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ON A PROJECTION THEOREM OF QUASI-VARIETIES IN ELIMINATION THEORY**

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Abstract

It is proved that the quasi-varieties in affine space is closed under the projection operation though it is not so for algebraic-varieties.

§1 Statement of Theorem

Let K be a field of characteristic 0 fixed in what follows. Consider a set PS of pols $P_i(X, Y)$, $i=1, \cdots, r$, over K in sets of variables $X=(X_1, \cdots, X_n)$ and $Y=(Y_1, \cdots, Y_m)$. The set Zero(PS) defines then an algebraic variety in the $(n+m)$-dimensional affine space $A_{nm}(X, Y)$ over K in coordinates X, Y. Many problems arising both from theory and practice lead to the problem of eliminating $Y_1, \cdots, Y_m$ from the equations $PS=0$, or $P_i(X, Y)=0$, $i=1, \cdots, r$. In particular, we may mention the determination of geometrical loci in terms of equations involving X alone from geometrial constraints given by the above equations in both X and Y. Let $A_n(X)$ be the n-dimensional affine space over K in coordinates X alone and Proj be the projection of $A_{nm}(X, Y)$ to $A_n(X)$ defined by

$$\text{Proj:} \quad (X_1, \cdots, X_n, Y_1, \cdots, Y_m)\to(X_1, \cdots, X_n).$$

Then the above elimination problem amounts to the determination of the set Proj Zero(PS). A variety of methods dealing with such a problem will lead to a system of polynomial equations $Q_j(X)=0$ in X alone to be satisfied by points in this projection, by the usual method of elimination (cf. e. g. [3], [4], etc). However, simple examples show that Proj Zero(PS) is generally not an algebraic variety at all so that it cannot be represented in the form Zero(QS) for any polset QS in X alone. This means that the algebraic varieties in an affine space are not closed under the operation of projection to a lower dimensional affine space. On the other hand, we shall prove in this paper that the projection operation will become closed if we enlarge the domain of algebraic varieties to quasi-varieties defined as follows.

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Definition. *For any finite number of polsets PS_i and pols G_i over K in variables $X=(X_1, \cdots, X_n)$ the set*

$$\mathrm{SUM}_i \, \mathrm{Zero}(PS_i/G_i)$$

is called a QUASI-VARIETY in the affine space $A_n(X)$.

Quasi-varieties in the affine space $A_{nm}(X, Y)$ will be defined in the same way. Then we have the following

Theorem P. *The projection of a quasi-variety in $A_{nm}(X, Y)$ to $A_n(X)$ is a quasi-variety in $A_n(X)$.*

As the projection of A_{nm} to A_n may be resolved into a series of consecutive projections to affine spaces lowering dimension by 1, the above theorem may be reduced to the following one:

Theorem P′. *The projection of a qaasi-variety in the $(n+1)$-dimensional affine space A_{n1} over K on coordinates $X_1, \cdots, X_n, Y$ to the n-dimensional affine space A_n over K on coordinates $X_1, \cdots, X_n$ is a qaasi-variety in A_n.*

The proof of Theorem P′, to be given in the next section, is based on our general method of mathematics mechanization. In particular the following theorem will be used (cf. e. g. [5] and the relavent papers of the author).

Zero Decomposition Theorem (in weak form). *Let us order the variables as*

$$X_1 < \cdots < X_n$$

and define asc-sets, etc. w. r. t. this ordering. Then for any polset PS over K in $X_1, \cdots, X_n$ we have

$$\mathrm{Zero}(PS) = \mathrm{SUM}_k \, \mathrm{Zero}(\mathrm{ASC}_k/J_k),$$

in which each ASC_k is an asc-set and J_k is the product of all the initials of pols in ASC_k.

We remark that the notion of quasi-variety appears already in usual treatise on algebraic geometry (cf. e. g. [2]). However, it seems that this notion is rarely studied in the literature. The quasi-varieties are, by the very definition, closed under the operations of union and intersection. The theorems P and P′ show that they are also closed under the operation of projection. The author is ignorant of whether these projection theorems are already known in some disguised form or not. In any way, our proof of the theorems is constructive in character, in contrast to the usual existential character of the modern treatment of algebraic geometry. The proof will thus not only show that the projection of a quasi-variety is still such a one but will also give the precise form of that projection as a quasi-variety. In this respect one may also compare with the corresponding theorem of semi-algebraic sets in the real case (cf. e. g. [1]).

§ 2. Proof of Projection Theorem

In what follows XS, XS_1, XS', etc. will denote polsets consisting of pols in variables $X_1, \cdots, X_n$ alone, while PS, PS_1, PS', etc. those in both varables $X_1, \cdots, X_n$ and Y. We shall fix the ordering of variables

$$X_1<\cdots<X_n<Y$$

in forming char-sets and asc-sets, etc. Besides, for a pol $F(X, Y)$ we shall denote by Deg F its degree in Y.

Definition. *In the affine space $A_{n1}(X, Y)$ the quasi-variety of the form*

$$Z=\text{Zero}(XS,\ F(X, Y)/G(X, Y)*H(X))$$

will be called a SIMPLE QUASI-VARIETY of TYPE $d=\text{Deg}\,F$.

We shall first reduce the study of any quasi-variety in $A_{n1}(X, Y)$ to that of simple ones, viz.

Lemma S. *Any quasi-variety*

$$\text{Zero}(PS/G(X, Y)*H(X))$$

is the union of simple quasi-varieties each of form (1) *for which $H(X)$ is divisible by the initial of $F(X, Y)$.*

Proof By the Zero Decomposition Theorem we have

$$\text{Zero}(PS)=SUM_k\ \text{Zero}(ASC_k/J_k).$$

In the formula each ASC_k is an asc-set which is of the form

$$ASC_k=XS_k+\{F_k(X, Y)\}.$$

The pol J_k is the product of all initials of pols in ASC_k including in particular the initial of $F_k(X, Y)$ and is a pol in X alone. It follows that

$$\text{Zero}(PS/G(X, Y)*H(X))=SUM_k\text{Zero}(XS_k,\ F_k(X, Y)/G(X, Y)*H_k(X)),$$

with $H_k(X)=H(X)*J_k(X)$ so that each member in the SUM is a simple quasi-variety in $A_{n1}(X, Y)$ with H_k divisible by the initial of $F_k(X, Y)$ as to be proved.

Let us now consider two particular cases of simple quasi-varieties (1) for which either F or G is lacking.

Lemma F. *For a simple quasi-variety of type* 0

$$Z_0=\text{Zero}(XS/G(X, Y)*H(X))$$

the projection Proj Z_0 is a quasi-variety in $A_n(X)$.

Proof Let us write $G(X, Y)$ as a pol in Y in the form

$$G=B_0(X)*Y^\wedge e+\cdots+B_e(X),\quad e=\text{Deg}G.$$

Let XS_i $(i=0, 1, \cdots, e-1)$ be the polset XS enlarged by adjoining to it the pols $B_1(X), \cdots, B_i(X)$. Then it is clear that

$$\text{Proj}\ Z_0=\text{Zero}(XS/B_0(X)*H(X))+SUM_i(XS_i/B_i(X)*H(X))$$

is the union of quasi-varieties in $A_n(X)$, the SUM_i running over $i=0, 1, \cdots, e-1$

with $j=i+1$.

Lemma G. *For the simple quasi-variety Z of* (1) *with G lacking:*

$$Z'=\text{Zero}(XS,\ F(X,\ Y)/H(X)),$$

the projection of Z' is a quasi-variety in $A_n(X)$.

Proof Let us write $F(X, Y)$ as a pol in Y in the form

$$F(X,\ Y)=A_0(X)*Y\wedge d+\cdots+A_d(X),\quad d=\text{Deg}\,F.$$

Set for $i=0, 1, \cdots, d$,

$$F_i(X,\ Y)=A_i(X)*Y\wedge(d-i)+\cdots+A_d(X),$$
$$XS_i=XS+[A_0(X),\ \cdots,\ A_j(X)\},$$

in which $j=i-1$. Then it is clear that

$$Z'=SUM_i\text{Zero}(XS_i,\ F_i(X,\ Y)/H_i(X)*A_i(X)),$$

in which SUM_i runs over $i=0, 1, \cdots, d-1$, the member corresponding to $i=d$ being empty. It is easy to see that for each i,

$$\text{Proj Zero}(XS_i,\ F_i(X,\ Y)/H(X)*A_i(X))=\text{Zero}(XS_i/H_1(X))$$

with $H_i(X)=H(X)*A_i(X)$ is a quasi-variety in $A_n(X)$ and the Lemma is thus proved.

We are now in a position of proving Theorem P′.

By Lemma S, it is only necessary to consider quasi-varieties Z as given in the form (1). By Lemma F Proj Z is a quasi-varity if F is lacking or if Z is of type 0. We shall thus prove Theorem P′ for Z of type $d=\text{Deg}\,F>0$. By Lemmas S, G and F we may further restrict ourselves to the case for which the following conditions are observed:

(a) $\text{Deg}\,G>0$,

(b) $H(X)$ is divisible by initial $I(X)$ of $F(X, Y)$.

Let us form the remainder$R(X, Y)$ of $G(X, Y)\wedge d$ w. r. t. $F(X, Y)$ so that for some non-negative s we have

$$[I(X)\wedge s]*[G(X,\ Y)\wedge d]=Q(X,\ Y)*F(X,\ Y)+R(X,\ Y),\tag{3}$$

with $\text{Deg}R<d$. Set

$$Z'=\text{Zero}(YS/G(X,\ Y)*H(X)*R(X,\ Y)).\tag{4}$$

Now for any zero(X', Y') of Z the value $H(X')$ is unequal to 0 and hence by (b) the value $I(X')$ is unequal to 0 too. On the other hand $F(X', Y')=0$. Hence by (3) the value $R(X', Y')$ is also unequal to 0 and so (X', Y') belongs to Z'. Therefore Z is contained in Z' so that

$$\text{Proj}\ Z-<\text{Proj}\ Z'.\tag{5}$$

Note that here and in what follows $-<$ stands for "is contained in". Similarly we shall use $<>$ to stand for "is unequal to".

To see the inclusion relation in the reverse direction let us consider first the case for which $R(X, Y)$ is identically 0. Then Z' is empty so that trivially we shall have $Z' -< Z$ and so

$$\text{Proj } Z' -< \text{Proj } Z. \tag{6}$$

Consider therefore the case $R(X, Y)$ not identically 0 and let X' be an arbitrary zero in Proj Z', if there are any. Then there will be some Y' with (X', Y') belonging to Z'. We have then in particular

$$R(X', Y') <> 0, \tag{7}$$

$$G(X', Y') <> 0. \tag{8}$$

We have also $H(X') <> 0$ so that by (b) we have

$$I(X') <> 0$$

too. It follows that the degree in Y of $F(X', Y)$ is the same as that of $F(X, Y)$, viz. d. If now any zero Y'' of $F(X', Y)$ is also a zero of $G(X', Y)$ then $F(X', Y)$ will be a factor of $G(X', Y)\wedge d$ which is non-zero by (8) so that by (3) $F(X', Y)$ is also a factor of $R(X', Y)$. As degree of $R(X', Y)$ is less than the degree d of $F(X', Y)$ this is only possible if $R(X', Y)$ is identically 0 which is absurd owing to (7). Consequently there is at least one zero Y'' of $F(X', Y)$ which is not a zero of $G(X', Y)$. Then (X', Y'') belongs to Z so that X' belongs to Proj$Z$. This proves again (6) and together with (5) we have therefore

$$\text{Proj } Z = \text{Proj } Z'.$$

By Lemma F Proj Z' is a quasi-variety in A_n. Hence Proj Z is also a quasi-variety in A_n and Theorem P′ is thus proved.

§ 3. Some Examples

We shall give below some examples to illustrate the significance of introducing the notion of quasi-varieties.

Let us consider for mechanical theorem proving the following

Theorem of Desargues. *Let ABC, A′B′C′ be two triangles with three pairs of corresponding sides parallel to each other. Then the three pairs of joining lines of corresponding vertices will be either intersect in the same point or be parallel to each other.*

Suppose that AA' and BB' meet in a point 0. Then it is sufficient to prove that CC' passes also through 0. For this sake let us take a coordinate system with AA', BB' as X- and Y-axis and the coordinates of various points to be:

$$A = (U_1, 0), \quad A' = (U_2, 0), \quad B = (0, U_3),$$

$$C = (U_4, U_5), \quad B' = (0, X_1), \quad C' = (X_2, X_3).$$

Then the hypothesis corresponding to the parallelism of side-pairs will be HYP $=0$, or

$$H_i = 0, \quad i = 1, 2, 3$$

will HYP$=\{H_1, H_2, H_3\}$, and

$$H_1=U_1*X_1-U_2*U_3,$$
$$H_2=U_4*(X_3-X_1)-(U_5-U_3)*X_2,$$
$$H_3=(U_4-U_1)*X_3-U_5*(X_2-U_2).$$

The conclusion that CC' passes through 0 is given by

$$\text{CONC}=0,$$

with

$$\text{CONC}=U_4*X_3-U_5*X_2.$$

Let us consider the particular configuration with values

$$\begin{matrix} U_1 & U_2 & U_3 & U_4 & U_5 & X_1 & X_2 & X_3 \\ =2 & 4 & 2 & 1 & 1 & 4 & 1 & 3 \end{matrix}$$

so that A, B, C are collinear. In this case we see that $H_i=0$, $i=1$, 2, 3, while CONC$=2\langle\ \rangle 0$. It follows that no power of CONC can be in the ideal (H_1, H_2, H_3) so that by Hilbert Zero Theorem:

Zero(HYP) is not contained in Zero(CONC).

This amounts to say that for the Desargues Theorem the hypothesis $H_i=0$ cannot unconditionally imply the conclusion CONC$=0$. It shows also the inconvenience of the notion of ideals in dealing with mechanical theorem proving since such situations occur in general in elementary and other kinds of geometries.

On the other hand our general method permits us to deduce from the polset HYP a char-set CS consisting of the pols

$$C_1=U_1*X_1-U_2*U_3,$$
$$C_2=(U_1*U_3-U_1*U_5-U_3*U_4)*X_2+(U_4-U_1)*U_4*X_1+U_2*U_4*U_5,$$
$$C_3=U_4*X_3-(U_5-U_3)*X_2-U_4*X_1.$$

The initials, i. e. the coefficients of the leading variables X_i, in C_i are resp.

$$I_1=U_1,$$
$$I_2=U_1*U_3-U_1*U_5-U_3*U_4,$$
$$I_3=U_4.$$

Let J be the product of all these initials. Then our general theory shows that

Zero(HYP$/J$)$=$Zero(CS/J) is contained in Zero(CONC),

though Zero(HYP) is not so. This shows that CONC$=0$ would follow from HYP$=0$ so far $J\langle\ \rangle 0$. By direct computation we have more precisely

$$I_2*\text{CONC}=I_2*C_3-U_3*C_2-U_4*U_5*C_1.$$

The Desargues Theorem is thus seen to be true under the non-degeneracy condition

$$I_2=U_1*U_3-U_1*U_5-U_3*U_4\langle\ \rangle 0.$$

which just means that the triangle ABC should not be degenerate into a line.

The above example shows clearly the significance of introducing the notion of quasi-variety, as the set Zero(HYP$/J$) or Zero(CS/J) in our case.

As a second example let us consider some problems arising from geometric modelling, e. g. the problem of turning a curve, a surface, or more generally a variety expressed in parametric form

$$X_1=G_1(Y),\ \cdots,\ X_n=G_n(Y),$$

with $Y=(Y_1,\ \cdots,\ Y_m)$ as parameters into usual implicit forms. The problem amounts to the elimination of Y from these equations and is a particular case of the problem of geometrical-loci determination. However, even in such particular case such an implicit form is usually impossible. To be precise, let us consider the following surface in the ordinary $(X_1,\ X_2,\ X_3)$-space defined parametrically by

$$X_1=Y_1*Y_2,$$
$$X_2=Y_1*Y_2\hat{}2,$$
$$X_3=Y_1\hat{}2.$$

Let PS be the polset consisting of the pols $X_1-Y_1*Y_2$, etc. *w. r. t.* the ordering $X_1<X_2<X_3<Y_1<Y_2$ we have then the decomposition

$$\mathrm{Zero}(PS)=\mathrm{Zero}(PS_1/X_1*X_2)+\mathrm{Zero}(PS_2)+\mathrm{Zero}(PS_3)$$

in which

$$PS_1=\{X_2\hat{}2*X_3-X_1\hat{}4,\ X_2*Y_1-X_1\hat{}2,\ X_1\hat{}2*Y_2-X_1*X_2\},$$
$$PS_2=\{X_1,\ X_2,\ Y_1\hat{}2-X_3,\ Y_2\},$$
$$\mathrm{PS}_3=\{X_1,\ X_2,\ X_3,\ Y_1\}.$$

It follows that

$$\mathrm{Proj\ Zero}(PS)=\mathrm{Zero}(X_2\hat{}2*X_3-X_1\hat{}4/X_1*X_2)+\mathrm{Zero}\ (X_1,\ X_2)$$

is a quasi-variety in $A_3(X)$. However, it is not a variety in the proper sense and cannot be written in an implicit form.

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