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EXTRAPOLATION OF BILINEAR FINITE ELEMENT SOLUTION WITH LOCAL REFINEMENT MESH

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Abstract

For the less smooth solution caused by the reentrant domain it is shown that one step of extrapolation increases the order of bilinear finite element solution from 2 to 3 when the rectangular mesh satisfies certain local refinement condition.

§ 1. Introduction

"Extrapolation techniques for finite element method" is a new topic developed in recent years. At the beginning it was tested with two point boundary value problem and shown that the order of linear finite element solution on an arbitrary mesh can be increased from 2 to 4 by one step of extrapolation. Subsequently, a finite element proof for the extrapolation was given for Poisson equation defined on a triangular domain. Then, an extrapolation technique was given for Poisson equation defined on the quadralateral domain. After 1983, Chinese and German groups were able to establish a systematic theory for extrapolation techniques of finite element method to elliptic problem defined on the polygonal domain and the curved domain. See Rannacher's survey for details. All of these works, except for the one dimensional case, are limited on the triangular elements and require that the triangulation satisfies the uniform condition or piecewise uniform condition even in the sense of certain transformed variant. These geometrical restrictions are contrary to the local refinement mesh used widely for the reentrant domain.

Is it possible to avoid these uniform conditions? This question did not have a positive answer since we had fallen into the restriction of triangular elements. We found only recently that the extrapolation technique can work with certain non-uniform partition when the finite element solution is defined on the quadralateral mesh instead of defining on a triangular mesh. We will prove in this paper that one step of extrapolation increases the order of bilinear finite element solution from 2 to

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3 when the rectangular mesh satisfies certain local refinement condition.

§ 2. Expansion for Integral of Interpolation

In order to discuss the expansion for finite element solution we need the expansion for integral of interpolation. Suppose that Ω is the union of several rectangles and each edge of Ω parallels to the coordinate axis. Consider the bilinear form

$$a(u,\ v)=\int_\Omega (\partial_x u(a_{11}\partial_n v+a_{12}\partial_y v)+\partial_y u(a_{21}\partial_x v+a_{22}\partial_y v))\,dx\,dy, \tag{2.1}$$

where the functions a_{ij} are appropriately smooth. We will use the following notations:

$$L_1^1=a_{11}\partial_x+a_{12}\partial_y,$$
$$L_2^1=a_{21}\partial_x+a_{22}\partial_y,$$
$$L_3^1=(\partial_x a_{11})\partial_x+(\partial_x a_{12})\partial_y,$$
$$L=(\partial_y a_{21})\partial_x+(\partial_y a_{22})\partial_y$$

and, for $D\subset\Omega$,

$$a_D(u,\ v)=\int_D (\partial_x u L_1^1 v+\partial_y u L_2^1 v)\,dx\,dy.$$

Let T^h be a rectangular partition, $e\in T^h$ an arbitrary rectangular element with a center $(x_e,\ y_e)$ and the length of $2h_e$/and $2k_e$ along x and y direction respectively, $d_e=\max(h_e,\ k_e)$. $h=\max\limits_e(d_e)$ and $S^h\subset H_0^1$ the bilinear finite element space over T^h.

For $u\in C(\bar\Omega)\cap H_0^1$, let $u^I\in S^h$ be the bilinear interpolant of u. We want to expand the integrals

$$a(u^I-u,\ v),\ \forall v\in S^h; \tag{2.2}$$
$$a_e(u^I-u,\ v),\ \forall v\in S. \tag{2.3}$$

In order to expand u^I we introduce the functions

$$E(x)=(h_e^2-(x-x_e)^2)/2,\ \forall x\in[x_e-k_e,\ x_e+h_e];$$
$$F(y)=(k_e^2-(y-y_e)^2)/2,\ \forall y\in[y_e-k_e,\ y_e+k_e];$$
$$A(x)=(E(x))^2/6;\ \ B(y)=(F(y))^2/6.$$

It is easy to check

$$E,\ F,\ A,\ B,\ A',\ B'\in C(\bar\Omega);$$
$$E,\ A,\ A'\in C_0^1([x_e-h_e,\ x_e+h_e]),\ \forall e\in T^h;$$
$$F,\ B,\ B'\in C_0^1([y_e-k_e,\ y_e+k_e]),\ \forall e\in T^h.$$

By the standard argument we can prove

Lemma 1. *If* $u\in C^3(e)$, *then*

$$u^I(x,\ y)-u(x,\ y)=E(x)\partial_x^2 u(x,\ y)+F(y)\partial_y^2 u(x,\ y)+R_{12}$$

If $u\in H^4(e)$, *then*

$$u^I(x, y)-u(x, y)=E(x)\partial_x^2u(x, y)+F(y)\partial_y^2u(x, y)+2A'(x)\partial_x^3u(x, y)$$
$$+2B'(y)\partial_y^3u(x, y)+R_2,$$

where

$$\|R_1\|_{s,\infty,e}\leqslant cd_e^{3-s}\|u\|_{3,\infty,e}, \quad s=0, 1 \tag{2.4}$$

$$\|R_2\|_{1,2,e}\leqslant cd_e^3\|u\|_{4,2,e}. \tag{2.5}$$

In order to expand the integrals (2.2) and (2.3) we need to estimate each term appeared in the expansions in Lemma 1.

Lemma 2. *For $v\in S^h$ and an auxiliary function $W\in W^{2,1}$ we have*

(i) *If $u\in C^3(\bar{\Omega})$ then*

$$|a(R_1, v)|\leqslant ch^3\|u\|_{3,\infty}(\|v-W\|_{1,1}+h\|W\|_{2,1});$$

(ii) *If $u\in H^4(e)$ then*

$$|a_e(R_2, v)|\leqslant cd_e^3\|u\|_{4,2,e}\|v\|_{1,2,e}.$$

Proof By (2.4),

$$|a(R_1, v)|\leqslant|a(R_1, v-W)|+|a(R_1, W)|$$
$$\leqslant ch^2\|u\|_{3,\infty}\|v-W\|_{1,1}+|a(R_1, W)|.$$

Integrating by parts we obtain

$$\int_\Omega \partial_x R_1 L_1^1 W=\int_{\partial\Omega} R_1 L_1^1 W(x\cdot n)\,ds-\int_\Omega R_1\partial_x L_1^1 W,$$

$$|a(R_1, W)|\leqslant C\|R\|_{0,\infty}(\|W\|_{2,1}+\|W\|_{1,1,\partial\Omega})$$
$$\leqslant ch^3\|u\|_{3,\infty}\|W\|_{2,1},$$

and (i) follows. The conclusion (ii) follows from (2.5).

Lemma 3. *For $\phi\in H^1(e)$ and $v\in S^h$ we have*

$$|a_e(A'(x)\phi, v)|\leqslant cd_e^3\|\phi\|_{1,2,e}\|v\|_{1,2,e}.$$

Proof Since

$$A'(x_e\pm h_e)=A(x_e\pm h_e)=0,$$

we have, by integral by parts,

$$\int_e \partial_x(A'(x)\phi)L_1^1 v=-\int_e A'(x)\phi\partial_x L_1^1 v$$
$$=\int_e A(x)\partial_x(c\phi\partial_x L_1^1 v)$$
$$=0(d_e^4)\|\phi\|_{1,2,e}\|v\|_{2,2,e}.$$

Note that

$$\|\partial_x\partial_y v\|_{0,2,e}\leqslant d_e^{-1}\|v\|_{1,2,e}, \quad \forall v\in S^h.$$

We obtain

$$\int_e \partial_x(A'(x)\phi)L_1^1 v=0(d_e^3)\|\phi\|_{1,2,e}\|v\|_{1,2,e}.$$

On the other hand,

$$\int_e \partial_y(A'(x)\phi)L_2^1 v=\int_e A'(x)\partial_y\phi L_2^1 v$$
$$=O(d_e^3)\|\phi\|_{1,2,e}\|v\|_{1,2,e}.$$

Lemma 3 follows.

Lemma 4. *For $v\in S^h$ and $W\in W^{2,1}$ we have*

$$a(E(x)\partial_x^2 u,\ v)=-\sum_e \frac{1}{3}h_e^2\int_e(\partial_x^2 u\partial_x L_1^1 v-\partial_y\partial_x^2 u L_2^1 v)+r,$$

$$|r|\leqslant ch^3\|u\|_{3,\infty}(\|v-W\|_{1,1}+h\|W\|_{2,1}).$$

Proof Since

$$E(x_e\pm h_e)=0,\quad E(x)=\frac{1}{3}h_e^2-A''(x),$$

we have, by integral by parts,

$$\int_e\partial_x(E(x)\partial_x^2 u)L_1^1 v=-\int_e E(x)\partial_x^2 u\partial_x L_1^1 v$$

$$=-\frac{1}{3}h_e^2\int_e\partial_x^2 u\partial_x L_1^1 v+\int_e A''(x)\partial_x^2 u\partial_x L_1^1 v$$

and

$$\left|\int_e A''(x)\partial_x^2 u\partial_x L_1^1 v\right|\leqslant cd_e^3\|u\|_{3,\infty,e}\|v\|_{2,1,e},$$

$$\|v\|_{2,1,e}\leqslant\|v-W^I\|_{2,1,e}+\|W^I\|_{2,1,e}$$

$$\leqslant cd_e^{-1}\|v-W^I\|_{1,1,e}+\|W\|_{2,1,e}.$$

On the other hand

$$\int_e\partial_y(E(x)\partial_x^2 u)L_2^1 v=\int_e E(x)\partial_y\partial_x^2 u L_2^1 v$$

$$=\frac{1}{3}h_e^2\int_e\partial_y\partial_x^2 u L_2^1 v-\int_e A''(x)\partial_y\partial_x^2 u L_2^1 v.$$

Integrating by parts we have, for the second term,

$$\int_\Omega A''(x)\partial_y\partial_x^2 u L_2^1 v=\int_\Omega A''(x)\partial_y\partial_x^2 u L_2^1 W+O(h^2)\|u\|_{3,\infty}\|v-W\|_{1,1}$$

$$=-\int_\Omega A'(x)\partial_y\partial_x^3 u L_2^1 W+0(h^2)\|u\|_{3,\infty}(\|v-W\|_{1,1}+h\|W\|_{2,1}),$$

$$-\int_\Omega A'(x)\partial_y\partial_x^3 u L_2^1 W_1$$

$$=\int_\Omega A'(x)\partial_x^3 u\partial_y L_2^1 W-\int_{\partial\Omega}A'(x)\partial_x^3 u L_2^1 W(n\cdot y)\,ds.$$

Lemma 4 follows.

Lemma 5. *For $u\in H^4(e)$ and $v\in S^h$ we have*

$$a_e E((x)\partial_x^2 u,v)=-\frac{1}{3}h_e^2\int_e(\partial_x^2 u\partial_x L_1^1 v-\partial_y\partial_x^2 u L_2^1 v)+O(d_e^3)\|u\|_{4,2,e}\|v\|_{1,2,e}.$$

Proof Similar to Lemma 4 we need only the following estimates:

$$\int_e A''(x)\partial_x^2 u\partial_x L_1^1 v=\int_e A(x)\partial_x^2(\partial_x^2 u\partial_x L_1^1 v)$$

$$=O(d_e^4)\|u\|_{4,2,e}\|v\|_{2,2,e}$$

$$=O(d_e^3)\|u\|_{4,2,e}\|v\|_{1,2,e},$$

$$\int_e A''(x)\partial_y\partial_x^2 u L_2^1 v = -\int_e A'(x)(\partial_y\partial_x^3 u L_2^1 v + \partial_y\partial_x^2 u \partial_x L_2^1 v)$$
$$= -\int_e A'(x)\partial_y\partial_x^3 u L_2^1 v + {}_eA(x)\partial_x(\partial_y\partial_x^2 u \partial_x L_2^1 v)$$
$$= O(d_e^3)\|u\|_{4,2,e}\|v\|_{1,2,e} + O(d_e^4)\|u\|_{4,2,e}\|v\|_{2,2,e}$$
$$= O(d_e^3)\|u\|_{4,2,e}\|v\|_{1,2,e}.$$

The above lemmas lead to the main result of this section.

Theorem 1. *For $v\in S^h$ and $W\in W^{2,1}$ we have*

(i) *If $u\in C^3(\overline{\Omega})$ then*

$$a(u^I-u,\ v)$$
$$= -\sum_e \frac{1}{3} h_e^2 \int_e (\partial_x^2 u \partial_x L_1^1 v - \partial_{yx}^2 u L_2^1 v) + \sum_e \frac{1}{3} k_e^2 \int_e (\partial_x \partial_y^2 u L_1^1 v - \partial_y^2 u \partial_y L_2^1 v)$$
$$+ O(h^2)\|u\|_{2,\infty}(\|v-W\|_{1,1} + h\|W\|_{2,1});$$

(ii) *If $u\in H^4(e)$, then*

$$a_e(u^I-u,\ v)$$
$$= -\frac{1}{3} h_e^2 \int_e (\partial_x^2 u \partial_x L_1^1 v - \partial_y \partial_x^2 u L_2^1 v) + \frac{1}{3} k_e^2 \int_e (\partial_x \partial_y^2 u L_1^1 v - \partial_y^2 u \partial_y L_2^1 v)$$
$$+ O(d_e^3)\|u\|_{4,2,e}\|v\|_{1,2,e}.$$

§ 3. Extrapolation for Smooth Solution

We assume in this section that Ω is a rectangular domain. Cansider the boundary value problem: find $u\in H_0^1$ such that

$$a(u,\ \phi) = (f,\ \phi),\quad \forall\phi\in H_0^1. \tag{3.1}$$

For simplicity we assume

$$a_{12} = a_{21},\ \text{i. e. } a(u,\ v) = a(v,\ u).$$

The rectangular partion is further assumed to satisfy the following conditions.

A1. There exists some constant $\gamma\geqslant 1$ such that

$$h_e \leqslant ch^\gamma,\ k_e \geqslant ch^\gamma,\ \forall e\in T^h;$$

A2. For any two abjacent elements e and e',

$$c_1 h_e \leqslant h_{e'} \leqslant c_2 h_e,\ c_1 k_e \leqslant k_{e'} \leqslant c_2 k_e.$$

We need the Green function $G_z\in W_0^{1,p}(p<2)$: for $z\in\Omega$,

$$a(G_z,\ \phi) = \phi(z)\quad \forall\phi\in C_0^\infty.$$

Let G_z^h be the finite element projection of G_z:

$$a(G_z^h,\ v) = v(z)\quad \forall v\in S^h.$$

Since G is singular at point z we need also the regularized Green function g_z. For $z\in\Omega$ there exists $e\in T^h$ such that $z\in e$. One can construct a regularized δ function $\hat{\delta}\in C_0^\infty(e)$ such that

$$\int_\Omega \hat{\delta} v = v(z)\quad \forall v\in S^h,$$

$$\|\hat{\delta}\|_{k,\infty}\leqslant c\rho_e^{-k-2},\ k=0,\ 1,\ 2,\ \cdots,$$
$$\operatorname{diam}(\operatorname{supp}\hat{\delta})\leqslant 2\rho_e,$$

with $\rho_e=\min(h_e,\ k_e)$. We define g_z as follows: $g_z\in H_0^1$ satisfies

$$a(g_z,\ \phi)=(g_z,\ \phi),\quad \forall\phi\in H_0^1.$$

Then G_z^h is also the finite element projection of g_z. One can prove (c. f. [3])

$$\|G_z^h-G_z\|_{1,1}+\|G_z^h-g_z\|_{1,1}+h\|g_z\|_{2,1}\leqslant ch|\ln h|.$$

Let $u^h\in S^h$ be the finite element projection of u. Taking $v=G_z^h$ and $W=g_z$ in Theorem 1 we obtain

$$\begin{aligned}(u^I-u^h)(z)&=a(u^I-u^h,\ G_z^h)=a(u^I-u,\ G_z^h)\\&=-\sum_e\frac{1}{3}h_e^2\int_e(\partial_x^2u\partial_xL_1^1G_z^h-\partial_y\partial_x^2uL_2^1G_z^h)\\&\quad+\sum_e\frac{1}{3}k_e^2\int_e(\partial_x\partial_y^2uL_1^1G_z^h-\partial_y^2u\partial_yL_2^1G_z^h)\\&\quad+O(h^3|\ln h|).\end{aligned}\tag{3.2}$$

Note that

$$\partial_xL_1^1G=L_3^1G_z^h+a_{12}\partial_x\partial_yG_z^h.\tag{3.3}$$

In order to estimate $\partial_x\partial_yG_z^h$ we need

Lemma 6. *For $W\in H$ and $v\in S$ we have*

$$\sum_e h_e^2\int_e W\partial_x\partial_yv=-\sum_e h_e^2\int_e\partial_yW\partial_xv.$$

Proof Set $\Omega=\bigcup\Omega_i$. If $e\subset\Omega_i$ then $h_e=c$ and $\partial_xv\in H^1(\Omega_i)$. Integrating by parts we have

$$\sum_{e\subset\Omega_i}h_e^2\int_e W\partial_x\partial_yv=h_e^2\int_{\Omega_i}W\partial_x\partial_yv=-h_e^2\int_{\Omega_i}\partial_yW\partial_yv.$$

Lemma 6 follows.

(3.2), (3.3) and Lemma 6 lead to

$$\begin{aligned}(u^I-u^h)(z)=&-\sum_e\frac{1}{3}h_e^2\int_e(\partial_x^2uL_3^1G_z^h-\partial_y\partial_x^2uL_2^1G_z^h)+\sum_e\frac{1}{3}k_e^2\int_e(\partial_x\partial_y^2uL_1^1G_z^h-\partial_y^2uL_4^1G_z^h)\\&+\sum_e\frac{1}{3}h_e^2\int_e\partial_y(a_{12}\partial_x^2u)\partial_xG_z^h+\sum_e\frac{1}{3}k_e^2\int_e\partial_x(a_{21}\partial_y^2u)\partial_yG_z^h\\&+O(h^3|\ln h|)\|u\|_{3,\infty},\end{aligned}\tag{3.4}$$

or, by using the abbreviation D,

$$(u^I-u^h)(z)=\sum_e h_e\int_e D^3uDG_z^h+\sum_e k_e^2\int_e D^3uDG_z^h+O(h^3|\ln h|).$$

We now divide each element in T^h into four equal rectangular elements and form a mew partion $T^{h/2}$. Let $S^{h/2}$ be the corresponding bilinear finite element projections of u and G_z respectively. Note that $e'\in T^{h/2}$, $e\in T^h$ and $e'\subset e$ imply $h_{e'}=\frac{1}{2}h_e$ and that

$$u^I(z)=u^{I/2}(z)=u(z)$$

for z being the nodal points of T^h. We have

$$(u-u^{h/2})(z)=\sum_{e}\sum_{e,\subset e} h_e^2\int_{e'} D^3uDG_z^h+\cdots+O(h^3|\ln h|)$$
$$=\frac{1}{4}\sum_{e}\sum_{e'\subset e} h_e^2\int_{e} D^3uDG_z^h+\cdots+O(h^3|\ln h|)$$
$$=\frac{1}{4}\sum_{e} h_e^2\int_{e} D^3uDG_z^h+\cdots+O(h^3|\ln h|),$$

and hence,

$$|4(u-u^{h/2})(z)-(u-u^h)(z)|$$
$$\leqslant ch^2\|G_z^h-G_z^{h/2}\|_{1,1}\|u\|_{3,\infty}+O(h^3|\ln h|)\|u\|_{3,\infty}$$
$$\leqslant ch^3|\ln h|\,\|u\|_{3,\infty},$$

which leads to

Theorem 2. *Assume that $u\in C^3(\bar{\Omega})$ and T^h satisfies condition A1 and A2. Then, for z being the nodal points of T^h,*

$$|(4u^{h/2}-u^h)(z)/3-u(z)|\leqslant ch^3|\ln h|.$$

§ 4. Extrapolation for Less Smooth Solution

We discuss in this section the polygonal domain with reetreent conners which cause certain singularities for the solution. In order to counteract the singularity and achieve the high accuracy estimate from extrapolation we will adopt certain local refinement partition.

Consider for simplicity the model problem

$$-\Delta u=f \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega \tag{4.1}$$

with the domain Ω consisting of several rectanges (for example an L-shap domain). We assume that each edge of Ω is parallel to the coodinate axis.

In order to describe the smoothness of the solution we need some notations. Let $\{V_j\}_{j=1}^m$ be the conners of Ω, a_j the cooresponding interior angles and $\beta_j=\pi/a_j$. For $x=(x_1, x_2)\in\Omega$ define the function

$$\Gamma_j(x)=|x-V_j|,$$
$$\Gamma^\tau(x)=\prod_j(\Gamma_j(x))^{\tau_j},\ \tau=(\tau_1, \tau_2, \cdots, \tau_m),$$

and the weight function space

$$H_\tau^k=\{\phi/\tau^{\tau-k+|\alpha|}\partial^\alpha\phi\in L^2,\ \forall|a|\leqslant k\}$$

with the norm

$$\|\phi\|_{k,\tau}=\Big(\sum_{|\alpha|\leqslant k}\|r^{\tau-k+|\alpha|}\partial^\alpha\phi\|_{L^2}^2\Big)^{1/2}.$$

By [1], for $\tau>k-1-\beta$ and $k\geqslant 2$, if $f\in H_\tau^{k-2}$ then there exists $u\in H_\tau^k$ satisfying (4.1) and

$$\|u\|_{k,\tau}\leqslant C\|f\|_{k-2,\tau}. \tag{4.2}$$

We now come to describe the rectangular partition T^h. Denote

$$d_j=\max\{d_e: e\in T^h,\ V_j\in e\},$$
$$T_j^h=\{e\in T^h/\min_{x\in e} r_j(x)<d_j\},$$
$$T_0^h=T^h\backslash\bigcup_j T_j^h,$$
$$\Omega_0=\bigcup_{e\in T_0^h} e,\ \Omega_j=\bigcup_{e\in T_j^h} e,\ r_e^\tau=\min_{x\in e} r^\tau(x).$$

Besides the conditions A1 and A2 in section 4, T^h satisfies the following conditions:

A3. $d_e\leqslant c r_e^\sigma h\ \forall e\in T_0^h$;

A4. For any two adjacent elements $e,\ e'\in T_0^h$, we have

$$|h_e-h_{e'}|\leqslant ch^2 r_e^{2\sigma-1},$$
$$|k_e-k_{e'}|\leqslant ch^2 r_e^{2\sigma-1},$$

A5. $d_j\leqslant ch^{1/(1-\sigma_j)}$,

where $\sigma=(\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_m)$ is called the refinement index.

Let us explain above conditions by an example. Suppose

$$\Omega_1=\{(x_1,\ x_2)/-1<x_1<1,\ 0<x_2<1\},$$
$$\Omega_2=\{(x_1,\ x_2)/0<x_1<1,\ -1<x_2<1\}$$

and $\Omega=\Omega_1\cup\Omega_2$. For $q\geqslant 1$ we construct

$$T=\{[S_i,\ S_{i+1}]\times[S_j,\ S_{j+1}]\},$$

where

$$S_i=\begin{cases} -\dfrac{2n-i}{2n}, & 0\leqslant i\leqslant n,\\ -\dfrac{1}{2}\left(\dfrac{2n-i}{n}\right)^q, & n\leqslant i\leqslant 2n,\\ \dfrac{1}{2}\left(\dfrac{i-2n}{n}\right)^q, & 2n\leqslant i\leqslant 3n,\\ \dfrac{i-2n}{2n}, & 3n\leqslant i\leqslant 4n.\end{cases}$$

Set $V_2=(0,\ 0)$, $\sigma_1=1-1/q$, $\sigma_j=0\,(j>0)$. Then T^h satisfies conditions A1-A5 for $\sigma=(\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_m)$.

One can prove (c. f. [3]) the auxiliary Green functions g_z and G_z^h satisfy

$$\|g_z\|_{1,2}+\|G_z^h\|_{1,2}\leqslant C|\ln h|^{1/2}.$$

To derive our main result we need some lemmas.

Lemma 7. *For $v\in S^h$ and $W\in H^1_{3\sigma-1}$ we have*

$$\sum_{e\in T_0^h} h_e^2\int_e W\partial_{x_1}\partial_{x_2}v=-\sum_{e\in T_0^h} h_e^2\int_e \partial_{x_2}W\partial_{x_1}V+O(h^3)\,\||W\||_{1,3\sigma-1}\|V\|_{1,2}.$$

Proof Consider the interior element $e\in T_0^h$. Let S_1 and S_1' be the edges of e parallel to x_1 axes. By integral by parts,

$$\sum_{e\in T_0^h} h_e^2\int_e W\partial_{x_1}\partial_{x_2}V=-\sum_{e\in T_0^h} h_e^2\int_e \partial_{x_2}W\partial_{x_1}V+\sum_{e\in T_0^h} h_e^2\left(\int_{S_1'}-\int_{S_1}\right)W\partial_{x_1}V\,dS. \qquad (4.3)$$

For two elements e and e' with the common side s_1, $\partial_{x_1} v$ is continuous in $e\cup e'$, $h_e=h_{e'}$ and

$$\partial_{x_1}v(x)=0\ \forall\ x\in s_1\subset\partial\Omega.$$

Hence, all the line integrals in (4.3) over the interior of Ω_0 are canceled and the boundary integrals remain:

$$\sum_{e\in T_0^h} h_e^2\Big(\int_{s_1'}-\int_{s_1}\Big)W\partial_{x_1}v\,ds=\sum_{s_1\subset\partial\Omega_0\backslash\partial\Omega} h_{s_1}^2 s\int_{s_1} W\partial_{x_1}v ds,\quad s=\pm 1.$$

For $s_1\subset\partial\Omega_0\backslash\partial\Omega$ there exist elements $e_1\in T_0^h$ and $e_2\in T_j^h$ such that $e_1\cap e_2=s_1$ and hence, by A5,

$$h_{s_1}=h_{e_1}=h_{e_2}\leqslant d_j\leqslant ch^{1/(1-\sigma_j)}.$$

Consider

$$T_{e_1}=\{e\in T_0^h/x_e=x_{e_1},\ |y_e-y_{e_1}|<h_{s_1}\},$$
$$D=\bigcap_{e\in T_{e_1}} e.$$

For any $e\in T_{e_1}$, we have $h_e=h_{e_1}=h_{s_1}$ and D is a rectange with one edge of length of h_{s_1} and another edge of length of

$$k=\sup\{|x_2-x_2'|:x,\ x'\in D\}\geqslant ch_{s_1}$$

and

$$\partial_{x_1}v\in H^1(D).$$

Note that

$$r_e\geqslant d_j\ \forall\ e\in T_{e_1}\subset T_0^h.$$

A trace theorem leads to

$$\begin{aligned}
h_{s_1}^2\Big|\int_{s_1} W\partial_{x_1}v\,ds\Big| &\leqslant ch_{s_1}\int_D|W\partial_{x_1}v|+ch_{s_1}^2\int_D|\nabla(W\partial_{x_1}v)|\\
&\leqslant ch_{s_1}\int_D|W\partial_{x_1}v|+ch_{s_1}^2\int_D|\nabla W|\,|\partial_{x_1}v|+ch_{s_1}^2\int_D|W|\,|\nabla v|\\
&\leqslant ch_{s_1}^{3-3\sigma_j}\int_P r^{3\sigma-2}|W\partial_{x_1}v|+ch_{s_1}^{3-3\sigma_j}\int r^{3-1}|\nabla W|\,|v|\\
&\quad+ch_{s_1}h_{s_1}^{3-3\sigma_j}\int r^{3\sigma_j-2}|W|\,|\nabla^2 v|\\
&\leqslant ch_{s_1}^{3-3\sigma_j}\|W\|_{0,3\sigma-2,D}\|v\|_{1,2,D}+ch_{s_1}^{3-3\sigma_j}\|W\|_{1,3\sigma-1,D}\|v\|_{1,2,D}\\
&\quad+ch_{s_1}h_{s_1}^{3-3\sigma_j}\|W\|_{0,3\sigma-2,D}\|v\|_{2,2,D}\\
&\leqslant ch_{s_1}^{3-3\sigma_j}\|W\|_{1,3\sigma-1,D}\|v\|_{1,2,D},
\end{aligned}$$

where we have used the inverse estimate since $h_e=h_{s_1}$. Hence

$$\sum_{s_1\subset\partial\Omega_0\backslash\partial\Omega} h_{s_1}^2\Big|\int_{s_1} W\partial_{x_1}v\,ds\Big|\leqslant ch^3\|W\|_{1,3\sigma-1}\|v\|_{1,2}.$$

Lemma 8. *For* $\phi\in H_0^1$ *and* $W\in H_{3\sigma}^1$, *we have*

$$\Big|\sum_{e\in T_0^h} h_e^2\int_e W\partial_{x_1}\phi\Big|\leqslant ch^2\|W\|_{1,3\sigma}(\|\phi\|_{0,-\sigma}+h\|\phi\|_{1,2}).$$

Proof For $e\in T_0^h$, s_2, s_2' being the edges of e parallel to x_2, by integral by part,

$$\sum_{e\in T_0^h} h_e^2\int_e W\partial_{\alpha_1}\phi=-\sum_{e\in T_0^h} h_e^2\int_e \partial_{\alpha_1}W\phi+\sum_{e\in T_0^h} h_e^2\Big(\int_{s_1'}-\int_{s_2}\Big)W\phi\,ds. \tag{4.4}$$

Since, for $e\in T_0^h$,

$$\Big|h_e^2\int_e \partial_{\alpha_1}W\cdot\phi\Big|\leqslant ch^2r_e^{2\sigma}\int_e|\nabla W|\,|\phi|$$

$$\leqslant ch^2\int_e r^{2\sigma}|\nabla W|\,|\phi|\leqslant ch^2|\!|\!| W|\!|\!|_{1,3\sigma,e}|\!|\!|\phi|\!|\!|_{0,-\sigma,e},$$

we have

$$\Big|\sum_{e\in T_0^h} h_e^2\int_e \partial_{\alpha_1}W\cdot\phi\Big|\leqslant ch^2|\!|\!| W|\!|\!|_{1,3\sigma}|\!|\!|\phi|\!|\!|_{0,-\sigma}.$$

We now consider the last term in (4.4). Let e and e' be the two elements with common side s_2. After a re-arrangement of summation we obtain

$$\Big|\sum_{e\in T_0^h} h_e^2\Big(\int_{s_1'}-\int_{s_2}\Big)W\phi\,ds\Big|\leqslant\Big|\sum_{s_2}(h_e^2-h_{e'}^2)\int_{s_2}W\phi\,ds\Big|+\sum_{s_2\subset\partial\Omega_0\backslash\partial\Omega}h_e^2\Big|\int_{s_2}W\phi\,ds\Big|. \tag{4.5}$$

Similar to Lemma 7 we can prove

$$\sum_{s_2\subset\partial\Omega_0\backslash\partial\Omega}h_e^2\Big|\int_{s_2}W\phi\,ds\Big|\leqslant ch^2|\!|\!| W|\!|\!|_{1,3\sigma}(|\!|\!|\phi|\!|\!|_{0,-\sigma}+h\|\phi\|_{1,2}).$$

By a trace theorem and conditions A4 and A3,

$$\Big|(h_e^2-h_{e'}^2)\int_{s_2}W\phi\,ds\Big|\leqslant ch_e^{-1}|h_e^2-h_{e'}^2|\int_e|W\phi|+c|h_e^2-h_{e'}^2|\int_e|\nabla(W\phi)|$$

$$\leqslant c|h_e-h_{e'}|\int_e|W\phi|+c|h_e^2-h_{e'}^2|\int_e|\nabla(W\phi)|$$

$$\leqslant ch^2\int_e r^{2\sigma-1}|W\phi|+ch^2\int r^{2\sigma}|\nabla W|\,|\phi|$$

$$+ch^3\int_e r^{3\sigma-1}|W|\,|\nabla\phi|$$

$$\leqslant ch|\!|\!| W|\!|\!|_{1,3\sigma,e}(|\!|\!|\phi|\!|\!|_{0,-\sigma,e}+h\|\phi\|_{1,2,e}).$$

Lemma 8 follows.

We now consider the estimate near the conners.

Lemma 9. *For $u\in H^2_{3\sigma-2}$ and $v\in S^h$ we have*

$$|a_{\Omega_j}(u^I-u,\ v)|\leqslant ch^3|\!|\!| u|\!|\!|_{2,3\sigma-2}\|v\|_{1,2}.$$

Proof For an appropriate $p<2$ and $q=2p/(2-p)$ we have

$$\|u^I-u\|_{1,2,\cdot}\leqslant c^{1-2/q}\|u^I-u\|_{1,q,\Omega_j}\leqslant cd^{1-2/q}\|u\|_{1,q,\Omega_j},$$

where $d=\operatorname{diam}\Omega_j$. Let D be the union of some squares near v_j such that

$$\Omega_j\subset D\subset\Omega;\quad \operatorname{diam}(D)\leqslant 2d.$$

Then $d_j\leqslant d\leqslant cd_j$. By an embedding theorem

$$\|u\|_{1,q,\Omega_j}\leqslant\|u\|_{1,q,D}\leqslant c\|u\|_{2,p,D}+cd^{-1}\|u\|_{1,p,D}.$$

Thus

$$\|u^I-u\|_{1,2,\Omega_j}\leqslant cd^{1-2/q}\|u\|_{1,q,D}\leqslant cd^{1-2/q}\|u\|_{2,p,D}+cd^{-2/q}\|u\|_{1,p,D}$$

$$\leqslant cd^{1-2/q}|\!|\!| u|\!|\!|_{2,3\sigma-2,D}d^{2/p-2+3-3\sigma_j}+cd^{-2/q}|\!|\!| u|\!|\!|_{1,3\sigma-3,D}d^{2/p-1+3-3\sigma_j}$$

$$\leqslant cd^{3-3\sigma_j}|\!|\!| u|\!|\!|_{2,3\sigma-2,D}\leqslant ch^3|\!|\!| u|\!|\!|_{2,3\sigma-2,D}$$

and hence

$$|a_{\Omega_j}(u^I-u,\ v)|\leqslant c\|u^I-u\|_{1,2,\Omega_j}\|v\|_{1,2,\Omega_j}\leqslant ch^3\|\!|u|\!\|_{2,3\sigma-2}\|v\|_{1,2}.$$

Lemma 10. *For $W\in H^2_\sigma$ and $\sigma>1-\beta$, we have*

(i) $\|W-W^I\|_{1,2}\leqslant ch\|\!|\Delta W|\!\|_{0,\sigma}$,

(ii) $\|\!|g_z-G^h_z|\!\|_{0,-\sigma}\leqslant ch|\ln h|^{1/2}$.

Proof (i) For interior element $e\in T^h_0$, by A3,

$$\|W-W^I\|_{1,2,e}\leqslant cd_e\|\nabla^2W\|_{0,2,e}\leqslant chr^\sigma_e\|\nabla^2W\|_{0,2,e}\leqslant ch\|r^\sigma\nabla^2W\|.$$

Similar to Lemma 9 we can prove

$$\|W-W^I\|_{1,2,\Omega_j}\leqslant ch\|\!|W|\!\|_{2,\sigma}.$$

Thus,

$$\|W-W^I\|_{1,2}\leqslant ch\|\!|W|\!\|_{2,\sigma}\leqslant ch\|\!|\Delta W|\!\|_{0,\sigma}.$$

In order to prove (ii) we need the dual argument. For $\phi\in L^2$, $r^{-\sigma}\phi\in H^0_\sigma$, by $\sigma>1-\beta$, there exists $W\in H^2_\sigma\cap H^1_0$ such that

$$-\Delta W=r^{-\sigma}\phi.$$

Thus

$$\begin{aligned}\int r^{-\sigma}(g_z-G^h_z)\phi=\int\nabla(g_z-G^h_z)\nabla W=\int\nabla(g_z-G^h_z)\nabla(W-W^I)\\ \leqslant\|g_z-G^h_z\|_{1,2}\|W-W^I\|_{1,2}\leqslant ch|\ln h|^{1/2}\|\!|\Delta W|\!\|_{0,-\sigma}\\ \leqslant ch|\ln h|^{1/2}\|\phi\|_{1,2}\end{aligned}$$

and hence

$$\|r^{-\sigma}(g_z-G^h_z)\|_{0,2}\leqslant ch|\ln h|^{1/2}.$$

We now come to discuss the error expansion of finite element solution. Taking $v=G^h_z$ in Theorem 1 and using Lemma 9 we have

$$\begin{aligned}&(u^I-u^h)(z)\\&=a(u^I-u,\ G^h_z)\\&=\sum_{e\in T^h_0}a_e(u^I-u,\ G^h_z)+O(h^3)\|\!|u|\!\|_{2,3\sigma-2}\|G^h_z\|_{1,2}\\&=-\sum_{e\in T^h_0}\frac{1}{3}h^2_e\int_e(\partial^2_{x_1}uL^1_3G^h_z-\partial_{x_1}\partial^2_{x_1}uL^1_2G^h_z)+\sum_{e\in T^h_0}\frac{1}{3}k^2_e\int_e(\partial_{x_1}\partial^2_{x_2}uL^1_1G^h_z-\partial^2_{x_2}uL^5_4G^h_z)\\&\quad+\sum_{e\in T^h_0}\frac{1}{3}h^2_e\int_e\partial_{x_2}(a_{12}\partial^2_{x_1}u)\partial_{x_2}G^h_z+\sum_{e\in T^h_0}\frac{1}{3}k^2_e\int_e\partial_{x_1}(a_{21}\partial^2_{x_2}u)\partial_{x_2}G^h_z\\&\quad+O(h^3|\ln h|^{1/2})(\|\!|u|\!\|_{2,3\sigma-2}+\|\!|u|\!\|_{4,3\sigma})+O(1)\sum_{e\in T^h_0}d^3_e\|u\|_{4,2,e}\|G^h_z\|_{1,2,e}.\end{aligned}$$

For $e\in T^h_0$, $d_e\leqslant chr^\sigma_e$,

$$d^3_e\|u\|_{4,2,e}\leqslant ch^3r^{3\sigma}_e\|u\|_{4,2,e}\leqslant ch^3\|\!|u|\!\|_{3,3\sigma}.$$

Since $a_{11}=a_{22}=1$ and $a_{12}=a_{21}=0$ in (4.1), we have

$$\begin{aligned}&(u^I-u^h)(z)\\&=\sum_{e\in T^h_0}\left(\frac{1}{3}h^2_e\int_e\partial_{x_2}\partial^2_{x_1}u\partial_{x_2}G^h_z+\frac{1}{3}k^2_e\int_e\partial_{x_1}\partial^2_{x_2}u\partial_{x_1}G^h_z\right)+O(h^3|\ln h|^{1/2})\|\!|u|\!\|_{4,3\sigma}.\end{aligned}$$

Using Lemma 8 and Lemma 10 we obtain

$$(u^I-u^h)(z)$$
$$=\sum_{e\in T_0^h}\left(\frac{1}{3}h_e^2\int_e \partial_{x_2}\partial_{x_1}^2 u\partial_{x_2}g_z+\frac{1}{3}k_e^2\int_e \partial_{x_2}\partial_{x_2}^2 u\partial_{x_1}g_z\right)+O(h^3|\ln h|^{1/2})|||u|||_{4,3\sigma}+R, \quad (4.6)$$

where

$$|R|\leqslant ch^2|||u|||_{4,3\sigma}(|||g_z-G_z^h|||_{0,-\sigma}+h\|G_z^h-g_z\|_{1,2})\leqslant ch^3|\ln h|^{1/2}|||u|||_{4,3\sigma}.$$

Similar to section 3 we have a new partition $T^{h/2}$. Let $T_0^{h/2}$ be the set of the elements refined from T_0^h. Then $T^{h/2}$ satisfies conditions A1–A5. Choose appropriately $\hat{\delta}$ such that g_z is also the regularized Green function corresponding to $T^{h/2}$. Thus, we have a similar expansion as (4.6) for $T^{h/2}$ and hence, for the nodal points z of T^h,

$$|4(u-u^{h/2})(z)-((u-u^h)(z)|\leqslant ch^3|\ln h|^{1/2}|||u|||_{4,3\sigma},$$

which leads to

Theorem 3. *Suppose that*

$$u\in H_{3\sigma}^4,\quad \sigma>1-\beta$$

and the partition T^h satisfies conditions A1–A5. Then, for z being the nodal points of T^h,

$$\left|\frac{1}{3}(4u^{h/2}-u^h)(z)-u(z)\right|\leqslant ch^3|\ln h|^{1/2}.$$

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