \delta(|\Delta x|^2 + |\Delta y|^2 + |\Delta t|) \quad \text{for } \Delta t \leqslant 0;$$
$$\text{and (3.24), (3.25) hold at } (x, y, t)\}.$$

Take a sequence $\delta_k \leqslant \delta_0$, $\delta_k \to 0$ $(k\to\infty)$. Let $z_k=(x_k, y_k, t_k)$ be a point in S_{δ_k} and set

$$X_k = M_1(x_k, y_k, t_k), \quad Y_k = M_2(x_k, y_k, t_k).$$

By the definition of S_δ we obtain

$$|D(w-\varphi)(z_k)| \leqslant \varepsilon(\delta_k) + 2\delta_k^2, \tag{3.26}$$

$$(w-\varphi)_t(z_k) \geqslant -\delta_k, \tag{3.27}$$

$$-\frac{2}{\varepsilon} I_{2n} \leqslant \begin{pmatrix} X_k & 0 \\ 0 & Y_k \end{pmatrix} \leqslant 2\delta_k + D^2\varphi(z_k), \tag{3.28}$$

$$[u_t^\varepsilon - F_\varepsilon(\cdot, \cdot, u^\varepsilon, Du^\varepsilon, X_k)]_{(x_k, t_k)} \leqslant 0,$$

$$[(v_\varepsilon)_t - F^\varepsilon(\cdot, \cdot, v_\varepsilon, Dv_\varepsilon, -Y_k)]_{(y_k, t_k)} \geqslant 0. \tag{3.29}$$

By (3.28) we conclude that $\{X_k\}$ and $\{Y_k\}$ are bounded sequences of matrix; therefore (extracting subsequence if necessary) X_k and Y_k converge respectively to matrices X and Y. Clearly

$$-\frac{2}{\varepsilon} I_{2n} \leqslant \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant D^2\varphi(\bar{x}, \bar{y}, \bar{t}).$$

Finally, combining (3.26), (3.27) and (3.29) and letting $i\to\infty$ we obtain inequality (3.20). This proves our theorem.

Remark. *It is possible to replace the domain $Q=\Omega\times(0, T]$ by*

$$Q = \{(x, t) \mid x \in \Omega_t, t \in (0, T]\}.$$

Next we will describe and prove our comparison and uniqueness results. More precisely, we want to compare viscosity subsolution and viscosity supersolution pair (u, v) with properties indicated above. We will denote

$$M = \sup_{\partial_p Q} (u-v)^+(x, t)$$

and the assertion we want to prove is

$$u(x, t) - v(x, t) \leqslant M \quad \text{for all } (x, t) \in Q. \tag{3.30}$$

We will use the following assumptions:

$$\forall R<\infty,\ \exists\gamma_R>-\infty,\ F(x,\ t,\ u,\ p,\ A)\leqslant F(x,\ t,\ v,\ p,\ A)-\gamma_R(u-v) \qquad (3.31)$$

for all $(x,\ t)\in Q,\ R\geqslant u\geqslant v\geqslant -R,\ p\in R^n,\ A\in S^n$.

$$|F(x,\ t,\ u,\ p,\ A)-F(y,\ s,\ u,\ p,\ A)|\leqslant\omega_R(|x-y|(1+|p|))+\bar{\omega}(|t-s|) \qquad (3.32)$$

if $(x,\ t),\ (y,\ s)\in Q,\ |u|\leqslant R,\ p\in R^n,\ A\in S^n$, for all $R<\infty$; or

$$|F(x,\ t,\ u,\ p,\ A)-F(y,\ s,\ u,\ p,\ A)|\leqslant\omega_R(|x-y|+|t-s|^{1/2}), \qquad (3.33)$$

if $(x,\ t),\ (y,\ s)\in Q,\ |u|\leqslant R,\ |p|\leqslant R,\ A\in S^n$, for all $R<\infty$, where $\omega_R(\sigma)\to 0$, $\bar{\omega}(\sigma)\to 0$ as $\sigma\to 0$ and $\bar{\omega}$ is independent of A.

Remark. By taking a change of dependent variable: $u\to e^{\gamma t}u$ we can assume that (3.31) holds for some $\gamma_R>0$.

Theorem 3.5. *Assume* (1.3) *and* (3.31). *If, in addition,* (3.32) *holds or if* (3.33) *holds and* u *or* v *is locally Lipschitz with respect to* X *in* Q, *then*

$$(u-v)^+(x,\ t)\leqslant e^{(-\gamma_R)^+}M, \quad in\ Q.$$

Proof We can assume w. l. o. g. that $\gamma_R>0$ and we will prove assertion (3.30).

This will be a proof by contradiction. Assume the theorem is false; then

$$\sup_Q\ (u-v)^+>\sup_{\partial_p Q}\ (u-v)^+=M. \qquad (3.35)$$

Since $u-v$ is upper-semicontinuous, it follows that $u-v$ achives its maximum over $\bar{Q}$ at a point $(x_0,\ t_0)\in Q$. w. l. o. g. we can assume that $(u-v)(x,\ t)<(u-v)(x_0,\ t_0)$ for all $(x,\ t)\in Q\cap\{0\leqslant t<t_0\}$ and $t_0=T$.

We choose $\varphi=\varphi(x,\ y,\ t)=\Phi_\delta(x-y,\ t)=\dfrac{1}{2\delta}\ |x-y|^2+\delta(T-t)$ and assume that $w-\varphi$ achives its maximum at a point $(\bar{x},\ \bar{y},\ \bar{t})=(x_{\varepsilon,\delta},\ y_{\varepsilon,\delta},\ t_{\varepsilon,\delta})$ with $(\bar{x},\ \bar{t}),\ (\bar{y},\ \bar{t})\in\bar{Q}^\varepsilon\subset\bar{Q}$. Letting ε go to zero, we see (extracting subsequences if necessary) that $(\bar{x},\ \bar{y},\ \bar{t})$ converges to a point $(x_\delta,\ y_\delta,\ t_\delta)$ with $(x_\delta,\ t_\delta),\ (y_\delta,\ t_\delta)\in\bar{Q}$. By the inequality

$$(w-\varphi)(x,\ y,\ t)\leqslant(w-\varphi)(\bar{x},\ \bar{y},\ \bar{t}) \quad \text{for all } (x,\ t),\ (y,\ t)\in\bar{Q}^\varepsilon \qquad (3.36)$$

we obtain from the upper-semicontinuity of $u,\ -v$

$$u(x,\ t)-v(y,\ t)-\varphi(x,\ y,\ t)\leqslant u(x_\delta,\ t_\delta)-v(y_\delta,\ t_\delta)-\varphi(x_\delta,\ y_\delta,\ t_\delta) \quad \text{for all } (x,\ t),\ (y,\ t)\in Q. \qquad (3.37)$$

From this and the definition of φ we deduce easily

$$\delta^{-1}|x_\delta-y_\delta|^2\leqslant u(x_\delta,\ t_\delta)-v(y_\delta,\ t_\delta)+(v-u)(x,\ T),\ \forall x. \qquad (3.38)$$

So $|x_\delta-y_\delta|=O(\delta)$. Consequently (extracting subsequences if necessary) we may assume that $(x_\delta,\ y_\delta,\ t_\delta)$ converges to a point $(\bar{x}_0,\ \bar{x}_0,\ \bar{t}_0)$ with $(\bar{x}_0,\ \bar{t}_0)\in\bar{Q}$. Clearly point $(\bar{x}_0,\ \bar{t}_0)$ is a maximum point of $u-v$ over Q, so $(\bar{x}_0,\ \bar{t}_0)\in Q$ and $\bar{t}_0=T$. In conclusion, we deduce that, for ε and δ small enough $w-\varphi$ admits a maximum point denoted by $(\bar{x},\ \bar{y},\ \bar{t})$ which satisfies $(\bar{x},\ \bar{t}),\ (\bar{y},\ \bar{t})\in Q^\varepsilon$.

We may now apply Theorem 3.4 and deduce the existence of X and $Y\in S^n$

such that

$$\varphi_t(\bar{x}, \bar{y}, \bar{t}) - F_\varepsilon(\bar{x}, \bar{t}, u^\varepsilon(\bar{x}, \bar{t}), D_x\varphi(\bar{x}, \bar{y}, \bar{t}), X) + F^\varepsilon(\bar{y}, \bar{t}, v_\varepsilon(\bar{y}, \bar{t}), -D_y\varphi(\bar{x}, \bar{y}, \bar{t}), -Y) \leq 0, \tag{3.39}$$

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq D^2\varphi(\bar{x}, \bar{y}, \bar{t}) = \frac{1}{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.40}$$

Notice that (3.40) implies in particular that $X+Y\leq 0$: indeed, in this case any point of the form ${}^T(\xi, \xi)$ for $\xi\in R^n$ is in kernel of $D^2\varphi$, so that by (3.40) $(X\xi, \xi)+(Y\xi, \xi)\leq 0$, which proves thus our claim. Therefore, we deduce from (1.3), (3.31) and (3.32) that for some $\gamma>0$

$$\gamma(u^\varepsilon(\bar{x}, \bar{t}) - v_\varepsilon(\bar{y}, \bar{t}))^+ \leq \bar{\omega}_\delta(2c_0\sqrt{\varepsilon}) + |\varphi_t(\bar{x}, \bar{y}, \bar{t})| + \omega[(2c_0\sqrt{\varepsilon} + |\bar{x}-\bar{y}|)(1+|D\Phi|)], \tag{3.41}$$

where $\bar{\omega}_\delta(\sigma)$, $\omega(\sigma)\to 0$ as $\sigma\to 0$.

Since u (v) is upper-semicontinuous (lower-semicontinuous), we have $u(x_\delta, t_\delta) \leq u(\bar{x}_0, T)+o(1)$ and $u(y_\delta, t_\delta)\geq v(\bar{x}_0, T)+o(1)$, and conclude easily from (3.38) that $|x_\delta - y_\delta|^2 = o(\delta)$. From (3.36) and (3.41), we get

$$\gamma(u^\varepsilon - v_\varepsilon)^+(\bar{x}_0, T) \leq \bar{\omega}_\delta(2c_0\sqrt{\varepsilon}) + \delta + \omega\left[(2c_0\sqrt{\varepsilon} + |\bar{x}-\bar{y}|)\left(1+\frac{1}{\varepsilon}\delta|\bar{x}-\bar{y}|\right)\right].$$

Letting ε go to zero we find that

$$\gamma(u-v)^+(\bar{x}_0, T) \leq \delta + \omega\left(|x_\delta - y_\delta| + \frac{1}{\delta}|x_\delta - y_\delta|^2\right).$$

This clearly contradicts (3.35) if δ is sufficiently small.

As it is standard in the theory of viscosity solutions, if u or v is Lipschitz, one can replace in the proof of (3.32) by the weaker condition (3.33) observing that $D\Phi_\delta(\bar{x}-\bar{y}, \bar{t})$ is bounded independently of ε and δ.

We next give another example of application of Theorem 3.4 above; we will explain briefly how Theorem 3.4 yields the uniqueness for parabolic type Isaacs-Bellman equations, i. e. we take

$$F(x, t, z, p, A) = \inf_{\alpha\in A}\sup_{\beta\in B}\{\mathrm{Tr}[a_{\alpha\beta}(x, t)\cdot {}^T a_{\alpha\beta}(x, t)\cdot A] + (b_{\alpha\beta}(x, t), p) + c_{\alpha\beta}(x, t)z + f_{\alpha\beta}(x, t)\}, \tag{3.42}$$

where A, B are two given sets, $a_{\alpha\beta}$ is (for all $(x, t)\in Q$, $\alpha\in A$, $\beta\in B$) an element of $S^{n,m}$ and $S^{n,m}$ will denote the space of $n\times m$ real matrices, $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$, $f_{\alpha\beta}$ are bounded uniformly continuous on $\bar{Q}$ uniformly in $\alpha\in A$, $\beta\in B$. Here we consider the following parabolic Isaacs-Bellman equation

$$u_t - F(x, t, u, Du, D^2u) = 0 \quad \text{in } Q. \tag{3.43}$$

We will use the following conditions

$$\exists c_0 > -\infty, c_{\alpha\beta}(x, t) \leq -c_0 \quad \text{for all } \alpha\in A, \beta\in B, \tag{3.44}$$

for all $(x, t)\in Q$;

$$\|a_{\alpha\beta}(x, t) - a_{\alpha\beta}(y, t)\| \leq C|x-y|^\theta, \quad \forall (x, t), (y, t)\in\bar{Q} \tag{3.45}$$

$\forall\alpha\in A,\ \beta\in B,$ for some $C\geqslant 0,\ \theta\in(0,1]$;

$$|b_{\alpha\beta}(x,t)-b_{\alpha\beta}(y,t)|\leqslant C|x-y|\ \forall(x,t),\ (y,t)\in\bar{Q}, \tag{3.46}$$

$\forall\alpha\in A,\ \beta\in B,$ for some $C\geqslant 0$.

Remark. *By taking a change of dependent variable: $u\to e^{\gamma t}u$,* we can assume that (3.44) holds for some $c_0>0$.

Theorem 3. 6. *Assume that F is given by* (3.42) *and that* (3.44) *holds. If, in addition,* (3.45) *and* (3.46) *hold with $\theta=1$ or if u or v is locally Lipschitz in Q and* (3.45) *holds with $\theta>1/2$, then*

$$(u-v)^+(x,t)\leqslant e^{(-c_0)^+}M,\quad in\ Q.$$

Sketch of proof We can assume w. l. o. g. that $c_0>0$. The main idea of the proof is the following consequence of (3.40). Choose $\varphi(x,y,t)=\frac{1}{2\delta}|x-y|^2+\delta(T-t)$ so that

$$D^2\varphi(\bar{x},\bar{y},\bar{t})=\frac{1}{\delta}\begin{pmatrix} I & -I \\ -I & I\end{pmatrix},$$

and then observing that if $a_1,\ a_1\in S^{n,m}$ the matrix

$$\begin{pmatrix} a_1\cdot{}^Ta_1 & a_1\cdot{}^Ta_2 \\ a_2\cdot{}^Ta_1 & a_2\cdot{}^Ta_2\end{pmatrix}$$

is nonnegative, we may multiply (3.40) by this matrix and take the trace. In this way, we obtain

$$\mathrm{Tr}\,(a_1\cdot{}^Ta_1\cdot X)+\mathrm{Tr}(a_2\cdot{}^Ta_2\cdot Y)\leqslant\frac{1}{\delta}\mathrm{Tr}[(a_1-a_2)\cdot({}^Ta_1-a_2)]. \tag{3.47}$$

Then, the rest of the proof is quite similar to the proof of Theorem 3.5 with the same notations, one obtains, using (3.44), (3.45), (3.46) and (3.47),

$$c_0(u-v)^+(\bar{x}_0,T)\leqslant\delta+\frac{c}{\delta}\ |x_\delta-y_\delta|^2+\omega(\sqrt{\delta})$$

where

$$\omega(R)=\sup_{\alpha,\beta}\ \sup_{|x-y|\leqslant R}\{|c_{\alpha\beta}(x,t)-c_{\alpha\beta}(y,t)|+|f_{\alpha\beta}(x,t)-f_{\alpha\beta}(y,t)|\}.$$

We deduce easily our assertion since $|x_\delta-y_\delta|^2=o(\delta)$.

The case when u or v is Lipschitz is a standard adaptation.

It is obvious that continuous viscosity solutions of problem (1.1)—(1.2) or (3.43)—(1.2) are unique by Theorem 3.5 and Theorem 3.6 under the assumptions on F in Theorem 3.5 or Theorem 3.6 respectively.

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