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HOLOMORPHIC VECTOR FIELDS AND CHARACTERISTIC FORMS ON A HERMITIAN MANIFOLD

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Abstract

Let M be a compact Hermtian manifold, $\dim_C M=m$, Ω be the curvature form of the Hermitian connection. F is a $U(m)$-invariant polynomial of degree $k<m$ defined on the Lie algebra $u(m)$, $F(\Omega)$ is a characteristic form of M defined by F. Let σ be arbitrary closed $(2m-2k)$-form M. In this paper, the author gives the integral formula

$$\int_M F(\Omega)\wedge\sigma=\sum_i\int_{N_i}\left\{\text{The coefficient of } t^{\nu_i} \text{ of the Taylor expansion of the expression } \frac{F(t\widetilde{\Omega}+A_v)}{E(t\widetilde{\Omega}^{\perp}+A^{\perp}_v)}\right\}\wedge\sigma',$$

where N_i are the connected components of the singular set N of $(m-k+1)$ involutive holomorphic vector fields $\{v_k, v_{k+1}, \cdots v_m\}$, defined on M, $\dim N_i=2m-2k+\nu_i$.

§1. Introduction

Let M be a compact Hermitian manifold, $\dim_C M=m$, the Hermitian metric be H. There exists unique connection ∇ associated to H, the corresponding connection form is $\omega=\partial H\cdot H^{-1}$, it is a form of (1, 0)-type; the corresponding curvature form is $\Omega=d\omega-\omega\wedge\omega=\bar\partial\omega$, it is a form of (1, 1)-type. The structure group of the tangent bundle $T(M)$ of M is complex linear group $\mathrm{GL}(m, C)$, the corresponding Lie algebra $\mathrm{gl}(m, C)$ is the algebra of complex $(m\times m)$-matrices. Let F be the $\mathrm{GL}(m, C)$-invariant complex valued polynomial of degree k defined on $\mathrm{gl}(m, C)$, $F(\Omega)$ be global closed $2k$-form of M, called the characteristic form of the tangent bundle $T(M)$ defined by F. The purpose of this paper is to calculate the integral

$$\int_M F(\Omega)\wedge\sigma,$$

where σ is any arbitrary closed $(2m-2k)$-form of M.

When $\deg F=m$, this problem was solved by R. Bott[1], but his method was failed to the case $\deg F<m$. In this paper, we hope to generalize Bott's result to the case $\deg F<m$, and give an integral formula.

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§ 2. Complex Vector Fields on the Hermitian Manifold

Given $(m-k+1)$ complex vector fields $\{u_k, u_{k+1}, \cdots, u_m\}$ on the Hermitian manifold M. Let S be the set of the singular points of $\{u_{k+1}, \cdots, u_m\}$:

$$S=\{x\in M: \text{the vectors } u_{k+1}(x), \cdots, u_m(x) \text{ are linearly dependent}\},$$

S be the support of the dual cycle DC_{k+1} of the Chern class C_{k+1} on M ([2, p. 411—413]).

$$\dim S \geqslant \dim C_{k+1}=2(m-k-1).$$

Weassume that the vector fields $\{u_{k+1}, \cdots, u_m\}$ are situated in general position. Then

$$\dim S=2(m-k-1).$$

Let $S(\varepsilon)$ be the ε-tubular neighbourhood of S in M. On $M\backslash S(\varepsilon)$, the vector fields $\{u_{k+1}, \cdots, u_m\}$ are linearly independent.

Lemma 1. *There exists another closed $(2m-2k)$-form σ' of M and sufficient small ε such that $\sigma-\sigma'=d\theta$ for some $(2m-2k-1)$-form θ of M and $\sigma'|\overline{S(\varepsilon)}=0$.*

Proof See [3, Lemma 1].

Corollary. $\int_M F(\Omega)\wedge\sigma=\int_M F(\Omega)\wedge\sigma'=\int_{M\backslash S(\varepsilon)} F(\Omega)\wedge\sigma'.$

Proof Since M is compact, $\partial M=\varnothing$, so that

$$\int_M F(\Omega)\wedge d\theta=\int_M d(F(\Omega)\wedge\theta)=\int_{\partial M} F(\Omega)\wedge\theta=0.$$

Let N be the set of the singular points of $\{u_k, u_{k+1}, \cdots, u_m\}$ on $M\backslash S(\varepsilon)$,

$$\dim N\geqslant 2m-2k,$$

$N=U_iN_i$, where N_i are the connected components of N. $\dim N_i=2m-2k+\nu_i$, where ν_i are non-negative integers.

Now we suppose $\{u_k, u_{k+1}, \cdots, u_m\}$ to be involutive holomorphic vector fields, where "involutive" means that on any open set of $M\backslash S(\varepsilon)$, $[u_\lambda, u_\mu]$ $(\lambda, \mu=k, k+1, \cdots, m)$ are linear combinations of $\{u_k, u_{k+1}, \cdots, u_m\}$ with holomorphic coefficients. On the singular set N, the vector fields $\{u_{k+1}, \cdots, u_m\}$ are involutive, the well-known theorem of Frobenius asser s that, at every point $p\in N$, there exists a coordinate chart U with holomorphic coordinate system $(z^1, \cdots, z^m)$ such that $u_\alpha(\alpha=k+1, \cdots, m)$ are linear combinations of $\left\{\frac{\partial}{\partial z^{k+1}}, \cdots, \frac{\partial}{\partial z^m}\right\}$ with holomorphic coefficients, or

$$dz^a(u_\alpha)=0, \quad (a=1, \cdots, k;\ \alpha=k+1,\cdots, m).$$

Consider another coordinate chart V with holomorphic coordinate system $(z'^1, \cdots,$

z'^m), $dz^a=0=dz'^a=0$ $(a=1, \cdots, k)$, so that

$$\frac{\partial z'^a}{\partial z^\alpha}=0, \quad (a=1, \cdots, k;\ \alpha=k+1, \cdots, m).$$

It means that on $U\cap V$, $z'^a(a=1, \cdots, k)$ are holomorphic functions of $(z^1, \cdots, z^k)$. So there exists a subbundle E of $T(M)$ on $N(\varepsilon)$, whose fibres are spanned by $\left\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^k}\right\}$ with transition functions $\left(\frac{\partial z'^a}{\partial z^b}\right)$ $(a, b=1, \cdots, k)$. E is a holomorphic vector bundle of complex dimension k.

Restricting the connection of $T(M)$ to E, we get the connection ∇^E of E as follows:

$$\nabla^E_{e_a}=\sum_{b=1}^{k}\omega_{ab}e_b \quad (a=1, \cdots, k),$$

where $e_a=\frac{\partial}{\partial z^a}$ $(a=1, \cdots, k)$. Extending this connection to $T(M)$, we get the connection ∇^E of $T(M)$ on $N(\varepsilon)$:

$$\nabla^E_{e_a}=\sum_{b=1}^{k}\omega_{ab}e_b \quad (a=1, \cdots, k),$$

$$\nabla^E_{u_\alpha}=0, \quad (\alpha=k+1, \cdots, m).$$

We can extend this connection ∇^E to the tangent bundle $T(M)$ on $M\backslash S(\varepsilon)$ as follows:

$$\nabla^E_{e_a}=\nabla_{e_a} \ (\mathrm{mod.}\ u_k, u_{k+1}, \cdots, u_m) \quad (a=1, \cdots, k),$$

$$\nabla^E u_\alpha=0, \quad (\alpha=k+1, \cdots, m).$$

Note that, on N, mod. $u_k, u_{k+1}, \cdots, u_m$ reduces to mod. $u_{k+1}, \cdots, u_m$, then this connection ∇^E becomes the connection of the bundle E.

Let the connection form of ∇^E be ω^E, it is still the form of the (1, 0)-type, the corresponding curvature form being $\nabla^E=\bar{\partial}\omega^E$. Constructing the homotopy between the connections ∇ and ∇^E of the tangent bundle $T(M)$ on $M\backslash S(\varepsilon)$, we have

Lemma 2. *There exists a $(2k-1)$-form τ on $M\backslash S(\varepsilon)$ such that*

$$F(\Omega)-F(\Omega^E)=d\tau.$$

Proof Let $u=\omega-\omega^E$, $\omega_t=\omega^E+tu$,

$$\Omega_t=d\omega_t=d\omega_t-\omega_t\wedge\omega_t.$$

According to the Weil Lemma [4, p. 39, (B)], we get

$$\tau=k\int_0^1 F(u, \underbrace{\Omega_t, \cdots, \Omega_t}_{k-1})dt.$$

Corollary.

$$\int_M F(\Omega)\wedge\sigma=\int_M F(\Omega^E)\wedge\sigma'.$$

Define an operator $\tilde{L}_X$ of $T(M)$ associated to the vector field X. For any $Y\in \Gamma(T(M))$,

$$\tilde{L}_X=[X, Y]\ (\mathrm{mod.}\ u_k, u_{k+1}, u_m).$$

Then define the third connection $\widetilde{\nabla}$ of $T(M)$ on $M\backslash S(\varepsilon)$ such that

$$\widetilde{\nabla}_{e_a}=\nabla^E_{e_a},\quad (a=1,\cdots,k),$$
$$\widetilde{\nabla}_{u_\alpha}=\widetilde{L}_{u_\alpha},\quad (\alpha=k+1,\cdots,m).$$

Let the connection form of $\widetilde{\nabla}$ be $\widetilde{\omega}$, it is a form of (1, 0)-type, and $\widetilde{\omega}(u_\alpha)$ $(\alpha=k+1,\cdots m)$ are holomorphic functions. The corresponding curvature form be $\widetilde{\Omega}=\widetilde{\Omega}_1+\widetilde{\Omega}_2$, where $\widetilde{\Omega}_1=\partial\widetilde{\omega}-\widetilde{\omega}\wedge\widetilde{\omega}$ is the form of (2, 0)-type, and $\widetilde{\Omega}_2=\bar{\partial}\widetilde{\omega}$ is the form of (1,1)-type.

Constructing the homotopy of the connections ∇^E and $\widetilde{\nabla}$ of the bundle $T(M)$ on $M\backslash S(\varepsilon)$, we have

Lemma 3.

$$\int_M F(\Omega^E)\wedge\sigma'=\int_M F(\widetilde{\Omega})\wedge\sigma'.$$

§ 3. The Vanishing Theorem

Define the fourth connection ∇' of $T(M)$ on $(M\backslash S(\varepsilon))\backslash N$ as follws:

$$\nabla'_{e_a}=\widetilde{\nabla}_{e_a}=\nabla^E_{e_a}=\nabla_{e_a}\ (\text{mod. } u_k, u_{k+1},\cdots,u_m)\quad (a=1,\cdots,k),$$
$$\nabla'_{u_k}=\widetilde{L}_{u_k},$$
$$\nabla'_{u_\alpha}=\widetilde{\nabla}_{u_\alpha}=\widetilde{L}_{u_\alpha},\quad (\alpha=k+1,\cdots,m).$$

Let the connection form of ∇' be ω', it is also a form of (1, 0)-type, and $\omega'(u_\lambda)$ $(\lambda=k, k+1,\cdots,m)$ are holomorphic functions. The curvature form is $\Omega'=\Omega'_1+\Omega'_2$, where $\Omega'_1=\partial\omega'-\omega'\wedge\omega'$ and $\Omega'_2=\bar{\partial}\omega'$ are the forms of (2, 0)- and (1, 1)-type respectively.

Lemma 4.

$$\Omega'(u_\lambda, u_\mu)=0.\quad (\lambda,\mu=k, k+1,\cdots,m).$$

For any vector field $\overline{X}$ of (0, 1)-type,

$$\Omega'(u_\lambda,\overline{X})=0.\quad (\lambda=k, k+1,\cdots,m).$$

Proof Let $R'(X, Y)Z$, $\forall X, Y, Z\in\Gamma(T(M))$, be the curvature tensor of the connection ∇'. We have

$$R'(X, Y)e_a=\sum_{b=1}^{k}\Omega_{ab}(X, Y)e_b,\quad (a=1,\cdots,k),$$

$$\begin{aligned}R'(u_\lambda, u_\mu)=e_a&=\nabla'_{u_\lambda}\nabla'_{u_\mu}e_a-\nabla'_{u_\mu}\nabla'_{u_\lambda}e_a-\nabla'_{[u_\lambda,u_\mu]}e_a\\&=\widetilde{L}_{u_\lambda}\widetilde{L}_{u_\mu}e_a-\widetilde{L}_{u_\mu}\widetilde{L}_{u_\lambda}e_a-\nabla'_{[u_\lambda,u_\mu]}e_a\\&=([u_\lambda,[u_\mu,e_a]\ (\text{mod.} u_k,\cdots,u_m)]\\&\quad-[u_\mu,[u_\lambda,e_a]\ (\text{mod. } u_k,\cdots,u_m)])\ (\text{mod. } u_k,\cdots,u_m)\\&\quad-\nabla'_{[u_\lambda,u_\mu]}e_a.\end{aligned}$$

Since the holomorphic vector fields $\{u_k, u_{k+1},\cdots,u_m\}$ are involutive

$$[u_\lambda,[u_\mu,e_a]\ (\text{mod. } u_k,\cdots,u_m)]\ (\text{mod. } u_k,\cdots,u_m)$$
$$=[u_\lambda,[u_\mu,e_a]]\ (\text{mod. } u_k,\cdots,u_m),$$

$$[u_\mu, [u_\lambda, e_a](\mathrm{mod.}\, u_k, \cdots, u_m)](\mathrm{mod.}\, u_k, \cdots, u_m)$$
$$= [u_\mu, [u_\lambda, e_a]](\mathrm{mod.}\, u_k, \cdots, u_m),$$
$$\nabla'_{[u_\lambda, u_\mu]} e_a = \widetilde{L}_{[u_\lambda, u_\mu]} e_a = [[u_\lambda, u_\mu], e_a](\mathrm{mod.}\, u_k, \cdots, u_m).$$

According to the Jacobi's identity,

$$R'(u_\lambda, u_\mu)e_a = ([u_\lambda, [u_\mu, e_a]] - [u_\mu, [u_\lambda, e_a]] - [(u_\lambda, u_\mu], e_a])(\mathrm{mod.}\, u_k, \cdots, u_m)$$
$$= ([u_\lambda, [u_\mu, e_a]] + [u_\mu, [e_a, u_\lambda]] + [e_a, [u_\lambda, u_\mu]])(\mathrm{mod.}\, u_k, \cdots, u_m)$$
$$= 0.$$
$$R'(u_\lambda, u_\mu) = 0, \quad \Omega'_{ab}(u_\lambda, u_\mu) = 0.$$
$$\Omega'(u_\lambda, \overline{X}) = \Omega'(u_\lambda, \overline{X}) = i(u_\lambda)\bar\partial\omega'(\overline{X}) = -\bar\partial(i(u_\lambda)\omega'(\overline{X}) = 0.$$

The Lemma is proved.

Theorem 1 (Vanishing Theorem). *On* $(M\backslash S(\varepsilon))\backslash N$, *we have*

$$F(\Omega') = 0.$$

Proof $F(\Omega') = F(\Omega'_1 + \Omega'_2) = \sum_{p=0}^{k} C_k^p F(\underbrace{\Omega'_1, \cdots, \Omega'_1}_{k-r}, \underbrace{\Omega'_2, \cdots, \Omega'_2}_{r}).$

Let $X_i (i=1, \cdots, m)$ and $\overline{Y}_i (i=1, \cdots, m)$ be vector fields of (1, 0)- and (0, 1)-types on $(M\backslash S(\varepsilon))\backslash N$ respectively.

$$F(\underbrace{\Omega'_2, \cdots, \Omega'_2}_{k})(X_1, \cdots, X_k; \overline{Y}_1, \cdots, \overline{Y}_k) = F(\Omega'_2(X_1, \overline{Y}_1), \cdots, \Omega'_2(X_k, \overline{Y}_k)).$$

Take $X_a = e_a (a=1, \cdots, k)$. Since $u_k, u_{k+1}, \cdots, u_m$ are linearly independent, e_k should be linearly dependent to $e_1, \cdots, e_{k-1}, u_k, u_{k+1}, \cdots, u_m$, so that one of the $\Omega'_2(X_a, \overline{Y}_a)$ should be $\Omega'_2(u_\lambda, \overline{Y}_a) = 9$. Then

$$F(\underbrace{\Omega'_2, \cdots, \Omega'_2}_{k}) = 0.$$

$$F(\underbrace{\Omega'_1, \cdots, \Omega'_1}_{k-p}, \underbrace{\Omega'_2, \cdots, \Omega'_2}_{p})(X_1, \cdots, X_{2k-p}, \overline{Y}_1, \cdots, \overline{Y}_p)$$
$$= F(\Omega'_2(X_1, \overline{Y}_1), \cdots, \Omega'_2(X_p, \overline{Y}_p), \Omega'_1(X_{p+1}, X_{p+2}), \cdots, \Omega'_1(X_{2k-p-1}, X_{2k-p})).$$

Take $X_a = e_a$ $(a=1, \cdots, p)$. Then at most $(k-p)$ of the remaining X's are $e_b (b=p+1, \cdots, k)$, so that at least one of the $\Omega'_1(X_i, X_j)$ is $\Omega'_1(e_k, u_\lambda) = 0$ or $\Omega'_1(u_\lambda, u_\mu) = 0$, $(\lambda, \mu = k, \cdots, m)$. We have also

$$F(\underbrace{\Omega'_1, \cdots, \Omega'_1}_{k-p}, \underbrace{\Omega'_2, \cdots, \Omega'_2}_{p}) = 0.$$

The theorem is proved.

Corollary. *There exists a* $(2k-1)$-*form* η *on* $(M\backslash S(\varepsilon))\backslash N$ *such that*

$$F(\Omega) = -d\eta.$$

Proof Construct the homotopy of the connections $\widetilde{\nabla}$ and ∇' of the tangent bundle $T(M)$ on $(M\backslash S(\varepsilon))\backslash N$. Let $u' = \omega' - \widetilde{\omega}$,

$$\omega'_t = \widetilde{\omega} + tu',$$
$$\Omega'_t = d\omega'_t - \omega'_t \wedge \omega'_t.$$

Then
$$-F(\tilde{\Omega})=F(\Omega')-F(\tilde{\Omega})=d\eta.$$

§ 4. The Transgressive Form

On $N(\varepsilon)\setminus N$, the tangent bundle $T(M)$ is the direct sum of the bundle E and E_1, where E_1 is the subbundle of $T(M)$ whose fibres are spanned by $\{u_{k+1}, \cdots, u_m\}$, it is a holomorphic vector bundle of complex dimension $m-k$. Let H^E and H^{E_1} be the restrictions of the Hermitian metric H to the subbundles E and E_1 respectively, Define another Hermitian metric H' of $T(M)$ on $N(\varepsilon)\setminus N$ as follows
$$H'=H^E+H^{E_1}.$$

On a coordinate chart U of $N(\varepsilon)$,
$$u_k=\sum_{a=1}^{k} v^a\frac{\partial}{\partial z^a}+\sum_{\alpha=k+1}^{m} v^\alpha u_\alpha,$$
where v^a and v^α are holomorphic functions of U. Let
$$v=\sum_{a=1}^{k} v^a e_a=u_k-\sum_{\alpha=k+1}^{m} v^\alpha u_\alpha,$$
where v is a vector field on $N(\varepsilon)$. Since $\{u_k, u_{k+1}, \cdots, u_m\}$ are linearly dependent on N, $v=0$ on N, i.e. N is the set of zero points of v, so that $v^a=0$ $(a=1, \cdots, k)$ on N. With respect to the new Hermitian metric H', we have
$$H'(v, v_\alpha)=0, \quad (\alpha=k+1, \cdots, m).$$
Choose a frame $\{v_1, \cdots, v_{k-1}, v\}$ of the bundle E such that
$$H'(v, v_a)=0, \quad (a=1, \cdots, k-1).$$
Then the definition of the connection $\tilde{\nabla}$ of $T(M)$ on $V(\varepsilon)$ can be rewritten in the following form:
$$\tilde{\nabla}_{v_a}=\nabla^E_{v_a}, \quad (a=1, \cdots, k-1),$$
$$\tilde{\nabla}_v=\nabla^E_v,$$
$$\tilde{\nabla}_{u_\alpha}=\tilde{L}_{u_\alpha}, \quad (\alpha=k+1, \cdots, m),$$
because v is the linear combination of $\{e_1, \cdots, e_k\}$ and $v_a(a=1, \cdots, k-1)$ is the linear combinations of $\{e_1, \cdots, e_{k-1}, v\}$.

Similarly, the definition of the connection ∇' of $T(M)$ on $N(\varepsilon)\setminus N$ can be rewritten in the following form:
$$\nabla'_{v_a}=\tilde{\nabla}_{v_a} \quad (a=1, \cdots, k-1),$$
$$\nabla'_v=\tilde{L}_v,$$
$$\nabla'_{u_\alpha}=\tilde{\nabla}_{u_\alpha}=\tilde{L}_{u_\alpha} \quad (\alpha=k+1, \cdots, m),$$
because
$$u'_v=\nabla'_{u_k-\sum_\alpha v^\alpha u_\alpha}=\nabla'_{u_k}-\sum_\alpha v^\xi\nabla'_{u_\alpha}=\tilde{L}_{u_k}-\sum_\alpha v^\alpha\tilde{L}_{u_\alpha}=\tilde{L}_{u_k-\sum_\alpha v^\alpha u_\alpha}=\tilde{L}_v.$$

Now we shall give an explicite expression of the transgressive form η on $N(\varepsilon)\setminus N$. Define a 1-form ϖ on $N(\varepsilon)\setminus N$ such that

$$\pi(X)=\frac{H'(X,\ v)}{H'(v,\ v)},\quad \forall X\in\Gamma(T(M)).$$

Then define another operator of the tangent bundle $T(M)$ as follows:

$$A_v=\tilde{L}_v-\tilde{\nabla}_v.$$

Lemma 5. *On $N(\varepsilon)\backslash N$, we have*

$$u'=\pi\otimes A_v.$$

Proof

$$\pi(v)\otimes A_v=A_v=\tilde{L}_v-\tilde{\nabla}_v=\omega'(v)-\tilde{\omega}(v)=u'(v).$$
$$\pi(v_a)\otimes A_v=0=\tilde{\nabla}_{v_a}-\tilde{\nabla}_{v_a}=\nabla'_{v_a}-\tilde{\nabla}_{v_a}=\omega'(v_a)-\tilde{\omega}(v_a)=u'(v_a).$$
$$\pi(u_a)\otimes A_v=0=\tilde{\nabla}_{u_a}-\tilde{\nabla}_{u_a}=\nabla'_{u_a}-\tilde{\nabla}_{u_a}=\omega'(v_a)-\tilde{\omega}(v_a)=u'(u_a).$$

Then

$$u'=\pi\otimes A_v.$$

Corollary. *On $N(\varepsilon)\backslash N$,*

η=The coefficient of t^{k-1} in the Taylor expansion of the expression $F(t\tilde{\Omega}+A_v)\dfrac{\pi}{1-td\pi}$.

Proof

$$\begin{aligned}\omega'_t&=d\omega'_t-\omega'_t\wedge\omega'_t=d(\tilde{\omega}+tu)-(\tilde{\omega}+tu)(\tilde{\omega}+tu)\\&=d(\tilde{\omega}+t\pi\otimes A_v)-(\tilde{\omega}+t\pi\otimes A_v)\wedge(\tilde{\omega}+t\pi\otimes A_v)\\&=\tilde{\Omega}+td\pi\otimes A_v-2\tilde{\omega}\wedge\pi\otimes A_v.\end{aligned}$$

$$\begin{aligned}\eta&=k\int_0^1F(u',\ \underbrace{\Omega'_t,\ \cdots,\ \Omega'_t}_{k-1})dt\\&=k\int_0^1F(\pi\otimes A_v,\ \underbrace{\tilde{\Omega}+t\,d\pi\otimes A_v-2\tilde{\omega}\wedge\pi\otimes A_v),\ \cdots\cdots}_{k-1})dt\\&=k\int_0^1F(\pi\otimes A_v,\ \underbrace{\tilde{\Omega}+t\,d\pi\otimes A_v,\ \cdots,\ \tilde{\Omega}+tr\pi\otimes A_v}_{k-p})dt\\&=k\sum_{p=0}^{k-1}C_{k-1}^pF(\underbrace{\tilde{\Omega},\ \cdots,\ \tilde{\Omega}}_{p},\ \underbrace{A_v,\ \cdots,\ A_v}_{k-p})\pi\wedge(d\pi)^{k-p-1}\int_0^1t^{k-p-1}dt\\&=\sum_{p=0}^{k}C_k^pF(\underbrace{\tilde{\Omega},\ \cdots,\ \tilde{\Omega}}_{p},\ \underbrace{A_v,\ \cdots,\ A_v}_{k-p})\cdot\pi\wedge(d\pi)^{k-p-1}\end{aligned}$$

= The coefficient of t^{k-1} in the Taylor expansion of the expression $F(t\tilde{\Omega}+A_v)\cdot\dfrac{\pi}{1-td\pi}$.

§ 5. The Integral Formula of Characteristic Forms

$$\begin{aligned}\int_MF(\Omega)\wedge\sigma&=\int_MF(\Omega^E)\wedge\sigma'=\int_MF(\tilde{\Omega})\wedge\sigma'=\lim_{\varepsilon\to0}\int_{M\backslash N(\varepsilon)}F(\tilde{\Omega})\wedge\sigma'\\&=\lim_{\varepsilon\to0}\int_{M\backslash N(\varepsilon)}-d(\eta\wedge\sigma')=\lim_{\varepsilon\to0}\int_{\partial N(\varepsilon)}\eta\wedge\sigma'.\end{aligned}$$

So we need only to consider the transgressive form η in $N(\varepsilon)$. We have just given the explicite expression of η, so that we are in the position to give the integral formula of characteristic forms.

Let N_i be the connected components of the singular set N, $\dim N_i = 2m-2k+2\nu_i$, $(\nu_i \geqslant 0)$. They are in fact the sets of zeros of the holomorphic vector field v in $N(\varepsilon)$. We suppose that N_i are non-degenerated zero sets. It means that: First, N_i are complex submanifolds of M, since they are closed sets, they are also compact and finite in number; Second, every tangent space T_xM, $x \in N_i$, splits into two parts: T_xN_i and $(T_xN_i)^\perp$, where $(T_xN_i)^\perp$ is the orthogonal complement of T_xN_i in T_xM with respect to the Hermitian metric H'. "Non-degenerated" requires that T_xN_i should be just the kernel of the operator $A_v = \tilde{L}_v$(since $v=0$ on N_i, $\tilde{\nabla}_v = 0$). Because $\{u_{k+1}, \cdots, u_m\} \subset \ker \tilde{L}_v$, $E_{1x} \subset T_xN_i$ and $E_x \supset (T_xN_i)^\perp$. Let $T^\perp N_i$ be the bundle on N_i with the fibres $(T_xN_i)^\perp$, $x \in N_i$; it is a holomorphic vector bundle of complex dimension $2k-2\nu_i$, $T^\perp N_i \subset E$. Denote the restriction of A_v to $T^\perp N_i$ by $A_v^\perp$, and the restriction of the curvature form $\tilde{\Omega}$ be $\tilde{\Omega}^\perp$. We can prove:

Lemma 6. *Let $S^\perp(\varepsilon)$ be the sphere with radius ε in the fibres of the holomorphic vector bundle $T^\perp N_i$. Then we have*

$$\lim_{\varepsilon\to 0}\int_{S^\nu(\varepsilon)} \frac{\pi}{1-t\,d\pi} = \frac{t^{k-\nu_i-1}}{E(t\tilde{\Omega}^\perp + A_v^\perp)'},$$

where E is the Euler characteristic form of the bundle $T^\perp N_i$.

Proof See Kobayashi [5, p. 120—121].

Theorem 2.

$$\int_L F(\Omega)\wedge\sigma = \sum_i \int_{N_i} \Big\{\textit{The coefficient of } t^{\nu_i} \textit{ in the Taylor expansion of the expression } \frac{F(t\tilde{\Omega}+A_v)}{E(t\tilde{\Omega}^\perp + A_v^\perp)}\Big\}\wedge\sigma'.$$

Proof

$$\int_M F(\Omega)\wedge\sigma = \lim_{\varepsilon\to 0}\int_{\partial N(\varepsilon)} \eta\wedge\sigma' = \sum_i \lim_{\varepsilon\to 0}\int_{\partial N_i(\varepsilon)} \eta\wedge\sigma'$$

$$= \sum_i \lim_{\varepsilon\to 0}\int_{\partial N_i(\varepsilon)} \Big\{\text{The coefficient of } t^{k-1} \text{ of the expression } F(t\tilde{\Omega}+A_v)\frac{\pi}{1-t\,d\pi}\Big\}\wedge\sigma'.$$

Note that $\partial N_i(\varepsilon)$ can be considered as a bundle on N_i with the fibres $S^\perp(\varepsilon)$. When $\varepsilon\to 0$, the factor $F(t\tilde{\Omega}+A_v)$ becomes the form of the base space N_i, so that

$$\int_M F(\Omega)\wedge\sigma = \sum_i \int_{N_i} \Big\{\text{The coefficient of } t^{k-1} \text{ in the Taylor expansion of the expression } F(t\tilde{\Omega}+A_v)\cdot\lim_{\varepsilon\to 0}\int_{S^\perp(\varepsilon)} \frac{\pi}{1-t\,d\pi}\Big\}\wedge\sigma'.$$

$$=\sum_i \int_{N_i} \text{The coefficient of } t^{\nu_i} \text{ in the Taylor expansion of the expression } \frac{F(t\tilde{\Omega}+A_v)}{E(t\tilde{\Omega}^{\perp}+A_v^{\perp})}\Bigg\} \wedge \sigma'.$$

The theorem is proved.

When deg $F=m$, there exists only one holomorphic vector field v. The zero set of v is $N=\bigcup_i N_i$, where N_i are the connected components of N. Let $\dim N_i=2\nu_i(\nu_i<m)$. We get the result of Bott [1] from Theorem 2:

$$\int_M F(\Omega)=\sum_i \int_{N_i}' \Bigg\{\text{The coefficient of } t^{\nu_i} \text{ in the Taylor expansion of the expression } \frac{F(t\tilde{\Omega}+A_v)}{E(t\tilde{\Omega}^{\perp}+A_v^{\perp})}\Bigg\}.$$

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