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A STRONG RESONANCE PROBLEM

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Abstract

Consider a functional $f(x, v) = (Ax, x)/2 + G(x, v)$, defined on a product space $H\times V$, where H is a Hilbert space and V is a compact manifold. Suppose that the linear part (Ax, x) is at resonance. In this paper, the strong resonance problem is studied in the variational approach, the existence of at least, cuplength $V+1$ critical points of f is proved. The abstract theorems are then applied to the existence problems of solutions for elliptic boundary value problems and Hamiltonian systems.

§ 0. Introduction

A boundary value problem with the linear part at resonance was firstly studied by Landesman-Lazer in 1970. They considered the boundary value problem:

$$y'' + y = g_0(y) + h(t), \quad t\in(0, \pi)$$
$$y(0) = y(\pi) = 0, \tag{0.1}$$

where $g_0\in C(\boldsymbol{R})$ is bounded. Under the condition that the function $G(y) = G_0(y) + h(t)y$, $G_0(y) = \int_0^y g_0(x)dx$, satisfies

$$\int_0^\pi G(\alpha \sin t)dt \to +\infty \quad (\text{or} \quad -\infty) \quad \text{as } |\alpha|\to\infty, \tag{0.2}$$

(0.1) possesses a solution.

Since then a vast literature extended and improved their results to various types of problems.

Because the linear part is at resonance, in general, there is neither a priori bound for solutions nor Palais-Smale condition for the variational approach. Roughly speaking, the difficulty lies in the lack of conpactness. However, the essence of the Landesman-Lazer condition is to provide such a compactness condition.

Of course, other kinds of resomance are of interest as well. The strong resonance condition:

$$g_0(y)\to 0 \text{ and } G_0(y)\to\beta, \text{ a const.} \quad \text{as } |y|\to\infty \tag{0.3}$$

or more generally,

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$$g_0,\ G_0 \text{ are bounded and uniformly continuous} \tag{0.3}$$

was posed by Bartolo-Benci-Furtodato[1]. But, they assumed $G_0(y)\to\beta<0$, and made many extra conditions, which enforced a compactness condition upon (0.1).

A main step in studying the lack of compactness problem in this direction was due to Ward[13]. He studied the case: $g_0(y)=\sin y$, and $\int_0^\pi h(t)\ \sin t\ dt=0$. The study was followed by Solimini[12], Lupo-Solimini[10] and Mawhin[11].

In this paper, the strong resonance problem is studied in the variational approach by a quite different method. The new ingredients are as follows:

(1) Although the Palais-Smale condition does not hold for some values, we could compactify our space by adding some infinity points, and extend our functional onto the new space.

(2) Deformations, and then the critical point theory are extended to the enlarged space.

(3) We distinguish the genuine and the fake critical points.

(4) Multiple solutions are obtained by the richness of the topology of the compactified space along with the critical groups of isolated critical points.[2]

Our work is set up in the functional analytic framework. The main results are Theorems 2.6, 2.8, and 3.3. In the applications to differential equations, Theorem 2.6 implies the results due to Ward, Solimini, Lupo-Soliimini, as special cases. Theorem 2.8 improves these results to obtain a nontrivial solution if there is a trivial solution with certain restrictions on its Morse index (cf. Theorem 4.1.). And Theorem 3.3 extends the Landesmann-Lazer type problem about the multiple periodic solutions for the Hamiltonian systems with periodic nonlinearities, studied in Chang[3], and Liu[7], to the strong resonance case.

The paper is organized as follows: We compactify the space and set up the deformation lemma in § 1. § 2 is devoted to the study of the existence and the multiplicity of solutions for semi-definite functionals. § 3 extends the results in § 2 to indefinite functionals via the Garlekin method. And § 4 deals with the applications to semi-linear elliptic boundary value problems and periodic solutions of Hamiltonian systems.

§ 1. Deformation Lemma

Let H be a Hilbert space and A be a bounded self-adjoint operator on H, which splits H into $H_+ + H_- + H_0$ according to its spectral decomposition. We denote by $P_\pm$ and P the orthogonal projections onto the positive/negative spectrum space H and the kernel of A, H, respectively. Set $Q=P_+ + P_-$. The following assumptions are made:

(A1) The restriction $A|H_\pm$ is invertible, i.e., $A|H_\pm$ has a bounded inverse on $H_\pm$.

(A2) $m_-:=\dim H_-$, $m_0:=\dim H_0$ are finite.

(G) Let $G: H\to R$ be a C^1-function and have a bounded, compact gradient dG. We assume that $G(x)\to 0$, $dG(x)\to 0$ as $|Px|\to\infty$, uniformly in bounded Qx sets.

We are concerned with the existence of critical points of the function

$$f(x)=\frac{1}{2}(Ax,\ x)+G(x) \tag{1.1}$$

which is related to the asymptotically linear operator equation with strong resonance at infinity: $Ax+dG(x)=\theta$. This problem has been attacked by many authors, but our approach is quite different from theirs and easily extended to the case of function with periodic nonlinearity.

Let us recall the $(P.S)_c$ condition:

We say a function f defined on a Hilbert space H satisfies $(P.S)_c$ condition for $c\in R$, if any sequence x_n along which $f(x_n)\to c$ and $df(x_n)\to\theta$ possesses a convergent subsequence.

The function defined by (1.1) fails to satisfy $(P.S)_c$ condition at the level $c=0$. In fact any sequence x_n of H_0, for which $|x_n|\to\infty$, satisfies $f(x_n)\to 0$, $df(x_n)\to\theta$ and can have no convergent subsequence. But f does satisfy $(P.S)_c$ condition for $c\neq 0$. Namely we have the following lemma.

Lemma 1.1. *Under the assumptions (A1), (A2) and (G) the function f satisfies $(P.S)_c$ condition for $c\neq 0$. Moreover if $f(x_n)\to 0$, $df(x_n)\to\theta$ for a sequence x_n, then we can select a subsequence (still denoted by x_n) with the property that either x_n converges or $|Qx_n|\to 0$ and $|Px_n|\to\infty$, as $n\to\infty$.*

Proof Suppose that

$$f(x_n)=\frac{1}{2}(Ax_n,\ x_n)+G(x_n)\to c, \tag{1.2}$$

$$df(x_n)=Ax_n+dG(x_n)\to 0,\ \text{as } n\to\infty. \tag{1.3}$$

Decompose x_n into $x_n^+ + x_n^- + x_n^0$, where $x_n^\pm=P_\pm x_n$, $x_n^0=Px$. Then

$$|(Ax_n^\pm,\ x_n^\pm)|=|(Ax_n,\ x_n^\pm)|=|(df(x_n)-dG(x_n),\ x_n^\pm)|\leqslant C|x_n^\pm|. \tag{1.4}$$

Since A is positively/negatively definite on $H_\pm$, (1.4) implies the boundness of $x_n^\pm$. If x_n^0 is bounded too, then x_n has a weakly convergent subsequence. By the compactness of dG and the finite-dimensional condition on H_0 we get a strongly convergent subsequence. Now suppose that $|Px_n|$ tends to infinity. Then $G(x_n)\to 0$, $dG(x_n)\to\theta$. From (1.3), $x_n^\pm\to\theta$. Finally (1.2) implies that $f(x_n)\to 0$, that is, $c=0$.

The remainder of this section is devoted to proving a deformation theorem, which is essential in min-max theory.

Lemma 1.2. *Let $\widetilde{H}$ be the set $\{x\in H \mid df(x)\neq\theta\}$. There is a locally Lipschitz continuous mapping V: $\widetilde{H}\to H$ (the so-called pseudo gradient (p.g.) vector field) with a form $V(x)=Ax+h(x)$, satisfying*

(1) $|(V(x), df(x))|\geqslant\frac{1}{2}|df(x)|^2$,

(2) $|V(x)|\leqslant 2|df(x)|$,

(3) $|h(x)|\to 0$, *as* $|Px|\to\infty$, *uniformly in bounded Qx sets.*

Proof Let $u\in\widetilde{H}$. Then $Au+dG(u)$ is a p. g. vector for f at u with strict inequality in (1) and (2). By the continuity of df

$$z(v)=Av+dG(u) \tag{1.5}$$

is a p. g. vector for f for all v in an open neighborhood $N(u)$ of u. The set of all such neighborhoods covers $\widetilde{H}$. Therefore there exists a locally finite refinement $(N(x_i),\ i\in\Lambda)$, where Λ is an index set. Let $\rho_i(x)$ denote the distance from x to the complement of $N(x_i)$. Then ρ_i is Lipschitz continuous and vanishes outside $N(x_i)$. Set

$$\beta_i(x)=\frac{\rho_i(x)}{\sum\rho_i(x)}.$$

Since $(N(x_i))$ is a locally finite covering, for each $x\in\widetilde{H}$, the denominator of $\beta_i(x)$ is only a finite sum. Finally let

$$V(x)=\Sigma\,\beta_i(x)z_i(x)=\Sigma\,\beta_i(x)(Ax+dG(x_i))=Ax+h(x),$$

where z_i is defined by (1.5) for $u=x_i$:

$$z_i(x)=Ax+dG(x_i),$$

and

$$h(x)=\Sigma\,\beta_i(x)dG(x_i).$$

For each $x\in\widetilde{H}$, V is a convex combination of p. g. vectors for f and hence is a p. g. vector. Moreover V is locally Lipschitz continuous. It remains to verify the condition (3). Of course we can assume the diameters of the neighborhoods $N(u)$ to be less than 1. Suppose that $|Qx|$ is bounded, say $|Qx|\leqslant C$. For a given $\varepsilon>0$, take M so large that $|dG(x)|\leqslant\varepsilon$ for any x such that $|Qx|\leqslant C+1$ and $|Px|\geqslant M-1$. Now if $|Qx_i|\geqslant C+1$ or $|Px_i|\leqslant M-1$, then $\mathrm{dist}(x, x_i)\geqslant 1$ and $\beta_i(x)=0$; otherwise $|dG(x_i)|\leqslant\varepsilon$ by the choice of M. Therefore

$$\begin{aligned}|h(x)|&\leqslant\Sigma\,\beta_i(x)\,|dG(x_i)|\\&\leqslant\Sigma\beta_i(x)\,\varepsilon=\varepsilon.\end{aligned}$$

The proof is complete.

Now we compactify H_0, the kernel of A, by adding an infinity point. Namely set $\Sigma=H_0\cup\{\infty\}\cong S^{m_0}$ and $E=H^{\perp}\times\Sigma$, where $H^{\perp}=H_{+}+H_{-}$. Along with the function f, we define its extension F to the space E by

$$F(x)=F(u,s)=\begin{cases} f(u,s), & (u,s)\in H^{\perp}\times H_0 \\ J(u), & u\in H^{\perp},\ s=\infty, \end{cases}$$

where $f(u, s)=f(x)$ and $J(u)=1/2(Au, u)$. Since $G(u, s)\to 0$ as $s\to\infty$, the function F is continuous on E. Though F is not differentiable in general, we can still work out the necessary deformation theorem.

Theorem 1. 3. *If $c\in R\backslash\{0\}$ and N is any neighborhood of $K_c=\{x\in H\,|\,f(x)=c, df(x)=\theta\}$, then there exist $\eta(t, x)\in C([0, 1]\times E, E)$ and constants $\bar{\varepsilon}>\varepsilon>0$ such that*

$$\left.\begin{aligned} &(1)\ \eta(0, x)=x \text{ for all } x\in E,\\ &(2)\ \eta(t, x)=x \text{ for all } x\notin F^{-1}[c-\bar{\varepsilon}, c+\bar{\varepsilon}] \text{ and all } t\in[0, 1],\\ &(3)\ \eta(t, \cdot) \text{ is a homeomophism of } E \text{ for all } t\in[0, 1],\\ &(4)\ F(\eta(t, x)) \text{ is decreasing in } t \text{ for all } x\in E,\ t\in[0, 1],\\ &(5)\ \eta(1, F_{c+\varepsilon}\backslash N)\subset F_{c-\varepsilon},\\ &(6)\ \text{if } K_c=\varnothing,\ \eta(1, F_{c+\varepsilon})\subset F_{c-\varepsilon}. \end{aligned}\right\} \qquad (1.6)$$

Proof The idea is to construct two flows on H and $H^{\perp}\times\{\infty\}$ respectively, and then glue them. By $(P.S)_c$ condition, K_c is compact. Hence for $0<\delta$ sufficiently small, $N(\delta)=\{x\,|\,\text{dist}(x, K_c)<\delta\}\subset N$, so it suffices to prove (5) of (1.6) with N replaced by $N(\delta)$.

There are constants b, $\bar{\varepsilon}>0$ such that

$$|df(x)|\geq b, \text{ for } x\in f_{c+\bar{\varepsilon}}\backslash(f_{c-\bar{\varepsilon}}\cup N(\delta/8)). \qquad (1.7)$$

Since (1.7) remains valid if $\bar{\varepsilon}$ is decreased, we can assume

$$0<\bar{\varepsilon}<\min(\delta b^2/4, \delta b/8). \qquad (1.8)$$

Let $\varepsilon\in(0, \bar{\varepsilon}/2)$, and define a smooth function P: $R\to[0, 1]$ by

$$p(s)=\begin{cases} 0, & s\geq c+\bar{\varepsilon} \text{ or } s\leq c-\bar{\varepsilon},\\ 1, & c-\varepsilon\leq s\leq c+\varepsilon,\\ \text{between 0 and 1}, & \text{otherwise}. \end{cases}$$

Let $A=H-N(\delta/4)$, $B=N(\delta/8)$. Define another function $g(x)=\text{dist}(x, B)/[\text{dist}(x, A)+\text{dist}(x, B)]$. g is Lipschitz continuous with $g=0$ on B and $g=1$ on A. Next define $q(s)=1$ if $s\in[0, 1]$, $q(s)=1/s$ if $s\geq 1$. Finally define

$$X(x)=-g(x)p(f(x))q(|V(x)|)V(x),$$

where $V(x)$ is the p.g. vector field for f as shown in Lemma 1.2. Then X is a locally Lipschitz continuous vector field with $|X(x)|\leq 1$.

Consider the ordinary differential equation

$$\frac{d}{dt}\xi=X(\xi), \quad \xi(0, x)=x \text{ for } x\in H. \qquad (1.9)$$

Since X is bounded, Lipschitz continuous, by the basic existence theorem for such equations, for all $x\in H$ there exists a unique solution $\xi(t, x)$ defined on the whole line $t\in(-\infty, +\infty)$.

It follows in particular that $\xi(t, x)\in C([0, 1]\times H, H)$ and satisfies (1)—(4) of

(1.6) with E replaced by H. It remains to check (5). Since f is decreasing along a trajectory, (5) needs only to be verified for $x\in f_{c+\varepsilon}\setminus(f_{c-\varepsilon}\cup N(\delta))$. The procedure is now very standard. We shall prove that $f(\xi(3\delta/4, x))\leqslant c-\varepsilon$. If it is not, for $t<3\delta/4$, $c-\varepsilon\leqslant f(\xi(t, x))\leqslant c+\varepsilon$, and $p(f(\xi(t, x)))=1$. Because $|X|\leqslant 1$, we have

$$|\xi(t, x)-\xi(0, x)|\leqslant t;$$

hence for $t<3\delta/4$,

$$\begin{aligned}&d(\xi(t, x), N(\delta/4))\\ &\geqslant d(\xi(0, x), N(\delta/4))-|\xi(t, x)-\xi(0, x)|\\ &>3\delta/4-3\delta/4=0,\end{aligned}$$

and $g(\xi(t, x))=1$. Now

$$\begin{aligned}\frac{d}{dt}f(\xi(t, x))&=\left(df(\xi(t, x)), \frac{d}{dt}\xi(t, x)\right)\\ &=-q(|V(\xi(t, x)|)(df(\xi(t, x)), V(\xi(t, x)))\\ &\leqslant-\frac{1}{2}q(|V(\xi(t, x)|)|df(\xi(t, x))|^2.\end{aligned}\tag{1.10}$$

If for some t, $|V(\xi(t, x)|\leqslant 1$, $q(|V(\xi(t, x))|)=1$, by (1.10), (1.7)

$$\frac{d}{dt}f(\xi(t, x))\leqslant-\frac{1}{2}|df(\xi(t, x))|^2\leqslant-b^2/2,\tag{1.11}$$

while if $|V(\xi(t, x))|\geqslant 1$, $q(|V(\xi(t, x))|)=|V(\xi(t, x))|^{-1}$, and by (1.10) and (2) of Lemma 1.2

$$\begin{aligned}\frac{d}{dt}f(\xi(t, x))&\leqslant-\frac{1}{2}|df(\xi(t, x)|^2|V(\xi(t, x))|^{-1}\\ &\leqslant-\frac{1}{4}|df(\xi(t, x))|<-b/4.\end{aligned}\tag{1.12}$$

Combining (1.11) and (1.12) produces

$$\frac{d}{dt}f(\xi(t, x))\leqslant-\min(b^2/2, b/4).\tag{1.13}$$

So

$$\begin{aligned}f(\xi(\delta/2, x))&\leqslant f(\xi(0, x))-\frac{1}{2}\delta\min(b^2/2, b/4)\\ &\leqslant c+\varepsilon-\frac{1}{2}\delta\min(b^2/2, b/4)<c-\varepsilon,\end{aligned}$$

which is a contradiction.

At the same time we construct another flow ζ on $H^\perp\times\{\infty\}$ by the ordinary differential equation:

$$\frac{d}{dt}\zeta=W(\zeta),\ \zeta(0, u)=u \text{ for } u\in H^\perp,\tag{1.14}$$

where $W(u)=-p(J(u))q(|Au|)Au$, $J(u)=\frac{1}{2}(Au, u)$. We have

$$|dJ(u)|\geqslant b, \text{ for } x\in J_{c+\bar{\varepsilon}}\setminus J_{c-\bar{\varepsilon}}.$$

Repeating the above argument we do for the solutions of equation (1.9), we get a flow $\zeta\in C([0,1]\times H^{\perp}, H^{\perp})$ satisfying (1)—(5) of (1.6) with E replaced by $H^{\perp}$. Now define for $t\in[0,1]$

$$\eta(t,u,s)=\begin{cases}\xi(t,u,s), & (u,s)\in H^{\perp}\times H_0,\\ (\zeta(t,u),\infty), & u\in H^{\perp},\ s=\infty,\end{cases}$$

which is the desired mapping. It is clear that this mapping satisfies all the conditions of (1.6), provided we can prove its continuity at points $x=(u,\infty)$, $u\in H$. We state it in the following two lemmas.

Lemma 1. 4. *For $x=(u,s)\in H^{\perp}\times H_0$, set $R(x)=X(x)-W(u)$. Then $R(x)\to 0$ as $|Px|\to\infty$, uniformly on bounded Qx sets.*

Proof First of all, $g(x)=1$ as $|x|$ is large enough; hence

$$X(x)=-p(f(x))q(|V(x)|)V(x).$$

Suppose Qx is bounded. Then Ax is bounded, $G(x)$ and $h(x)$ tend to zero as $|Px|$ tends to infinity. Hence

$$|f(x)-J(u)|=|G(x)|=o(1),$$
$$|V(x)-Au|=|h(x)|=o(1).$$

It follows that

$$|p(f(x))-p(J(u))|\leqslant C|f(x)-J(u)|=o(1),$$
$$|q(|V(x)|)-q(|Au|)|\leqslant C|V(x)-Au|=o(1),$$

and finally $R(x)=X(x)-W(u)=o(1)$. The above symbol $o(1)$ denotes quantities which tend to zero as $|Px|$ tends to infinity uniformly on bounded Qx sets.

Lemma 1. 5. *Suppose that $t_n\to t$, $u_n\to u$, $s_n\to\infty$, where $t_n\in[0,1]$, $u_n\in H^{\perp}$, $s_n\in H_0$. Then*

$$P\xi(t_n,u_n,s_n)\to\infty,\quad Q\xi(t_n,u_n,s_n)\to\zeta(t,u).$$

Proof Consider the equation

$$d/dt\,\xi=X(\xi),\quad \xi(0,u_n,s_n)=(u_n,s_n).$$

Since $|X|\leqslant 1$, $|\xi(t,u_n,s_n)-\xi(0,u_n,s_n)|\leqslant 1$, hence $|Q\xi(t,u_n,s_n)|\leqslant|u_n|+1\leqslant C$, $|P\xi(t,u_n,s_n)|\geqslant|s_n|-1\to\infty$, and by Lemma 1.4 $R(\xi(t,u_n,s_n))$ tends to zero as $n\to\infty$. We have

$$\begin{aligned}&\frac{d}{dt}|\zeta(t,u)-Q\xi(t,u_n,s_n)|\\ &\leqslant|W(\zeta(t,u))-QX(\xi(t,u_n,s_n))|\\ &\leqslant|W(\zeta(t,u))-W(Q\xi(t,u_n,s_n))|+|QR(\xi(t,u_n,s_n))|\\ &\leqslant C|\zeta(t,u)-Q\xi(t,u_n,s_n)|+o(1).\end{aligned}$$

By Gronwall inequality,

$$\begin{aligned}&|\zeta(t,u)-Q\xi(t,u_n,s_n)|\\ &\leqslant C|\zeta(0,u)-Q\xi(0,u_n,s_n)|+o(1)\\ &=C|u-u_n|+o(1).\end{aligned}$$

But

$$|\xi(t, u_n, s_n)-\xi(t_n, u_n, s_n)|\leqslant|t-t_n|,$$

so

$$|\zeta(t, u)-Q\xi(t_n, u_n, s_n)|\leqslant C|u-u_n|+|t-t_n|+o(1).$$

The proof of Lemma 1.5, hence of Theorem 1.3, is complete.

Noticing that the constant $\bar{\varepsilon}$ depends on b and δ only, we can rewrite Theorem 1.3 as the following corollary, which is very useful later.

Corollary 1.6. *Let N and N' be two bounded subsets of H such that $N'\subset N$ and* $\mathrm{dist}(N', \partial N)\geqslant 7\delta/8$. *Suppose that there exist constants b and $\bar{\varepsilon}$ such that*

$$|df(x)|\geqslant b, \text{ for } x\in f_{c+\bar{\varepsilon}}\backslash(f_{c-\bar{\varepsilon}}\cup N'),$$

$$0<\bar{\varepsilon}<\min(\delta b^2/4, \delta b/8).$$

Then for $0<\varepsilon<\bar{\varepsilon}/2$, *there exists* $\eta(t, x)\in C([0, 1]\times E, E)$ *satisfying* (1)—(6) *of Theorem* 1.3.

Now we turn to the case of $c=0$. The point $x=(\theta, \infty)$ plays a role of critical points in some sense. We should exclude a neighborhood of this point. But first we give a few notations. Set

$$D(\delta)=\{u\in H^\perp, |u|\leqslant\delta\}, L(R)=\{s\in H_0, |s|\geqslant R\},$$

$$M(\delta, R)=D(\delta)\times L(R), M(\delta)=D(\delta)\times\{\infty\},$$

$$N(\delta)=\{x\in H, \mathrm{dist}(x, K_0)\leqslant\delta\}.$$

We have the following theorem.

Theorem 1.7. *Assume that the critical set K_0 is bounded. For any neighborhood N of $K_0\cup\{\theta, \infty)\}$ in E, the conclusion of Theorem 1.3 still holds, that is, there exist* $\eta(t, x)\in C([0, 1]\times E, E)$ *and constants* $\bar{\varepsilon}>\varepsilon>0$ *which satisfy* (1.6).

Proof We can assume that $N=N(\delta)\cup M(\delta, R)\cup M(\delta)$ and R is sufficiently large that $N(\delta)\cap M(\delta, R)=\varnothing$. Again we construct two flows ξ, ζ on $H^\perp\times H_0$ and $H^\perp$ respectively. By Lemma 1.1 there are constants b and $\bar{\varepsilon}>0$ such that

$$|df(x)|\geqslant b, \text{ for } x\in f_{\bar{\varepsilon}}\backslash(f_{-\bar{\varepsilon}}\cup N(\delta/8)\cup M(\delta/8, R+1)),$$

for otherwise there exist sequences $b_n\to 0$, $\varepsilon_n\to 0$ and x_n belongs to $f_{\varepsilon_n}\backslash(f_{-\varepsilon_n}\cup N(1/8\delta)\cup M(\delta/8, R))$ with $|df(x_n)|\leqslant b_n$. By Lemma 1.1 either x_n converges to a point x, satisfying $f(x)=0$ $df(x)=\theta$ and $x\overline{\in}N(\delta/8)$; or $Qx_n\to 0$ and $Px_n\to\infty$, hence x_n belsngs to $M(\delta/8, R+1)$eventually. In both cases we arrive at a contradictio n.

Let

$$A=H\backslash(N(\delta/4)\cup M(\delta/4, (R+1)/2)),$$

$$B=N(\delta/8)\cup M(\delta/8, R+1),$$

$$g(x)=\mathrm{dist}(x, B)/[\mathrm{dist}(x, A)+\mathrm{dist}(x, B)],$$

$$X(x)=-g(x)p(f(x))q(|V(x)|)V(x),$$

$$A_\infty=H^\perp\backslash D(\delta/4), B_\infty=D(\delta/8),$$

$$g_\infty(u)=\mathrm{dist}(x, B_\infty)/[\mathrm{dist}(x, A_\infty)+\mathrm{dist}(x, B_\infty)],$$

$$W(u) = -g_\infty(u)p(J(u))q(|Au|)Au.$$

Again we consider equations (1.9) and (1.14). Everything remains the same, provided we can prove the conclusion of Lemma 1.4, that is, $R(x) = X(x) - W(u) \to 0$ as $|Px| \to \infty$, uniformly in $u = Qx$ being in bounded sets. But it is clear that as Qx is bounded and Px tends to infinity, $\mathrm{dist}(x, A) = \mathrm{dist}(u, A_\infty)$ and $\mathrm{dist}(x, B) = \mathrm{dist}(u, B_\infty)$; hence $g(x) = g_\infty(u)$. As in Lemma 1.4 we get $|R(x)| = o(1)$. The proof is complete.

In parallel with Corollary 1.6 the following corollary holds.

Corollary 1.8. *Let N, N' be two bounded subsets of H such that $N' \subset N$ and* $\mathrm{dist}(N', \partial N) \geqslant 7\delta/8$. *Suppose that there exist constants b, R and $\bar{\varepsilon}$ such that*

$$|df(x)| \geqslant b, \text{ for } x \in f \setminus (f_- \cup N' \cup M(\delta/8, R+1)),$$
$$0 < \bar{\varepsilon} < \min(\delta b^2/4, \delta b/8).$$

Then for $0 < \varepsilon < \bar{\varepsilon}$ there exists $\eta(t, x) \in C([0, 1] \times E, E)$ satisfying (1.6) *with N replaced* by $N \cup M(\delta, R) \cup M(\delta)$.

More generally, one may extend the above discussion to the following problem.

Let V be a finite dimensional compact C^2 Riemannian manifold without boundary. Let H and A be defined above, and let G: $H \times V \to \mathbf{R}^1$ be a C^1 function satisfying the assumption:

($\mathrm{G}v$) G has a bounded compact gradient dG such that $G(x, v) \to 0$ and $dG(x, v) \to \theta$ as $|Px| \to \infty$ uniformly in (Qx, v), where $v \in V$ and Qx are bounded.

We are looking for critical points of the functional

$$f(x, v) = \frac{1}{2}(Ax, x) + G(x, v), \quad (x, v) \in H \times V. \tag{1.15}$$

Similarly, we introduce a new functional on the space $E = H^\perp \times \Sigma \times V$:

$$F(\zeta) = F(u, s, v) = \begin{cases} f(u, s, v) & (u, s, v) \in H^\perp \times H_0 \times V, \\ \frac{1}{2}(Au, u) & (u, v) \in H^\perp \times V, \ s = \infty. \end{cases} \tag{1.16}$$

Then Theorems 1.3 and 1.7 extend to the following

Theorem 1.9. *Let N and N' be two bounded subsets of $H \times V$ such that $N' \subset N$ and* $\mathrm{dist}(N', N) \geqslant 7\delta/8$. *Suppose that there exist positive constants b, R, and $\bar{\varepsilon}$ such that*

$$|df(\zeta)| \geqslant b \tag{1.17}$$

for $\zeta \in f_{c+} \setminus (f_{c-} \cup N \cup U)$, where $U = \varnothing$, if $c \neq 0$, and $U = M(\delta/8, R+1) \times V$, if $c = 0$; and

$$0 < \bar{\varepsilon} < \mathrm{Min}(\delta b^2/4, \delta b/8). \tag{1.18}$$

Then for any $0 < \varepsilon < \bar{\varepsilon}$, there exists $\eta \in C([0, 1] \times E, E)$ satisfying

(1) $\eta(0, \cdot) = id$,

(2) $\eta(t, \zeta) = \zeta$, $\forall t \in [0, 1]$, $\forall \zeta \notin F^{-1}[c - \bar{\varepsilon}, c + \bar{\varepsilon}]$,

(3) $\eta(t, \cdot)$ *is a homeomorphism of E*, $\forall t \in [0, 1]$,

(4) $F(\eta(t, \zeta))$ *is nonincreasing in* t, $\forall \zeta \in E$,

(5) $\eta(1, F_{c+\varepsilon} \backslash N) \subset F_{c-\varepsilon}$, *if* $c \neq 0$, *and*

$$\eta(1, F_{\varepsilon} \backslash (N \cup (M(\delta, R) \cup M(\delta)) \times V)) \subset F_{-\varepsilon}, \quad \text{if } c=0,$$

(6) *If* $c \neq 0$ *and* $K_c = \varnothing$, *then* $\eta(1, F_{c+\varepsilon}) \subset F_{c-\varepsilon}$.

§ 2. Existence and Multiplicity

We want to study the existence and multiplicity of critical points of the function f via its extension F defined in (1.16). Noticing that the function F is not differentiable at points on the set $\{(u, \infty, v) \mid u \in H^1, v \in V\}$, it is necessary to single out some specified points of F, which play the role of critical points in some sense.

Let K denote the critical set of f, we call the set $\hat{K} = K \cup ((\theta, \infty) \times V)$ the fake critical set of F. Each point $\zeta \in \hat{K}$ is called a fake critical point. A point, which is not a fake critical point, is called a fake regular point.

We denote $\hat{K}_c = \hat{K} \cap F^{-1}(c)$. In case $\hat{K}_c \neq \varnothing$, c is called a fake critical value. Otherwise, c is called a fake regular value.

It is easily seen that $\hat{K}_c = K_c$ if $c \neq 0$, and $\hat{K}_0 = K_0 \cup (0, \infty) \times V$.

Theorem 1.9 now turns out to be the following

Theorem 2. 1. *Assume that* K_0 *is bounded. Then* $\forall c \in \mathbf{R}^1$, $\forall$ *neighbourhood* N *of* $\hat{K}_c$, $\exists \delta > 0$ *such that* $\forall\ 0 < \varepsilon < \bar{\varepsilon} < \delta$, *there exists a deformation* $\eta \in C([0, 1] \times E, E)$, *satisfying*

(1) $\eta(0, \cdot) = id$,

(2) $\eta(t, \zeta) = \zeta$, $\forall (t, \zeta) \in [0, 1] \times (E \backslash F^{-1}[c - \bar{\varepsilon}, c + \bar{\varepsilon}])$,

(3) $F(\eta(t, \cdot))$ *is nonincreasing in* t,

(4) $\eta(t, \cdot)$ *is a homeomorphism of* E, $\forall t \in [0, 1]$,

(5) $\eta(1, F_{c+\varepsilon} \backslash N) \subset F_{c-\varepsilon}$,

(6) *If* $\hat{K}_c = \varnothing$, *then* $\eta(1, F_{c+\varepsilon}) \subset F_{c-\varepsilon}$.

Lemma 2. 2. *Let* $\alpha \in H_*(F_b, F_a)$ *be a nontrivial singular relative homology class, where* $a < b$ *are two fake regular values. Then*

$$c = \inf_{z \in \alpha} \sup_{x \in |z|} F(x)$$

is a fake critical value of F, *with* $a < c < b$.

Proof A standard Minimax Principle can be applied, provided by the conclusions (1), (3), and (6).

Definition 2. 3. *Let* X *be a topological space,* $Y \subset X$, *and let* $\alpha_1, \alpha_2 \in H_*(X, Y)$ *be two nontrivial singular homology classes. We say that* α_1 *is subordinated to* α_2, *and denoted by* $\alpha_1 < \alpha_2$, *if there exists* $\omega \in H^*(X)$, *with* $\dim \omega > 0$ *such that* $\alpha_1 = \alpha_2 \cap \omega$, *where* $\cap$ *is the cap product.*

Lemma 2. 4. *Let $\alpha_1<\alpha_2$ be two nontrivial singular homology classes in $H_*(F_d, F_a)$ with $\alpha_1=\alpha_2\cap\omega$, where $a<d$ are fake regular values of F. And let*

$$c_i=\inf_{z\in\alpha_i}\sup_{x\in|z|}F(x),\quad i=1,2.$$

Assume that there exists a neighbourhood N' of $\hat{K}_{c_2}$ and a singular cochain $\hat{\omega}\in\omega$ such that $\operatorname{supp}\omega\cap N'=\varnothing$. *Then $c_1<c_2$.*

Proof By definition, $\forall\varepsilon>0$, $\exists$ a singular relative closed chain $z_2\in\alpha_2$ such that $|z_2|\subset F_{c_2+\varepsilon}$. We choose a neighbourhood N of $\hat{K}_c$ such that $N\subset\bar{N}\subset N'$, and subdivide z_2 into $z_2'+z_2''$ such that $|z_2'|\subset N'$ and $|z_2''|\subset F_{c_2+\varepsilon}\backslash N$. By the assumption, we have

$$z_1=z_2\cap\hat{\omega}=z_2''\cap\hat{\omega},$$

which implies $|z_1|\subset F_{c_2+\varepsilon}\backslash N$.

Since $c_1\leqslant c_2$ are fake critical values of F, we have $a<c_1$ and $c_2<d$. We choose $0<\bar{\varepsilon}<\operatorname{Min}(d-c_2, c_1-a)$, and $0<\varepsilon<\bar{\varepsilon}$. According to Theorem 2.1, there exists $\eta\in C([0,1]\times E, E)$ satisfying (1)—(6), which imply that $\eta(1, F_{c_2+\varepsilon}\backslash N)\subset F_{c_2-\varepsilon}$ and $\eta(1,\cdot)\simeq id$, in (F_d, F_a). Therefore $\eta(1, |z_1|)\subset F_{c_2-\varepsilon}$. However, $\eta(1, z_1)\in\alpha_1$. It follows that $c_1\leqslant c_2-\varepsilon<c_2$.

Lemma 2.4 is an extension of a result due to Chang[2].

The following corollary emphasizes the fact that the positive lower bound of the difference c_2-c_1 depends only upon the constants: b, δ, $\bar{\varepsilon}$, and α.

Corollary 2. 5. *Let $c\in\boldsymbol{R}$, $d>0$ and $a<0$. Suppose that K_0 is bounded. Let α_i, c_i, $i=1, 2$, be defined as in Lemma 2.4. Suppose that there exist neighbourhoods $N\subset N'$ of $\hat{K}_c$ and constants b, $\bar{\varepsilon}$, $\delta>0$, satisfying* $\operatorname{dist}(N', \partial N)\geqslant 7\delta/8$, $|df(\zeta)|\geqslant b>0$, $\forall\zeta\in F_{c+\bar{\varepsilon}}\backslash(F_{c-\bar{\varepsilon}}\cup N')$ *and* $0<\bar{\varepsilon}<\operatorname{Min}(\delta b/4, \delta b^2/8, d/2)$. *Then $\forall\,\varepsilon\in(0,\bar{\varepsilon})$, $c_1<c-\varepsilon/3$ whenexer $c_2\leqslant c+\varepsilon/3$.*

The proof is the same as above.

Theorem 2. 6. *If K_0 is bounded, then there exist at least cuplength $(V)+1$ critical points of the function f.*

Proof One chooses d and $a\pm$ large enough which aro fake regular. By the same argument applied in the proof of Theorem 8.3 in Chang[2], we obtain

$$H_*(F_d, F_a)\cong H_{*-m_0}(\Sigma\times V),$$

and

$$H^*(F_d)\cong H^*(\Sigma\times V).$$

The later equivalence is in the sense of ring isomorphism. Assume that curlength $(V)=p$; this means that $\exists\omega_1, \cdots, \omega_p\in H^*(V)$ with din $\omega_i>0$, $i=1, \cdots, p$, such that $\omega_1\cup\omega_2\cup\cdots\cup\omega_p\neq0$.

Since $\Sigma=S^{m_0}$, there is $\omega^*\in H^*(\Sigma)$, with $\dim\omega^*=m_0$. Let π_1: $\Sigma\times V\to\Sigma$ and π_2: $\Sigma\times V\to V$ be the two projections. We have $\pi_2^*\omega_1$, $\pi_2^*\omega_2$, $\cdots$, $\pi_2^*\omega_p$ and $\pi_1^*\omega^*\in H^*(\Sigma\times V)$. Thus

$$\pi_1^*\omega^* \cup \pi_2^*\omega_1 \cup \cdots \cup \pi_2^*\omega_p \neq 0.$$

In the following we shall not distinguish the cohomology classes between $H^*(\Sigma\times V)$ and $H^*(F_d)$ in notations. Accordingly, there exists $\alpha_{p+1}\in H_*(F_d, F_a)$ such that

$$[\alpha_{p+1}, \pi_1^*\omega^* \cup \pi_2^*\omega_1 \cup \cdots \cup \pi_2^*\omega_p] \neq 0.$$

Let

$$\alpha_i=\alpha_{i+1}\cap\omega_i, \quad i=1, 2, \cdots, p,$$

and let

$$\alpha_i^*=\alpha_i\cap\omega^*, \quad i=1, 2, \cdots, p+1.$$

Then we define

$$c_i=\inf_{z\in\alpha_i}\sup_{x\in|z|}F(x), \tag{2.1}$$

and

$$c_i^*=\inf_{z\in\alpha_i^*}\sup_{x\in|z|}F(x), \tag{2.2}$$

$i=1, 2, \cdots, p+1$.

Because of the special choice of ω^*, it followt from Lemma 2, 4, that $c_i^*<c_i$, $i=1, 2, \cdots, p+1$. Moreover, by definition, we have

$$c_1\leqslant c_2\leqslant\cdots\leqslant c_{p+1},$$

and

$$c_i^*\leqslant c_2^*\leqslant\cdots\leqslant c_{p+1}^*.$$

We conclude that there must be at least $p+1$ nonzero values among these $2(p+1)$ values. In fact, if $0\leqslant c_1^*$, then $0<c_1$, therefore all c_j, $j=1, 2, \cdots, p+1$, are positive, and if $c_{p+1}^*<0$, then all c_j^*, $j=1, 2, \cdots, p+1$, are negative. Otherwise, there exists $j\in[1, p]$, such that $c_j^*<0\leqslant c_{j+1}^*$. As we have shown above, it follows that

$$0<c_{j+1}\leqslant c_{j+2}\leqslant\cdots\leqslant c_{p+1}.$$

Again we obtain $p+1$ nonzero values.

Provided by Lemma 2.4, if f has only isolated critical points with nonzero critical values, then f has at least $p+1$ critical points. Otherwise, the proof is through.

Under additional conditions, we may obtain one more solution. Let us define two index sets. Set

$$I=\left\{q\in\boldsymbol{Z}\mid q=\sum_{i=1}^{j}\dim\omega_i,\ j=1, 2, \cdots, p\right\}.$$

and set

$$I_-=m_-+I, \quad I_+=m_-+m_0+I.$$

Lemma 2. 7. *Suppose that $c<0$ (or >0) is a critical value of f obtained by the Minimax Principle via a nontrivial singular homology class $\alpha=\alpha_i$ (or α_i^* respectively) for some i in Theorem 2.6. Assume that f has only isolated critical points, then $\exists\, x_0\in K_c$ and $q\in I_-$ (or I_+ resp.) such that*

$$C_q(x_0, f)\neq 0,$$

where $C_*(x_0, f)$ *is the critical group of* x_0

Proof We only prove it for $c<0$. Set $0<\bar{\varepsilon}<\mathrm{Min}(d-c, c-a)$, where d and a are large enough fake regular values. If the conclusion is not true, then $\forall x\in K_c$ and $\forall q\in I_-$, $C_q(x, f)=0$. Since now $K_c=\hat{K}_c$, we have $H_q(F_{c+\varepsilon}, F_{c-\varepsilon})=0$ for sufficiently small $\varepsilon\in(0, \bar{\varepsilon})$. On the other hand, there is a relative singular closed chain $z\in\alpha$ with support $|z|\subset F_{c+\varepsilon}$. Let $y=\partial z$ be the boundary of z. Then $|y|\subset F_a\subset F_{c-\varepsilon}$, and then $\sigma=[y]\in H_{q-1}(F_{c-\varepsilon})$ is nontrivial, provided by the definition of c. In fact, if σ is trivial in $H_{q-1}(F_{c-\varepsilon})$, then $\exists$a singular $q-$chain τ such that $|\tau|\subset F_{c-\varepsilon}$, and $\partial\tau=\sigma$, which implies that $\tau\in\alpha$. In this case,

$$c=\inf_{z\in\alpha}\sup_{x\in|z|}F(x)\leqslant\mathop{\mathrm{Sup}}_{x\in|\tau|}F(x)\leqslant c-\varepsilon.$$

This is a contradiction. Noticing the exact sequence

$$\longrightarrow H_q(F_{c+\varepsilon}, F_{c-\varepsilon})\xrightarrow{\partial_*}H_{q-1}(F_{c-\varepsilon})\xrightarrow{i_*}H_{q-1}(F_{c+\varepsilon})\longrightarrow\cdots,$$

where i: $F_{c-\varepsilon}\to F_{c+\varepsilon}$ is the injection and $i_*(\sigma)=0$, i. e., $\sigma\in\ker i_*$, we have $\beta\in H_q(F_{c+\varepsilon}, F_{c-\varepsilon})$ such that $\partial_*\beta=\sigma$. This is a contradiction.

A similar result for homology link was obtained by Liu[6].

Theorem 2. 8. *Suppose that the function* f, *defined in* (1.15), *satisfies the assumptions* (A_1), (A_2), *and* (Gv). *Moreover, assume that* (x_0, v_0) *is an isolated critical point of* f. *Then* f *has at least cuplength*$(V)+2$ *critical points, if either*

(1) $f(x_0, v_0)=0$, *or*

(2) $f(x_0, v_0)<0$, *and* $C_q((x_0, v_0), f)=0$, $\forall q\in I_-$, *or*

(3) $f(x_0, v_0)>0$, *and* $C_q((x_0, v_0), f)=0$, $\forall q\in I_+$.

This is a direct consequence of Theorem 2.6 and Lemma 2.7. In the argument of the proof of Theorem 2.6, if the condition that f has only isolated critical points is dropped out, then our result may be improved as follows.

Theorem 2. 9. *Suppose that* $c>0$ (*or* $c<0$) *is a critical point of* f *obtained by the Minimax Principle via a nontrivial singular relative homology class* $\alpha=\alpha_i$ (or α_i^*) *for some* i *in* (2.1) (*or* (2.2) *resp.*). *If* c *is of multiplicity* $k+1$, *i.e.*, $\exists\alpha_i, \alpha_{i+1}, \cdots, \alpha_{i+k}$ (*or* $\alpha_i^*, \alpha_{i+1}^*, \cdots, \alpha_{i+k-1}^*$ *resp.*) *such that*

$$c=c_i=c_{i+1}=\cdots=c_{i+k}, \ (\text{or } c_i^*=c_{i+1}^*=\cdots=c_{i+k}^* \text{ resp.}),$$

then

$$\mathrm{cat}(K_c)\geqslant k+1.$$

Proof We may choose neighbourhoods $N'\subset N\subset N''$ of K_c with $\mathrm{cat}(N'')=\mathrm{cat}(K_c)$, constants $0<\varepsilon<\bar{\varepsilon}$, and η: $E\to E$ continuous such that $\eta|_{F_{c-\bar{\varepsilon}}}=id_{F_{c-\bar{\varepsilon}}}$, $\eta(F_{c+\varepsilon}\setminus N)\subset F_{c-\varepsilon}$ and $\eta\cong id$, provided by Theorem 1.8.

If the conclusion is not true, i. e., $\mathrm{cat}(K_c)\leqslant k$, then $\exists$ k contractible sets B_j, $j=1, i+1, \cdots, i+k-1$, covering N''. We choose $z\in\alpha_{i+k-1}$, with support $|z|\subset F_{c+\varepsilon}$. Since dim $\omega_j>0$, one may choose cochains $\hat{\omega}_j\in\omega_j$ with supports $|\hat{\omega}_j|\cap B_j=\varnothing$, $j=i$,

$i+1, \cdots, i+k-1$. Subdividing z into $z=z_0+z_i+\cdots+z_{i+k-1}$, such that $|z_0|\subset F_{c+\varepsilon}\backslash N$ and $|z_j|\subset B_j$, $j=i, i+1, \cdots, i+k-1$, one has

$$z=z\cap(\omega_i\cup\omega_{i+1}\cup\cdots\cup\hat{\omega}_{i+k-1})=z_0\cap(\hat{\omega}_i\cup\hat{\omega}_{i+1}\cup\cdots\cup\hat{\omega}_{i+k-1}).$$

Hence $|z'|\subset F_{c+\varepsilon}\backslash N$ and $z''=\eta(z')\subset F_{c-\varepsilon}$. However, z'' is a singular chain in α_i, one obtains $c_i\leqslant c-\varepsilon$. This is a contradiction.

Similarly, we prove for $c>0$.

Corollary 2. 10. *Let $N'\subset N$ be defined in Theorem* 1.8, *such that* (1.17) *and* (1.18) *hold. If either*

$$c-\varepsilon\leqslant c_i\leqslant\cdots\leqslant c_j\leqslant c+\varepsilon, \text{ or}$$
$$c-\varepsilon\leqslant c_i^*\leqslant\cdots\leqslant c_j^*\leqslant c+\varepsilon,$$

then $\operatorname{cat}(N'')\geqslant j-i+1$, *for any bounded set N'' containing the closure $\overline{N}$.*

§ 3. Indefinite Functions

If we want to apply the abstract theorems obtained in the previous section to general Hamiltonian systems, the restriction on the dimension of H_- should be dropped, i. e., we shall extend our absract theorems to the case of indefinte functions, the Galerkin approximation method will be applied (Cf. Li and Liu [9]). Instead of (A2), we assume

(A2)′ H_- is seperate, and dim $H_0<\infty$.

Let H_-^n be a sequence of finite-dimensional subspaces on H_-. Denote by P_-^n the orthogonal projection from H to H_-^n. The following assumption on the approximation scheme is made.

(Γ) H_-^n is invariant under the action of A. P_-^n strongly converges to P_-.

It is clear that (Γ) is fulfilled, if $A=P_+-P_-$, and a suitable sequence of subspaces is chosen.

Now let us list the notations to be used in the sequel:

(1) Space.

H, a Hilbert space as in the section 1,

V, a manifold as in the section 1,

$$H=H_++H_-+H_0,\ H^\perp=H_++H_-,\ E=H^\perp\times\Sigma\times V,$$
$$H_n=H_++H_-^n+H_0,\ H_n^\perp=H_++H_-^n,\ E_n=H_n^\perp\times\Sigma\times V,$$
$$\Sigma=H_0\cup(\infty),$$

$Q_n=P_++P_-^n$, the orthogonal projection onto $H_n^\perp$,

$P_n=Q_n+P$, the orthogonal projection onto H_n,

$$D=\{u\in H_-,\ |u|\leqslant R\},\ S=\partial D=\{u\in H_-,\ |u|=R\},$$
$$D_n=\{u\in H_-^n,\ |u|\leqslant R\},\ S_n=\partial D_n=\{u\in H_-^n,\ |u|=R\},$$
$$D_n=D\cap H_n,\ S_n=S\cap H_n.$$

(2) Functions.

$$f(x, v)=1/2(Ax, x)+G(x, v), \ (x, v)\in H\times V,$$

$$F(x)=F(u, s, v)=\begin{cases} f(u, s, v), & (u, s, v)\in H^{\perp}\times H_0\times V, \\ J(u), & (u, v)\in H^{\perp}\times V, \ s=\infty, \end{cases}$$

$$f_n=f\,|\,H_n\times V, \text{ the restriction of } f \text{ to } H_n\times V,$$

$$F_n=F\,|\,E_n, \text{ the restriction of } F \text{ to } E,$$

$$\alpha=\sup_{x\in S\times\Sigma\times V} F(x), \ \beta=\inf_{x\in H\times\Sigma\times V} F(x), \ \gamma=\sup_{x\in D\times\Sigma\times V} F(x).$$

Clearly $\beta\leqslant\gamma$. Also we have $\alpha<\beta$ for R large enough.

(3) Cohomology and homology classes.

$$\omega_n^*, \ \omega_{1,n}, \cdots, \omega_{p,n}\in H^*(H_n^{\perp}\times\Sigma\times V),$$

$$\omega_n^*\cup\omega_{1,n}\cup\cdots\cup\omega_{p,n}\neq 0,$$

$$\dim\omega_n^*>0, \ \omega_n^* \text{ corresponds to } H^*(\Sigma),$$

$$\alpha_{p+1,n}\in H_*(H_n^{\perp}\times\Sigma\times V, \ S_n\times\Sigma\times V),$$

$$[\alpha_{p+1,n}, \ \omega_n^*\cup\omega_{1,n}\cdots\cup\omega_{p,n}]\neq 0,$$

$$\alpha_{i,n}=\alpha_{i+1,n}\cap\omega_{i,n}, \quad i=1, \cdots, p,$$

$$\alpha_{i,n}^*=\alpha_{i,n}\cap\omega_n^*, \quad i=1, \cdots, p, p+1.$$

(4) Critical values.

$$C_{i,n}=\inf_{z\in\alpha_{i,n}^*}\sup_{x\in|z|} F(x), \quad i=1, \cdots, p+1,$$

$$C_{i,n}^*=\inf_{z\in\alpha_{i,n}^*}\sup_{x\in|z|} F(x), \quad i=1, \cdots, p+1,$$

$$C_i=\lim_{n\to\infty} C_{i,n}, \ C_i^*=\lim_{n\to\infty} C_{i,n}^*, \quad i=1, \cdots, p+1.$$

Since

$$\beta\leqslant C_{1,n}\leqslant\cdots\leqslant C_{p+1,n}\leqslant\gamma,$$

$$\beta\leqslant C_{1,n}^*\leqslant\cdots\leqslant C_{p+1,n}^*\leqslant\gamma,$$

the existence of C_i and C_i^* is ensured at least for a subsequence.

It turns out that these C_i and C_i^* enjoy the properties stated in Theorems 2.6, 2.9.

Lemma 3. 1. *Under the assumptions* (A1), (A2)′, (G.v) *and* (Γ) *the function* f *satisfies* $(P.S)_c^*$ *for* $c\neq 0$. *More precisely any sequence* (x_n, v_n), *such that* $(x_n, v_n)\in H_n\times V$, $f(x_n, v_n)\to c$, $df_n(x_n, v_n)\to 0$, *possesses a subsequence (still denoted by* (x_n, v_n)) *with the property that either* (x_n, v_n) *strongly converges to a critical point of* f *in* $H\times V$ *or* $c=0$ *and* $Qx_n\to 0$, $|Px_n|\to\infty$ *and* $v_n\to v\in V$.

Proof (Compare with Lemma 1.1.) First of all we can assume that v_n converges to v, since V is compact. Suppose that

$$f(x_n, v_n)=\frac{1}{2}(Ax_n, x_n)+G(x_n, v_n)\to c, \tag{3.1}$$

$$df(x_n, v_n)=(Ax_n+P_nd_xG(x_n, v_n), \ d_vG(x_n, v_n))\to 0. \tag{3.2}$$

From the boundness of dG it follows that Ax_n is bounded; therefore Qx_n is bounded.

If Px_n is bounded too, then x_n has a weakly convergent subsequence. By the compactness of dG and the finite dimension condition on H_0, this subsequence strongly converges in H. Since P_n strongly converges to the indentity mapping, the limit of this subsequence is a critical point. Now suppose $|Px_n|\to\infty$; then $dG(x_n, v_n)\to 0$. From (3.2), $Qx_n\to 0$. Finally from (3.2), $f(x_n, v_n)\to 0$.

Proposition 3. 2. *Assume that the conditions* $(A1)$, $(A2)$, $(G.v)$ *and* (Γ) *are satisfied.*

(1) *If* $C_i\neq 0$ *for some* i, *then it is a critical value of* f. *Moreover if* $C_i=\cdots=C_j=C\neq 0$, $1\leqslant i<j\leqslant p+1$, *then* $\mathrm{Cat}(K_c)\geqslant j-i+1$.

(2) *If* $C_i^*\neq 0$ *for some* i, *then it is a critical value of* f. *Moreoxer if* $C_i^*=\cdots=C_j^*=C\neq 0$, $1\leqslant i<j\leqslant p+1$, *then* $\mathrm{Cat}(K_c)\geqslant j-i+1$.

(3) *Suppose that* K_0 *is a compact subset of* $H\times V$. *There is a constant* ε *such that* $C_i\leqslant\varepsilon$ *implies* $C_i^*\leqslant-\varepsilon$.

Proof (1) Choose a neighborhoods N'' of K_c, $N''=N(2\delta)$, $\mathrm{Cat}(N'')=\mathrm{Cat}(K_c)$. Also set $N=N(\delta)$, $N'=N(\delta/8)$, $N_n=N\cap(H_n\times V)$, $N_n''=N''\cap(H_n\times V)$, $N_n'=N'\cap(H_n\times V)$. Then $N_n''\supset\bar N_n\supset N_n\supset N_n'$, $\mathrm{dist}(N_n', \partial N_n)\geqslant 7/8\delta$. There must be constants b, $\bar\varepsilon$ and an integer n_0 independent of n and satisfying

$$|df(x_n)|\geqslant b,\ \text{for } x_n\in f_{n,c+\bar\varepsilon}-f_{n,c-\bar\varepsilon}-N_n',\ \text{for } n\geqslant n_0$$
$$0<\bar\varepsilon<\min(\delta b^2/4,\ \delta b/8).$$

For otherwise, there is a sequence x_n such that $x_n\in H_n\times V$, $f(x_n)\to C$, $df_n(x_n)\to 0$ and $x_n\overline{\in}N'$. By Lemma 3.1, $\exists$ a subsequence of x_n converges to x with $f(x)=C$, $df(x)=\theta$ and $x\overline{\in}N'$, a contradiction. Take n large enough so that

$$C-\varepsilon\leqslant C_{i,n}\leqslant\cdots\leqslant C_{j,n}\leqslant C+\varepsilon.$$

By Corollary 2.10, $\mathrm{Cat}_{H_n\times V}(N'')\geqslant j-i+1$. For a subset of $H_n\times V$ is contractible in $H\times V$, if and only if it is contractible in $H_n\times V$. Hence

$$\mathrm{Cat}(K_c)=\mathrm{Cat}(N'')\geqslant\mathrm{Cat}(N_n'')=\mathrm{Cat}_{H_n\times V}(N'')\geqslant j-i+1.$$

(2) can be proved in the same way.

(3) Choose R large enough so that $M(\delta/8, R+1)\times V\cap N''=\varnothing$. Set

$$x_n=M(\delta/8,\ R+1)\times V\cap(H_n\times V)$$
$$=\{(x,\ v)\in H_n\times V,\ |Qx|\leqslant\delta/8,\ |Px|\geqslant R+1\}.$$

There exist an integer n_0 and constants b, $\bar\varepsilon$, independent of n and satisfying

$$|df(\mathrm{x})|\geqslant b,\ \text{for } x_n\in f_{n,\bar\varepsilon}\backslash(f_{n,-\bar\varepsilon}\cup N_n'\cup M_n),$$
$$0<\bar\varepsilon<\min(\delta b^2/4,\ \delta b/8).$$

For otherwise, there is a sequence (x_n, v_n) such that $(x_n, v_n)\in H_n\times V$, $f(x_n, v_n)\to 0$, $df_n(x_n, v_n)\to\theta$ and $(x_n, v_n)\overline{\in}N_n'\cup M_n$. By Lemma 3.1, turning to a subsequence, either (x_n, v_n) converges to (x, v) with $f(x, v)=0$, $df(x, v)=\theta$ and $(x, v)\overline{\in}N'$; or $|Qx_n|\to 0$, $|Px_n|\to\infty$, $v_n\to v$, and (x_n, v_n) belong to M_n eventually. In both cases we arrive at a contradiction. Take n large enough so that $C_{i,n}<s\varepsilon$, for one given $s>1$.

According to Corollary 2.5, $C^*_{i,n} \leqslant -s\varepsilon$, hence $C^*_i = \lim C^*_{i,n} \leqslant -\varepsilon$.

Theorem 3.3. *Suppose that the operator A and the function G satisfy the conditions (A1), (A2)′ and (Gv). Then the function f has at least cuplength $(V)+1$ critical points.*

Proof We first prove the theorem under the additional condition (Γ). Suppose that

$$C_1 \leqslant \cdots \leqslant C_i = 0 < C_{i+1} \leqslant \cdots \leqslant C_{p+1}.$$

By Proposition 3.2 (1), f has at least $p-i+1$ critical points corresponding to positive critical values. We can assume that K_0 is a compact set, otherwise we are done. By Proposition 3.2 (3), $C^*_1 \leqslant \cdots \leqslant C^*_i < C_j = 0$. Finally by Proposition 3.2 (2), f has at least i critical points corresponding to negative critical values. Altogether we obtain at least $p+1$ critical points.

Next we are going to drop the restriction (Γ). Define a new function $g(y, v) = \frac{1}{2}(A_1 y, y) - G_1(y, v)$, where $A_1 = P_+ - P_-$ and $G_1(y, v) = G(A_+^{-1/2} P_+ y + (-A_-)^{-1/2} P_- y + Py, v)$. A_1 and G_1 satisfy the assumptions (A_1), $(A_2)'$, (Gv) and (Γ), hence have at least $p+1$ critical points (y_i, v_i), $i=1, 2, \cdots, p+1$. Set

$$x_i = A_+^{-1/2} P_+ y_i + (-A_-)^{-1/2} P_- y_i + P y_i.$$

(x_i, v_i), $i=1, 2, \cdots, p+1$, are critical points of f. The proof is complete.

§ 4. Applications

In the last section, we apply the abstract theorems to some problems in differential equations.

1. Semilinear elliptic boundary value problems.

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We study the following Dirichlet BVP, with strong resonance.

$$\begin{aligned} -\Delta u(x) &= \hat{\lambda} u(x) + \phi(x, u(x)) + h(x), \quad x \in \Omega, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{4.1}$$

It is assumed that

(H_1) $\hat{\lambda}$ is an eigenvalue of $-\Delta$ with m_0 multiplicity,

(H_2) $h \in L^2(\Omega)$, with $h \in \ker(-\Delta - \hat{\lambda} I)$.

Let $\phi(x, \zeta): \Omega \times \boldsymbol{R}^1 \to \boldsymbol{R}^1$ be a Caratheodory function, and let $\Phi(x, \zeta)$ be a primitive of ϕ *w. r. t.* ζ.

(H_3) $\forall \zeta_j \in \boldsymbol{R}^{m_0}$, $|\zeta_j| \to \infty$, $\forall u_j \to u$ in $H_0^1(\Omega)$ and $\forall v \in H_0^1(\Omega)$, we have

$$\lim_{j\to\infty} \int_\Omega \phi(x, u_j(x) + \sum_{i=1}^{m_0} \zeta_j^i e_i(x)) v(x) dx = 0,$$

and

$$\lim_{j\to\infty}\int_\Omega \Phi(x,\ u_j(x)+\sum_{i=1}^{m_0}\zeta_j^i e_i(x))\,v(x)dx=0,$$

where $\{e_i(x)\}_1^{m_0}$ is an orthonormal basis of the eigenspace $\ker(-\Delta-\hat{\lambda}I)$, and $\zeta_j=(\zeta_j^1, \zeta_j^2, \cdots, \zeta_j^{m_0})$.

There are several sufficient conditions ensuring (H_3). Namely,

(A) $\phi(x,\zeta)\to 0$, and $\Phi(x,\ \zeta)\to 0$ as $|\zeta|\to\infty$. or

(B) ϕ and Φ, which depend on ζ only, are bounded and uniformly continuous on $\boldsymbol{R}^1$

(see for instance Ward [13] and Mawhin [11]).

Let us define a functional on $H_0^1(\Omega)$:

$$G(v)=\int_\Omega \Phi(x,\ v(x)+u_0(x))dx,$$

where $u_0(x)$ is a solution of the equation:

$$-\Delta u_0(x)=\hat{\lambda}u_0(x)+h(x)\quad \text{in } \Omega,$$
$$u_0(x)|_{\partial\Omega}=0.$$

Equation (4.1) turns out to be

$$\begin{aligned}-\Delta v(x)=\hat{\lambda}v(x)+\phi(x,\ v(x)+u_0(x))\quad &\text{in } \Omega,\\ v(x)|_{\partial\Omega}=0,&\end{aligned}\tag{4.2}$$

with $u=v+u_0$.

However, (4.2) is of the form (1.1), satisfying the assumptions (A_1) (A_2) and (G). According to Theorem 2.6, in which $V=\{\theta\}$, we proved that (4.2) possesses at least one solution.

The results due to Ward ($n=1$, $\hat{\lambda}=\lambda_1$), Solimini ($n=1$, $\hat{\lambda}=\lambda_k$), Lupo-Solimini are all included as special cases of the above conclusion.

In addition, if $u=\theta$ is a trivial solution, and if some conditions are imposed on the Morse index at $u=\theta$, we may improve the above conclution to obtain a nontrivial solution.

We consider the following problem:

$$-\Delta u(x)=\lambda_k u(x)+\phi(u(x)),\quad x\in\Omega,$$
$$u(x)|_{\partial\Omega}=0,$$

where we assume that $\Phi\in C^2(\boldsymbol{R}^1)$, and $\phi=\Phi'$, satisfying

$$\Phi(\zeta)\to 0,\ \phi(\zeta)\to 0\quad \text{as } |\zeta|\to\infty, \text{ and } \phi(0)=0.$$

The associated functional reads as follows

$$f(u)=\frac{1}{2}\int_\Omega(|\nabla u|^2-\lambda_k u^2)dx-\int_\Omega\Phi(u)dx.$$

Hence

$$f(\theta)=\Phi(0)\operatorname{meas}(\Omega),\quad f'(\theta)=\theta,$$

and

$$f''(\theta)=Id-(\lambda_k+\phi'(0))(-\Delta)^{-1}.$$

It follows that

$$C_q(\theta, f)=\delta_{qm} \quad \text{if } \underline{\lambda}<\lambda_k+\phi(0)<\bar{\lambda},$$
$$=0 \quad \text{if } q\notin[m, \bar{m}] \text{ and } \lambda_k+\phi(0)=\bar{\lambda},$$

where $\underline{\lambda}<\bar{\lambda}$ is a pair of consecutive eigenvalues of $-\varDelta$, and

$$\underline{m}=\sum_{\lambda_j\leqslant\lambda}\dim\ker(-\varDelta-\lambda_j\,Id), \quad \bar{m}=m+\text{demker}(-\varDelta-\bar{\lambda}\,Id).$$

Set ting $m_-=\sum_{j<k}\dim\ker(-\varDelta-\lambda_j Id)$ and $m_0=\dim\ker(-\varDelta-\lambda_k Id)$, we have

$$C_{m_-}(\theta, f)=0 \quad \text{if } \phi'(0)<-\lambda_k+\lambda_{k-1} \text{ or } \phi'(0)>0,$$
$$C_{m_-+m_0}(\theta, f)=0 \quad \text{if } \phi'(0)<0 \text{ or } \phi'(0)>\lambda_{k+1}-\lambda_k.$$

Theorem 4. 1. *Equation* (4.3) *possesses a nontrivial solution provided either*

(1) $\Phi(0)=0$, *or*

(2) $\Phi(0)<0$, *and* $\phi'(0)\notin[-\lambda_k+\lambda_{k-1}, 0]$, *or*

(3) $\Phi(0)>0$, *and* $\phi'(0)\notin[0, \lambda_{k+1}-\lambda_k]$.

Proof This is an application of Theorem 2.8.

For the special case $\Phi(u)=a\exp\{-u^2\}$, if either $a<-\lambda_k+\lambda_{k-1}$ or $a>\lambda_{k+1}-\lambda_k$, then equation (4.3) possesses a nontrivial solution. Similarly, the equation

$$u''(x)+k^2u(x)=a\sin u(x), \quad x\in(0, \pi),$$
$$u(0)=u(\pi)=0$$

possesses a nontrivial solution provided $a\notin[-2k+1, 2k+1]$.

2. Periodic solutions of Hamiltonian systems.

Let $H=H(t, p, q)$: $\boldsymbol{R}^1\times\boldsymbol{R}^n\times\boldsymbol{R}^n\to\boldsymbol{R}^1$ be a C^1 function. We partition the variables $(p, q)\in\boldsymbol{R}^n\times\boldsymbol{R}^n$ into several groups as follows:

$$\bar{p}=(p_1, p_2, \cdots, p_i), \quad \bar{q}=(q_1, q_2, \cdots, q_i),$$
$$\tilde{p}=(p_{i+1}, \cdots, p_j), \quad \tilde{q}=(q_{i+1}, \cdots, q_j),$$
$$\hat{p}=(p_{j+1}, \cdots, p_k), \quad \hat{q}=(q_{j+1}, \cdots, q_k),$$

and

$$\check{p}=(p_{k+1}, \cdots, p_n), \quad \check{q}=(q_{k+1}, \cdots, q_n),$$

$1\leqslant i\leqslant j\leqslant k\leqslant n$.

Assume that

(HS 1) H is periodic in the variables: t, $\bar{p}$, $\bar{q}$, $\hat{p}$ and $\tilde{q}$,

(HS 2) H and $\text{grad}\,H=\left(\frac{\partial H}{\partial p}, \frac{\partial H}{\partial q}\right)$ tend to zero as $|\hat{p}|+|\hat{q}|+|\check{q}|+|\check{p}|\to\infty$, uniformly in t, $\bar{p}$, $\bar{q}$, $\hat{p}$ and $\tilde{q}$.

Theorem 4. 2. *The Hamiltonian system*

$$p'=\frac{\partial H}{\partial q}(t, p, q),$$
$$q'=-\frac{\partial H}{\partial p}(t, p, q)$$

possesses at least $i+k+1$ periodic solutions.

Proof Let us denote $z=(p, q)$, $J=\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$, the Sobolev space $\widetilde{H}=H^{1/2}(S^1, \boldsymbol{R}^{2n})$. Define a bounded self-adjoint operator A on $\widetilde{H}$ satisfying

$$(Az, z)=\int_{S^1}-Jz'(t)z(t)dt, \quad \forall z\in C^1(S^1, \boldsymbol{R}^{2n}),$$

where $(\cdot, \cdot)$ is the scalar product of the space $\widetilde{H}$. We have $\ker A=\boldsymbol{R}^{2n}$. According to the assumption (HS 1), the functional

$$f(z)=1/2(Az, z)-\int_{S^1} H(t, z(t))dt$$

is well defined on $H^+\oplus H^-\oplus\boldsymbol{R}^{2n-k-i}\times T^{k+i}$, where $H^\pm$ is the positive / negative invariant subspace according to the spectral decomposition of A. Set $H_0=\boldsymbol{R}^{2n-k-i}$ and $H=H^+\oplus H^-\oplus H_0$. It is easy to verify the conditions (A_1), (A_2) and (Gv). According to Theorem 3.3 there exist at least cuplength $(T^{k+i})+1=k+i+1$ periodic solutions.

3. Other applications.

Theorems 2.6, 2.8 and 3.3 may be applied to a variety of semilinear problems with strong resonance, for instance, the semilinear elliptic systems (cf. Chang[3]), the semilinear forced oscillation of strings (cf. Rabinowitz [R]), the semilinear beam oscillation equations (cf. Chang-Sanchez[5] and Liu[8]) as well as the semilinear spherical wave equations, (cf. Chang-Hong[4]).

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