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t-PBIB DESIGNS

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Abstract

A new type of design, called a t-PBIB design, is introduced by combining the notion of a t-design and the one of PBIB design. Some basic properties of a t-PBIB design are given, and a class of 3-PBIB designs is constructed by means of finite vector spaces.

An incidence structure is a triple $\mathbf{D}=(S, \mathbf{B}, I)$, where S and $\mathbf{B}$ are two disjoint sets, and I a binary relation between S and $\mathbf{B}$, i. e. $I\subseteq S\times\mathbf{B}$. The elements of S are called points, and those of $\mathbf{B}$ blocks. A BIB design is an incidence structure $\mathbf{D}=(S, \mathbf{B}, I)$ satisfying

1. 1. For an arbitary $B\in\mathbf{B}$, $|\{s\in S|sIB\}$ is a constant indopendent of the choice of B;

1. 2. For an arbitary 2-subset $\{s_1, s_2\}$ of S, $|\{B\in\mathbf{B}|s_1IB \text{ and } s_2IB\}|$ is also a constant independent of the choice of $\{s_1, s_2\}$. If the constants in the conditions 1.1 and 1.2 are k and λ respectively, then $\mathbf{D}$ is called a (v, k, λ) BIB design, where $v=|S|$.

A t-design, as a generalization of a BIB design, is an incidence structure $\mathbf{D}=(S, \mathbf{B}, I)$ satisfying the condition 1.1 and the condition 1.2 with "2-subset" replaced by "t-subset". Clearly, 2-designs are BIB designs. A PBIB design is another generalization of a BIB design. For its definition we need the notion of an association scheme.

Let S be a set of v points, and

$$S^{(2)}=\{(s_1, s_2)|s_1, s_2\in S, s_1\neq s_2\}.$$

Let $[a, b]$ denote the set of integers between a and b. Let R_i ($i\in[1, m]$) be m binary relation on S, i. e. $R_i\subseteq S\times S$, satisfying the following conditions.

2. 1. $R_i\cap R_j\begin{cases}\neq\varnothing, & \text{if } 1\leqslant i=j\leqslant m.\\ =\varnothing, & \text{if } 1\leqslant i\neq j\leqslant m.\end{cases}$

2. 2. $S^{(2)}=\bigcup_{i=1}^{m} R_i$.

2. 3. For every i, R_i is symmetric, i. e. if $(s_1, s_2)\in R_i$, then $(s_2, s_1)\in R_i$.

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2. 4. Let $i\in[1, m]$. For every $s\in S$,

$$|\{s'\in S\,|\,(s', s)\in R_i\}|$$

is a constant independent of the choice of s.

2. 5. Let i, j, l be given integers, $1\leqslant i, j, l\leqslant m$. For an arbitary $(s_1, s_2)\in R_i$,

$$|\{s\in S\,|\,(s, s_1)\in R_j,\ (s, s_2)\in R_l\}|$$

is a constant independent of the choice of s_1 and s_2.

Then we call S with such R_i ($i\in[1, m]$) an association scheme with m associate classes $R_1, R_2, \cdots, R_m$. If the constants in the conditions 2.4) and 2.5 are denoted by n_i and p^i_{jl} respectively, then v, n_i, p^i_{jl} ($i, j, l\in[1, m]$) are called the parameters of the association scheme.

Based on association schemes we have the definition of PBIB designs.

Let S with $R_1, R_2, \cdots, R_m$ be an association scheme and $\mathbf{D}=(S, \mathbf{B}, I)$ an incidence structure. $\mathbf{D}$ is called a PBIB design with m association classes if it satisfies the following conditions.

3. 1. For an arbitary $B\in\mathbf{B}$,

$$|\{s\in S\,|\,sIB\}|$$

is a constant independent of the choice of B.

3. 2. Let $i\in[1, m]$. For an arbitary 2-subset $\{s_1, s_2\}$ with $(s_1, s_2)\in R_i$

$$|\{B\in\mathbf{B}\,|\,s_1\,IB \text{ and } s_2\,IB\}|$$

is also a constant independent of the choice of s_1 and s_2.

If the constants in conditions 3.1 and 3.2 are k and λ_i respectively, then the numbers

$$v,\ n_i,\ p^i_{jl},\ k,\ \lambda_i\ (i, j, l\in[1, m])$$

are called the parameters of the PBIB design.

As everyone knows that t-designs, especialy BIB designs, and PBIB designs have been studied extensively and fruitfully (see, e. g., [1-5]). In this paper we will generalize t-designs and PBIB designs and introduce the notion of t-PBIB designs, prove some basic properties of a t-PBIB design and construct a class of 3-PBIB designs by means of finite vector spaces.

We first give the definition of a t-PBIB design.

Let S be a set of v points, and

$$S^{(t)}=\{(s_1, s_2, \cdots, s_t)\,|\,s_i\in S,\ s_i\neq s_j\quad(1\leqslant i\neq j\leqslant t)\}.$$

Let $R_i's$ $(1\leqslant i\leqslant m)$ be m t-ary relations on S, and they satisfy the following conditions.

4. 1. $R_i\cap R_j\begin{cases}\neq\varnothing, & \text{if } 1\leqslant i=j\leqslant m,\\ =\varnothing, & \text{if } 1\leqslant i\neq j\leqslant m.\end{cases}$

4. 2. $s^{(t)}=\bigcup\limits_{i=1}^{m} R_i.$

4. 3. Every R_i ($i\in[1, m]$) is totally symmetric, i.e., if $(s_1, s_2,\cdots,s_t)\in R_i$, then

$$(s_{j_1}, s_{j_2}, \cdots, s_{j_t})\in R_i,$$

where $j_1, j_2, \cdots, j_t$ is an arbitary permutation of $1, 2, \cdots, t$.

4. 4. Let $i\in[1, m]$. For every $(s_1, s_2, \cdots, s_{t-1})\in S^{(t-1)}$,

$$|\{s\in S|(s_1, s_2, \cdots, s_{t-1}, s)\in R_i\}|$$

is a constant independent of the choice of $s_1, s_2, \cdots, s_{t-1}$.

4. 5. Let $i, j_1, j_2, \cdots, j_t\in[1, m]$. For an arbitary $(s_1, s_2, \cdots, s_t)\in R_i$,

$$|\{s\in S|(s_1, \cdots, s_{h-1}, s, s_{h+1}, \cdots, s_t)\in R_{j_h} \text{ for all } h\in[1,t]\}|$$

is also a constant independent of the choice of $s_1, s_2, \cdots$ and s_t.

Then S with such R_i's($i\in[1, m]$) is called a t-association scheme with t-associate classes $R_1, R_2, \cdots, R_m$. If the constants in conditions (4.4) and (4.5) are denoted by n_i, $p^i_{j_1j_2\cdots j_t}$ ($i, j_1, \cdots, j_t\in[1\ m]$) respectively, then

$$v,\ n_i,\ p^i_{j_1j_2\cdots j_t}$$

are called the parameters of the t-association scheme.

Let $\mathbf{D}=(S, \mathbf{B}, I)$ be an incidence structure, S with $R_1, R_2, \cdots, R_m$ be a t-association scheme, and the following two conditions hold.

5. 1. For an arbitary $B\in\mathbf{B}$,

$$|\{s\in S|s\,IB\}|$$

is a constant independent of the choice of B.

5. 2. For an arbitary $(s_1, s_2,\cdots,s_t)\in R_i$,

$$|\{B\in\mathbf{B}|s_j\,IB \text{ for all } j\in[1, t]\}|$$

is also a constant independent of the choice of $(s_1, s_2,\cdots,s_t)$.

Then $\mathbf{D}$ is called a t-PBIB design with m associate classes. If the constants in conditions 5.1 and 5.2 are denoted by k and λ_i, then

$$v,\ k,\ \lambda_i,\ n_i,\ p^i_{j_1j_2\cdots j_t}\quad (i, j_1, j_2, \cdots, j_t\in[1, m]) \tag{1}$$

are called the parameters of $\mathbf{D}$.

Clearly, t-PBIB designs with $t=2$ are PBIB designs, and t-PBIB designs with one associate class are t-designs.

We now prove some properties of a t-PBIB design. They are similar to these of a PBIB design or of a t-design.

Theorem 1. *Let* $\mathbf{D}$ *be a t-PBIB design with the parameters in* (1). *Then we have*

$$v=\sum_{i=1}^{m} n_i+t-1, \tag{2}$$

$$p^i_{j_1j_2\cdots j_t}=p^i_{j_{\sigma(1)}j_{\sigma(2)}\cdots j_{\sigma(t)}} \text{ for any permutation } \sigma \text{ of } 1, 2, \cdots, t, \text{ and } i, j_1,\cdots, j_t\in[1, m], \tag{3}$$

$$\sum_{j_2\cdots j_t=1}^{m} p^i_{j_1j_2\cdots j_t}=\begin{cases} n_i-1, & \text{if } 1\leqslant j_1=i\leqslant m,\\ n_{j_1}, & \text{if } 1\leqslant j_1\neq i\leqslant m,\end{cases} \tag{4}$$

$$n_i p^i_{j_1\cdots j_t}=n_{j_1}\cdot p^{j_1}_{ij_2\cdots j_t},\quad i, j_1,\cdots,j_t\in[1, m]. \tag{5}$$

Proof For a given $(s_1, s_2, \cdots, s_{t-1}) \in S^{(t-1)}$, there are $v-(t-1)$ $(s_1, s_2, \cdots, s_{t-1}, s) \in S^{(t)}$. On the other hand, by conditions 4.2 and 4.4 these $v-(t-1)$ elements of $S^{(t)}$ can be partitioned into m groups with n_i elements isn the i th group. This proves (2). As for (3), it is very clear by condition 4.3. For a given $(s_1, s_2, \cdots, s_t) \in R_i$, there are n_i-1 elements s' of S different from s_1 such that $(s', s_2, \cdots s_t) \in R_i$. On the other hand, these n_i-1 t-tuple $(s', s_2, \cdots, s_t)$ of R_i can be partitioned into m^{t-1} groups with $p^i_{ij_2\cdots j_t}$ t-tuples in the $(j_2, \cdots, j_t)$th group $(j_2, \cdots, j_t \in [1, m])$. This proves the first relation in (4). The second one is then clear by the same argument. To prove (5), we count the set

$$W=\left\{(s_1, s)\in S^{(2)}\left|\begin{array}{l}(s_1, s_2, \cdots, s_t)\in R_i \text{ and}\\ (s, \cdots, s_{h-1}, s, s_{h+1}, \cdots, s_t)\in R_{j_h}\\ \text{for all } h\in[1, t]\end{array}\right.\right\}$$

for a given $(s_2, s_3, \cdots, s_t) \in S^{(t-1)}$. The number of elements of W can be counted in two ways. There are n_i ways of choosing $s_1 \in S$ such that $(s_1, s_2, \cdots, s_t) \in R_1$, and for each such s_1 there are $p^i_{j_1j_2\cdots j_t}$ ways of choosing $s \in S$ such that $(s_1, \cdots, s_{h-1}, s, s_{h+1}, \cdots, s_t) \in R_{j_h}$ for all $h \in [1, t]$. So

$$|W| = n_i p^i_{j_1j_2\cdots j_t}. \tag{6}$$

On the other hand, there are n_{j_1} ways of choosing $s \in S$ such that

$$(s, s_2, \cdots, s_t) \in R_{j_1}. \tag{7}$$

For each such s there are $p^{j_1}_{ij_2\cdots j_t}$ ways of choosing s_1 such that

$$\begin{gathered}(s_1, s_2, \cdots, s_t) \in R_i,\\ (s, s_2, \cdots, s_{h-1}, s_1, s_{h+1}, \cdots, s_t) \in R_{j_a} \quad \text{for all } h \in [2, t].\end{gathered} \tag{8}$$

The latter is

$$(s_1, s_2, \cdots, s_{h-1}, s, s_{h+1}, \cdots, s_t) \in R_{j_h} \quad \text{for all } h \in [2, t]. \tag{9}$$

Clearly, (7)—(9) are all defining conditions for W. Therefore, we have

$$|W| = n_{j_1} p^{j_1}_{ij_2\cdots j_t}. \tag{10}$$

Combining (6) and (10) we get (5). This completes the proof.

Theorem 2. *A t-PBIB design* $\mathbf{D}=(S, \mathbf{B}, I)$ *with the parameters given in* (1) *is also a* BIB $(t-1)$-*design with the parameters* v, k *and*

$$\lambda = \sum_{i=1}^{m} n_i\lambda_i/(k-t+1), \tag{11}$$

which are independent of $p^i_{j_1j_2\cdots j_t}$.

Proof Let $(s_1, s_2, \cdots, s_{t-1}) \in S^{(t-1)}$, and

$$\mathscr{T} = \{\{s_1, s_2, \cdots, s_{t-1}, s\} \mid s \neq s_i \ \text{ for all } i \in [1, t-1]\}.$$

Then $|\mathscr{T}| = v-t+1$. We can calculate

$$u=\left|\left\{(\{s_1, s_2, \cdots, s_{t-1}, s\}, B)\left|\begin{array}{l}\{s_1, \cdots, s_{t-1}, s\}\in\mathscr{T}\\ sIB \text{ and } s_iIB\ (i\in[1, t-1]),\\ B\in\mathbf{B}\end{array}\right.\right\}\right|$$

in the following two ways. Let

$$\mathscr{V}=\{B\in\mathbf{B}\mid s_i IB(i\in[1,\ t-1])\}.$$

For each $B\in\mathscr{V}$, there are $k-t+1$ t-subsets $\in\mathscr{T}$. So

$$u=(k-t+1)\,|v|. \tag{12}$$

On the other hand, by the definition of n_i we know that there are n_i t-subsets $\{s_1,\ s_2,\ \cdots,\ s_{t-1},\ s\}$ in $\mathscr{T}$ such that $(s_1,\ s_2,\ \cdots,\ s_{t-1},\ s)\in R_i$, so these n_i t-subsets are included exactly $n_i\lambda_i$ times in the blocks of $\mathbf{D}$ $(i\in[1,\ m])$. Hence

$$u=\sum_{i=1}^{m} n_i\lambda_i. \tag{13}$$

Combining (12) and (13) gives

$$|\mathscr{V}|=\frac{1}{k-t+1}\sum_{i=1}^{m} n_i\lambda_i,$$

which is a constant independent of the choice of $(s_1,\ s_2,\ \cdots,\ s_{t-1})\in S^{(t-1)}$. Then $\mathbf{D}$ is a $(t-1)$-design, and $|\mathscr{V}|$ is the value of λ. Clearly, the value of $|\mathscr{V}|$ is independent of $p^{i}_{j_1j_2\cdots j_t}$. This completes the proof.

From this theorem and some known results on t-designs, we know that

$$b=|\mathbf{B}|=\lambda\binom{v}{t-1}\Big/\binom{k}{t-1},$$

$$r=|\{B\in\mathbf{B}\mid s|B\}|=\lambda\binom{v}{t-2}\Big/\binom{k}{t-2}\quad\text{for any } s\in S,$$

$$bk=vr.$$

Finally, we construct a class of 3-association schemes and 3-PBIB designs by using the finite vector spaces.

Let q be a prime power, F_q the finite field with q elements, and $V_n(F_q)$ the n-dimensional vector space over F_q. Let S be the set of 1-dimensional subspace of $V_n(F_q)$, and

$$R_1=\{(s_1,\ s_2,\ s_3)\in S^{(3)}\mid \dim(s_1\cup s_2\cup s_3)=3\},$$

$$R_2=\{(s_1,\ s_2,\ s_3)\in S^{(3)}\mid \dim(s_1\cup s_2\cup s_3)=2\},$$

where $s_1\cup s_2\cup s_3$ denotes the subspace spanned by s_1, s_2 and s_3. Clearly,

R_1 and R_2 are totally symmetric ternary relations on S.

$$S^{(3)}=R_1\cup R_2,\ R_1\cap R_2=\varnothing,\ R_1\neq\varnothing,\ R_2\neq\varnothing,$$

Let $(s_1,\ s_2,\ s_3)$, $(s_1^*,\ s_2^*,\ s_3^*)\in R_1$. Then we can find two $(n-3)\times n$ matrices P, P^* over F_q such that

$$\begin{pmatrix} s_1\\ s_2\\ s_3\\ P\end{pmatrix}\quad\text{and}\quad\begin{pmatrix} s_1^*\\ s_2^*\\ s_3^*\\ P^*\end{pmatrix}$$

are both nonsingular.

Put

$$T=\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ P \end{pmatrix}^{-1} \begin{pmatrix} s_1^* \\ s_2^* \\ s_3^* \\ P^* \end{pmatrix}.$$

Then $T\in GL_n(F_q)$, the linear group of order n over F_q, and

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ P \end{pmatrix} T=\begin{pmatrix} s_1^* \\ s_2^* \\ s_3^* \\ P^* \end{pmatrix}.$$

So

$$s_iT=s_i^*, \quad 1\leqslant i\leqslant 3,$$

which mean that $GL_n(F_q)$ transitively acts on R_1.

Let (s_1, s_2, s_3), $(s_0^*, s_2^*, s_3^*)\in R_2$. Then $s_3\subset s_1\cup s_2$, $s_3^*\subset s^*\cup s_2^*$, and after suitably choosing the vectors that represent the 1-dimensional subspaces s_1, s_2, s_1^* and s_2^* if necessary, we have the vector equations

$$s_3=s_1+s_2, \quad s_3^*=s_1^*+s_2^*. \tag{14}$$

Noting that both $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ and $\begin{pmatrix} s_1^* \\ s_2^* \end{pmatrix}$ are $2\times n$ matrices with rank 2, we can find $(n-2)\times n$ matrices Q and Q^* over F_q such that

$$\begin{pmatrix} s_1 \\ s_2 \\ Q \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s_1^* \\ s_2^* \\ Q^* \end{pmatrix}$$

are nonsingular matrices of order n. Therefore, there exists $T\in GL_n(F_q)$ such that $s_1T=s_1^*$, $s_2T=s_2^*$. Thus by (14) we have $s_3T=s_3^*$, which means that $GL_n(F_q)$ transitively acts on R_2. Since $GL_n(F_q)$ transitively acts on R_i $(i=1, 2)$, it follows that $p^i_{j_1j_2j_3}$'s $(1\leqslant i, j_1, j_2, j_3\leqslant 2)$ are all constants. And clearly, n_1 and n_2 are both constants. So we certainly obtain a 3-associatoon scheme with two associate classes. We now calculate its parameters.

By Theoreem 1, we only need to calculate the values of v, n_1, p^1_{111}, p^1_{112}, p^1_{222} and p^2_{222}, from which the other parameters are determined. Clearly

$$v=N(1, n)=\frac{q^n-1}{q-1},$$

$$n_1=N(1, n)-N(1, 2)=\frac{q^n-1}{q-1}-\frac{q^2-1}{q-1}=\frac{q^n-q^2}{q-1},$$

where $N(m, n)$ denotes the number of m-dimensional subspaces of $V_n(F_q)$[5]. Let $(s_1, s_2, s_3)\in R_1$, $\mathscr{T}_1$ be the set of 1-dimensional subspaces s of $V_n(F_q)$ such that $s\cup s_2\cup s_3$, $s_1\cup s\cup s_3$ and $s_1\cup s_2\cup s$ are all 3-dimensional subspaces, and $\mathscr{T}_2$ be the set of 1-dimensional subspaces s' of $V_n(F_q)$ such that $s'\cup s_2\cup s_3$, $s_1\cup s'\cup s_3$ are 3-

dimensional subspaces and $s_1 \cup s_2 \cup s'$ is a 2-dimensional subspaces. Then

$$p^1_{111} = |\mathscr{T}_1|, \quad p^1_{112} = |\mathscr{T}_2|.$$

$\mathscr{T}_1$ can be partitioned into $\mathscr{T}_{11}$ and $\mathscr{T}_{22}$, wheae $\mathscr{T}_{11}$ consists of the 1-dimensional subspaces in $\mathscr{T}_1$ that are not included in $s_1 \cup s_2 \cup s_3$, and $\mathscr{T}_{12}$ consists of those that are included in $s_1 \cup s_2 \cup s_3$. Then $|\mathscr{T}_{11}| = N(1, n) - N(1, 3)$. For an arbitary element s of $\mathscr{T}_{12}$, we have $s = as_1 + bs_2 + cs_3$ with a, b, $c \in F_q$ and $abc \neq 0$. Then

$$|\mathscr{T}_{12}| = \frac{(q-1)^3}{q-1} = (q-1)^2.$$

Therefore

$$p^1_{111} = N(1, n) - N(1, 3) + (q-1)^2 = \frac{q^n - q^3}{q-1} + (q-1)^2.$$

And $s' \in \mathscr{T}_2$ if and only if $s = as_1 + bs_2$ with a, $b \in F_q$ and $ab \neq 0$. Then

$$p^1_{112} = \frac{(q-1)^2}{q-1} = q-1.$$

For $(s_1, s_2, s_3) \in R_1$, if s is a 1-dimensional subspace of $V_n(F_q)$ with $\dim(s \cup s_2 \cup s_3) = 2 = \dim(s_1 \cup s \cup s_3)$, then $s \subset s_2 \cup s_3$, $s \subset s_1 \cup s_3$, and so $s \sqsubseteq (s_2 \cup s_3) \cap (s_1 \cup s_3) = s_3$. Thus, $\dim(s_1 \cup s_2 \cup s) = 3 \neq 2$. Therefore,

$$p^1_{222} = 0.$$

Now let $(s_1, s_2, s_3) \in R_2$, and s be a 1-dimensional subspace of $V_n(F_q)$. Then $\dim(s \cup s_2 \cup s_3) = \dim(s_1 \cup s \cup s_3) = \dim(s_1 \cup s_2 \cup s) = 2$ if and only if

$$s \subset s_2 \cup s_3, \quad s \subset s_1 \cup s_3, \quad s \cup s_1 \cup s_2, \quad s \neq s_1, s_2, s_3.$$

Therefore,

$$p^2_{222} = N(1, 2) - 3 = q-2.$$

Thus we have proved

Theorem 3. *Taking as treatments the 1-dimensional subspaces of $V_n(F_q)$, and defining three distinct treatments to be the first (resp. second) associates if they span a 3-dimensional (resp. 2-dimensional) subspace, we obtain a 3-association scheme with two associate classes and with the following parameters:*

$$v = \frac{q^n - 1}{q-1}, \quad n_1 = \frac{q^n - q^2}{q-1},$$

$$p^1_{111} = \frac{q^n - q^3}{q-1} + (q-1)^2, \quad p^1_{112} = q-1, \tag{15}$$

$$p^1_{222} = 0, \quad p^2_{222} = q-2.$$

Based on the association scheme in Theorem 3, we can construct a class of 3-PBIB designs.

Theorem 4. *Let $3 \leqslant u \leqslant n-1$. Adopt the association scheme in Theorem 3. Take as blocks the u-dimensional subspaces of $V_n(F_q)$, and define a treatment to be arranged in a block if the latter includes the former both as subspaces. Then we obtain a 3-PBIB design with two 3-associate classes and with the parameters in (15) and in the following:*

$$b=N(s, u),$$
$$k=N(1, u), \quad r=N^T(1, u),$$
$$\lambda_1=N^T(3, u), \quad \lambda_2=N^T(2, u),$$

where $N^T(x, u)$ denotes the number of u-dimensional subspaces including a fixed x-dimensional subspace in $V_n(F_q)$[5].

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