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GLOBAL EXISTENCE OF THE SOLUTIONS TO NONLINEAR HYPERBOLIC EQUATIONS IN EXTERIOR DOMAINS

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Abstract

This paper deals with the following IBV problem of nonlinear hyperbolic equations

$$\begin{cases} u_{tt}-\sum_{i,j=1}^{n} a_{ij}(u, Du)u_{x_ix_j}=b(u, Du), & t>0, x\in\Omega, \\ u(0, x)=u^0(x),\ u_t(0, x)=u^1(v), & x\in\Omega, \\ u(t, x)=0 & t>0, x\in\partial\Omega, \end{cases}$$

where Ω is the exterior domain of a compact set in R^n, and $|a_{ij}(y)-\delta_{ij}|=0(|y|^k)$, $|b(y)|=0(|y|^{k+1})$, near $y=0$. It is proved that under suitable assumptions on the smoothness, compatibility conditions and the shape of Ω, the above problem has a unique global smooth solution for small initial data, in the case that $k=1$ add $n\geqslant 7$ or that $k=2$ and $n\geqslant 4$. Moreover, the solution has some decay properties as $t\to+\infty$.

§ 1. Introduction

In this paper, we shall consider the following IBV problem for nonlinear hyperbolic equations

$$u_{tt}-\sum_{i,j=1}^{n} a_{ij}(u, Du)u_{x_ix_j}=b(u, Du),\quad (t, x)\in Q, \tag{1.1}$$

$$u(0, x)=u^0(x),\ u_t(0, x)=u^1(x),\quad x\in\Omega, \tag{1.2}$$

$$u|_\Sigma=0, \tag{1.3}$$

where the domain Ω is an exterior domain of a compact set in R^n with smooth boundary $\partial\Omega$, $Q=(0, +\infty)\times\Omega$, $\Sigma=(0, +\infty)\times\partial\Omega$ and $Du=(u_t, u_{x_1}, \cdots u_{x_n})$.

In this paper we suppose that

$$a_{ij}(y)\ (i, j=1, 2, \cdots n),\ b(y)\in C^\infty(R^{n+2}), \tag{1.4}$$

$$a_{ij}(y)=a_{ji}(y)\ (i, j=1, 2, \cdots n),\ y\in R^{n+2}, \tag{1.5}$$

There exist two positive constants r_0 and $r_1(<1)$, such that if $|y|<r_0$, we have

$$\begin{aligned} &r_1|z|^2\leqslant\sum_{i,j=1}^{n} a_{ij}(y)z_iz_j\leqslant r_1^{-1}|z|^2, \quad \text{for all } z\in R^n, \\ &|a_{ij}(y)-\delta_{ij}|=O(|y|^k),\ i, j=1, 2, \cdots n, \\ &|b(y)|=O(|y|^{k+1}), \end{aligned} \tag{1.6}$$

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where $k\geqslant 1$ is an integer and $\delta_{ij}=1$ when $i=j$ or $=0$ when $i\neq j$.

In this paper, we shall prove the existence and uniqueness of the global solutions to problem (1.1)-(1.3) and the asymptotic behavior of the solution. It plays an important part in studying the scattering of a reflecting object for the nonlinear wave equations.

There have been many papers for the Cauchy problem of equation (1.1). A. Matsumura[7] has proved that when $k=3$ and $n\geqslant 3$ or $k=4$ and $n\geqslant 2$, the Cauchy problem has a unique global solution for small initial data. If a_{ij} ($i, j=1, 2, \cdots n$) and b do not depend on u explicitly, S. Klainerman[5] has found that if $(n-1)/2>(1+1/k)/k$, the Cauchy problem has a unique global solution for small initial data.

For the following nonlinear hyperbolic equation with dissipative term in an exterior domain

$$\begin{cases} u_{tt}-\Delta u+u_t=F(u, \Lambda u), & (t, x)\in Q, \quad (1.7)\\ u(0, x)=u^0(x),\ u_t(0, x)=u^1(x),\ x\in\Omega, & (1.8)\\ u|_{\Sigma}=0, & (1.9)\end{cases}$$

where $\Lambda u=(Du, D^2u)$, $F(z)=F_1(\tilde{z})+F_2(\tilde{\tilde{z}})$, $z=(\tilde{z}, \tilde{\tilde{z}})$, Y. Shibata[10] has proved that if $F_1(z)=O(|z|^3)$, $F_2(\tilde{\tilde{z}})=O(|\tilde{\tilde{z}}|^2)$ near $z=0$ and $n\geqslant 3$, problem (1.7)-(1.9) has a unique global solution for small initial data.

In this paper our aim is to study the case that the nonlinear term contains u and the equation does not involve dissipative term. Our main result is the following

Main Theorem. *Assume that* (1.4)-(1.6) *hold and* Ω *is non-trapping* (*see Remark* 1.1). *Let the space demension* n *satisfy*

$$n\geqslant\begin{cases}7, & k=1,\\ 4, & k\geqslant 2.\end{cases} \tag{1.10}$$

Then, for every integer $m\geqslant 4n+6$, *there exists a positive constant* β, *such that if the initial data* $u^0\in H^m(\Omega)\cap W^{m,r}(\Omega)$ *and* $u^1\in H^{m-1}(\Omega)\cap W^{m-1,r}(\Omega)$ *satisfy*

$$\|u^0\|_{H^m(\Omega)}+\|u^1\|_{H^{m-1}(\Omega)}+\|u^0\|_{W^{m,r}(\Omega)}+\|u^1\|_{W^{m-1,r}(\Omega)}<\beta, \tag{1.11}$$

and the suitable compatibility condition, problem (1.1)-(1.3) *has a unique global solution* $u(t, x)$,

$$u\in\prod_{k=0}^{m-1}C^k([0, +\infty), H^{m-k}(\Omega)\cap H_0^1(\Omega))\cap C^m([0, +\infty), L^2(\Omega)). \tag{1.12}$$

Furthermore, the solution has the decay property

$$\|u(t)\|_{L^\infty(\Omega)}=0(t^{-\bar{a}}),\quad as\ t\to+\infty, \tag{1.13}$$

where

$$r=\begin{cases}1, & k\geqslant 2,\\ 4(n-1)/(3n-2), & k=1,\end{cases}\quad \bar{a}=\begin{cases}(n-1)/2, & k\geqslant 2,\\ (n-2)/4, & k=1.\end{cases} \tag{1.14}$$

Remark 1.1. Ω is said to be non-trapping if for some (any) $R>0$ such that $|x|<R$ on $\partial\Omega$ there exists T_R such that no generalized geodesic[8] of length T_R lies

completely within the ball $B_R=\{x\in\Omega,\ |x|\leqslant R\}$.

Remark 1.2. The compatibility condition means that

$$\partial_t^j u|_{t=0}\in H_0^1(\Omega),\ (j=3,\ 4,\ \cdots m-1). \tag{1.15}$$

Remark 1.3. If $b=b(Du)$, the assumption (1.10) can be replaced by

$$n\geqslant\begin{cases}6, & k=1,\\ 4, & k\geqslant 2.\end{cases} \tag{1.16}$$

Remark 1.4. By a linear change of coordinates, the assumption on quasilinear terms a_{ij} can be replaced by more general form

$$|a_{ij}(y)-a_{ij}(0)|=O(|y|^k).$$

The proof of Main Theorem is divided into two main steps. The first step is to get a local existence theorem. We adopt an appropriate iterative method to overcome the difficulty of 'derivatives loss because the nonlinear term includes second order derivatives. The second step is to establish an a priori estimate. For this purpose, we improve the usual energy estimates and derive an L^p—L^q decay estimate for the local solution, especially, for the case of $k=1$.

§ 2. Local Existence and Uniqueness

In this section, we shall prove the following local existence theorem.

Choose a positive constant $E_0\leqslant\frac{1}{2}$ such that if $f\in H^{[n/2]+3}(\Omega)$ satisfies $\|f\|_H^{[n/2]+3}\leqslant E_0$, we have $\|f\|_{W^{2,\infty}}\leqslant r_0$ (r_0 is defined by (1.6)).

Let $m\geqslant 2[n/2]+3$ be an integer, the solution of problem (1.1)-(1.3) is sought in the space $X(T,\ E)$ for some $E\leqslant E_0$ and $T>0$, where $X(T,\ E)$ is defined by

$$X(T,\ E)=\left\{\begin{array}{l}v\,|\,v\in\prod_{k=0}^{m-1}C^k([0,\ T],\ H^{m-k}(\Omega)\cap H_0^1(\Omega))\cap C^m([0,\ T],\ L^2(\Omega)),\\ D_m(v)\triangleq\max_{t\in[0,T]}\sum_{|\alpha|+|\beta|\leqslant m}\|D_x^\alpha\partial_t^\beta v(t,\ \cdot)\|_{L^2(\Omega)}\leqslant E.\end{array}\right\} \tag{2.1}$$

Theorem 2.1. *Assume that assumptions (1.4)-(1.6) hold, and $u^0\in H^m(\Omega)$ and $u^1\in H^{m-1}(\Omega)$, where $m\geqslant 2[n/2]+3$ is an integer. Then, there exist two positive constants T and b_0 (<1), such that for suitably small E^* $(\leqslant E_0)$, if u^0 and u^1 satisfy*

$$\|u^0\|_{H^m}+\|u^1\|_{H^{m-1}}\leqslant b_0E,\quad E\leqslant E^*, \tag{2.2}$$

and the suitable compatibility condition, problem (1.1)-(1.3) has a unique local solution $u\in X(T,\ E)$ (where T and b_0 depend on E_0 and m, but not on E).

Lemma 2.1. *Let Ω be an exterior domain of a compact set in R^n, and $u\in H_0^1(\Omega)$ be a solution of the elliptic equation*

$$\begin{cases}\Delta u=f, & in\ \Omega,\\ u|_{\partial\Omega}=0.\end{cases} \tag{2.3}$$

Then, for any integer $L\geqslant 0$, if $f\in H^L(\Omega)$, we have $u\in H^{L+2}(\Omega)$ and

$$\|u\|_{H^{L+2}}\leqslant C_0(\|u\|_{L^2}+\|f\|_{H^L}), \tag{2.4}$$

where C_0 is a positive constant depending only on n, L and Ω.

The above lemma is well known (see F. E. Browder [2]).

Now, we consider the following linear problem

$$\begin{cases} u_{tt}-\sum_{i,j=1}^{n} a_{ij}(t, x)u_{x_ix_j}=f(t, x), \quad (t, x)\in Q_T, & (2.5)\\ u(0, x)=u^0(x), \quad u_t(0, x)=u^1(x), & (2.6)\\ u|_{\Sigma_T}=0, & (2.7)\end{cases}$$

where Ω is the same as in the problem (1.1)-(1.3), $Q_T=(0, T)\times\Omega$, $\Sigma_T=(0, T)\times\partial\Omega$. In this problem we suppose that

$$a_{ij}(t, x)=a_{ji}(t, x), \quad i, j=1, 2, \cdots n. \tag{2.8}$$

$$r_2|z|^2\leqslant\sum_{i,j=1}^{n} a_{ij}(t, x)\, z_iz_j\leqslant r_2^{-1}|z|^2 \tag{2.9}$$

for some positive constant $r_2(\leqslant 1)$ and all $z\in R^n$, $t\in[0, T]$, $x\in R^n$.

$$a_{ij}(t, x)\,(i, j=1, 2, \cdots n),\ f(t, x)\in\prod_{k=0}^{m-1} W^{k,\infty}(0, T, H^{m-k-1}(\Omega)), \tag{2.10}$$

where $m\geqslant 2[n/2]+3$ is an integer.

$$\sum_{i,j=1}^{n}|a_{ij}(t, x)-\delta_{ij}|\leqslant(2C_0)^{-1}, \tag{2.11}$$

where $C_0>0$ is defined by Lemma 2.1 (take $L=m$).

Lemma 2.2. *Assume that assumption (2.8)-(2.11) hold and $u^0\in H^m(\Omega)$, $u^1\in H^{m-1}(\Omega)$ satisfy the suitable compatibility condition. Then, problem (2.5)-(2.7) has a unique solution u satisfying*

$$u\in\prod_{k=0}^{m-1} C^k([0, T], H^{m-k}(\Omega)\cap H_0^1(\Omega))\cap C^m([0, T], L^2(\Omega)) \tag{2.12}$$

and the following energy inequality holds in $[0, T]$,

$$\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(t)\|_{L^2}^2\leqslant C(M)\exp(C(M)T)\Big\{\|u^0\|_{H^m}^2+\|u^1\|_{H^{m-1}}^2 + \sum_{|a|+b\leqslant m-2}\|D_x^a\partial_t^b f(0)\|_{L^2}^2+\int_0^T\sum_{|a|+b\leqslant m-1}\|D_x^a\partial_t^b f(s)\|_{L^2}^2 ds\Big\}, \tag{2.13}$$

where

$$M=\sup_{t\in(0,T)}\sum_{i,j=1}^{n}\sum_{|a|+b\leqslant m-1}\|D_x^a\partial_t^b a_{ij}(t, \cdot)\|_{L^2} \tag{2.14}$$

and $C(M)$ is a positive constant depending on M, but not no T.

Proof Lemma 2.2 may be proved in four steps.

The first step, we shall prove that if

$$u\in C([0, T], H^m(\Omega)\cap H_0^1(\Omega))\cap C^{m+1}([0, T], L^2(\Omega))\cap \prod_{k=1}^{m} C^k([0, T], H^{m+1-k}(\Omega)\cap H_0^1(\Omega)) \tag{2.15}$$

is a solution of problem (2.5)-(2.7), then (2.13) holds for u.

In fact, estimating the equality

$$\sum_{k=0}^{m-1}\int_\Omega \partial_t^k(Lu)\partial_t^k u_t\,dx=\sum_{k=0}^{m-1}\int_\Omega \partial_t^k f\partial_t^k u_t dx, \tag{2.16}$$

where $Lu=u_{tt}-\sum_{i,j=0}^{n} a_{ij}(t,x)u_{x_ix_j}$, and noting assumptions (2.8)-(2.10) and

$$\|(a_{ij}\partial_t^k u_{x_ix_j}-\partial_t^k(a_{ij}u_{x_ix_j}))\|_{L^2}\leqslant CM\sum_{|a|+b\leqslant k+1}\|D_x^a\partial_t^b u(t)\|_{L^2},\quad 1\leqslant k\leqslant m-1, \tag{2.17}$$

$$d/dt\cdot\|u(t)\|_{L^2}^2\leqslant\|u(t)\|_{L^2}^2+\|u_t(t)\|_{L^2}^2, \tag{2.18}$$

we have

$$\|u(t)\|_{L^2}^2+\sum_{k=0}^{m-1}\|\partial_t^{k+1}u(t)\|_{L^2}^2+\|\Delta\partial_t^k u(t)\|_{L^2}^2\leqslant C(M)\Big\{\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(0)\|_{L^2}^2$$
$$+\int_0^t\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(t)\|_{L^2}^2+\sum_{k=0}^{m-1}\|\partial_t^k f(t)\|_{L^2}^2dt\Big\}. \tag{2.19}$$

Next, by induction we shall show that for each integer k with $0\leqslant k\leqslant m$, we have

$$\sum_{|a|+b\leqslant k}\|D_x^a\partial_t^b u(t)\|_{L^2}^2\leqslant C(M)\Big\{\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(0)\|_{L^2}^2+\sum_{|a|+b\leqslant m-2}\|D_x^a\partial_t^b f(t)\|_{L^2}^2$$
$$+\int_0^t\sum_{|a|+b\leqslant m-1}\|D_x^a\partial_t^b f(s)\|_{L^2}^2ds+\int_0^t\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(s)\|_{L^2}^2ds\Big\}\triangleq A(t). \tag{2.20}$$

In fact, it follows from (2.19) that (2.20) holds for $k=0$. Now, we show that if (2.20) holds for all integers $k\leqslant m-1$, then (2.20) also holds for $k=m$. Noting the elliptic equation which $\partial_t^{m-J-2}u$ satisfies

$$-\Delta\partial_t^{m-J-2}u=\sum_{i,j=1}^{n}(a_{ij}-\delta_{ij})\partial_t^{m-J-2}u_{x_ix_j}+\partial_t^{m-J-2}f$$
$$+\sum_{i,j=1}^{n}(\partial_t^{m+J-2}(a_{ij}u_{x_ix_j})-a_{ij}\partial_t^{m-J-2}u_{x_ix_j})-\partial_t^{m-J}u, \tag{2.21}$$

for every fixed $t\in[0,T]$ and each integer J with $0\leqslant J\leqslant m-2$, by using Lemma 2.1 for $\partial_t^{m-J-2}u$ and assumption (2.11) we conclude that

$$\|\partial_t^{m-J-2}u(t)\|_{H^{J+2}}^2\leqslant C\{\|\partial_t^{m-J-2}u(t)\|_{L^2}^2+\|\partial_t^{m-J-2}f(t)\|_{H^J}^2$$
$$+\|\partial_t^{m-J}u(t)\|_{H^J}^2+\sum_{|a|+b\leqslant m-1}\|D_x^a\partial_t^b u(t)\|_{L^2}^2\}. \tag{2.22}$$

From (2.19) and (2.20), by induction with respect to J we find

$$\sum_{|a|\leqslant J}\|D_x^a\partial_t^{m-J}u(t)\|_{L^2}^2\leqslant A(t) \tag{3.23}$$

holds for each integer J with $0\leqslant J\leqslant m$. So, (2.20) is valid for $k=m$. Applying Gronwall's inequality for (2.20) and noticing

$$\sum_{|a|+b\leqslant m-2}\|D_x^a\partial_t^b f(t)\|_{L^2}^2\leqslant\int_0^T\sum_{|a|+b\leqslant m-1}\|D_x^a\partial_t^b f(s)\|_{L^2}^2ds$$
$$+\sum_{|a|+b\leqslant m-2}\|D_x^a\partial_t^b f(0)\|_{L^2}^2, \tag{2.24}$$

$$\|\partial_t^k u(0)\|_{H^{m-k}}\leqslant C\{\|u^0\|_{H^m}+\|u^1\|_{H^{m-1}}+\sum_{|a|+b\leqslant m-2}\|D_x^a\partial_t^b f(0)\|_{L^2}\}, \tag{2.25}$$

we obtain (2.13).

The second step, from the above process we know that, using Galerkin's method, there exists a solution u of problem (2.5)-(2.7) satisfying

$$u\in\prod_{k=0}^{m-1}W^{k,\infty}(0,\ T,\ H^{m-k}(\Omega)\cap H_0^1(\Omega))\cap W^{m,\infty}(0,\ T,\ L^2(\Omega))$$
$$\cap\prod_{k=0}^{m-2}C^k([0,\ T],\ H^{m-k-1}(\Omega)\cap H_0^1(\Omega))\cap C^{m-1}([0,\ T],\ L^2(\Omega)). \tag{2.26}$$

The third step, we try to show (2.13) is valid even though u satisfies (2.12), but not (2.15). Set

$$u_\delta=\phi_\delta*u,\ \ f_\delta=\phi_\delta*f, \tag{2.27}$$

where $\phi_\delta*$ denotes the Friedrich's mollifier with respect to t. From (2.5), u satisfies

$$u_{\delta tt}-\sum_{i,j=1}^{n}a_{ij}u_{\delta x_ix_j}=f_\delta+g_\delta,\ \ \delta\leqslant t\leqslant T-\delta, \tag{2.28}$$

where

$$g_\delta=\sum_{i,j=1}^{n}\left(a_{ij}(\phi_\delta*u_{x_ix_j})-\phi_\delta*(a_{ij}u_{x_ix_j})\right). \tag{2.29}$$

We easily see that u_δ satisfies (2.15). Applying (2.13) to u_δ, we get

$$\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u_\delta(t)\|_{L^2}^2\leqslant C(M)\exp^{(C(M)T)}\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u_\delta(\varepsilon)\|_{L^2}^2$$
$$+\sum_{|a|+b\leqslant m-2}\|D_x^a\partial_t^b f_\delta(\varepsilon)\|_{L^2}^2+\sum_{|a|+b\leqslant m-2}\|D_x^a\partial_t^b g_\delta(\varepsilon)\|_{L^2}^2$$
$$+\int_\varepsilon^{T-\varepsilon}\sum_{|a|+b\leqslant m-1}\|D_x^a\partial_t^b f_\delta(t)\|_{L^2}^2ds+\int_\varepsilon^{T-\varepsilon}\sum_{|a|+b\leqslant m-1}\|D_x^a\partial_t^b g_\delta(s)\|_{L^2}^2, \tag{2.30}$$

where $\varepsilon<\delta$ is a small positive constant. Applying Friedrich's lemma and noticing (2.10) and (2.12), we can get (see [3])

$$\int_\varepsilon^{T-\varepsilon}\sum_{|a|+b\leqslant m-1}\|D_x^a\partial_t^b g_\delta(s)\|_{L^2}^2ds\to0,\ \text{ as }\ \delta\to0,$$
$$\sum_{|a|+b\leqslant m-2}\|D_x^a\partial_t^b g_\delta(s)\|_{L^2}^2\to0,\ \text{ as }\ \delta\to0. \tag{2.31}$$

Thus, in (2.30) first let ε be fixed and $\delta\to0$, then let $\varepsilon\to0$, we obtain (2.13).

The last step, we shall prove that the solution u obtained in the second step satisfies (2.12). Consequently, by the third step u also satisfies (2.13). In fact, we extend $a_{ij}(\cdot,\ x)$ $(i,\ j=1,\ 2,\ \cdots n)$ and $f(\cdot,\ x)$ from $[0,\ T]$ to $[-\delta_0,\ T+\delta_0]$ in the same regularity class (these extended functions are denoted by the original notations), where δ_0 is a positive constant. Note that relevant Sobolev norms of extended functions can be bounded by corresponding Sobolev norms of original functions multiplied by a constant independent of $a_{ij}(\cdot,\ x)$, $f(\cdot,\ x)$ and t. So, we can obtain a solution of problem (2.5)-(2.7) satisfying (2.26) on $(-\delta_0,\ T+\delta_0)$. From (2.5) we know that if δ and δ' are sufficiently small, the following equation holds for any compact set in $(-\delta_0,\ T+\delta_0)$,

$$(u_\delta-u_{\delta'})_{tt}-\sum_{i,j=1}^{n}a_{ij}(u_\delta-u_{\delta'})_{x_ix_j}=f_\delta-f_{\delta'}+g_\delta-g_{\delta'}, \tag{2.32}$$

where u_δ, $u_{\delta'}$, f_δ, $f_{\delta'}$ and g_δ, $g_{\delta'}$ are defined by (2.27) and (2.29), respectively. Limiting (2.32) in $[0,\ T]\times\Omega$, similar to (2.13) we have

$$\sum_{|a|+b\leq m}\|D_x^a\partial_t^b(u_\delta-u_{\delta'})(t)\|_{L^2}^2\leq C(M)\exp(C(M)T)$$

$$\Big\{\sum_{|a|+b\leq m}\|D_x^a\partial_t^b(u_\delta-u_{\delta'})(0)\|_{L^2}^2+\sum_{|a|+b\leq m-2}(\|D_x^a\partial_t^b(f_\delta-f_{\delta'})(0)\|_{L^2}^2$$

$$+\|D_x^a\partial_t^b(g_\delta-g_{\delta'})(0)\|_{L^2}^2)+\int_0^T\sum_{|a|+b\leq m-1}(\|D_x^a\partial_t^b(f_\delta-f_{\delta'})(s)\|_{L^2}^2$$

$$+\|D_x^a\partial_t^b(g_\delta-g_{\delta'})(s)\|_{L^2}^2)ds\Big\}. \tag{2.33}$$

Using the analogous estimate in the third step, we can get

$$\sup_{t\in(0,T)}\sum_{|a|+b\leq m}\|D_x^a\partial_t^b(u_\delta-u_{\delta'})(t)\|_{L^2}^2\to 0,\quad \text{as } \delta,\ \delta\to 0. \tag{2.34}$$

So, the solution u satisfies (2.12). The uniqueness follows from (2.13). The proof of Lemma 2.2 is complete.

Proof of Theorem 2.1 Let us construct the local solution of (1.1)-(1.3). For some positive numbers T and E (determined later), we construct the approximate sequence $\{u^{(m)}\}_0^\infty$ as follows:

$$u^{(0)}=0,\quad k=0,$$

$$\begin{cases} u_{tt}^{(k)}-\sum_{i,j=1}^n a_{ij}(u^{(k-1)},\ Du^{(k-1)})u_{x_ix_j}^{(k)}=b(u^{(k-1)},\ Du^{(k-1)}),\ (t,\ x)\in Q_T, \\ u^{(k)}(0,\ x)=u^0(x),\ u_t^{(k)}(0,\ x)=u^1(x), \\ u^{(k)}|_{\Sigma_T}=0,\quad k\geq 1. \end{cases} \tag{2.35}$$

It evidently holds that

$$u^{(0)}\in X(T,\ E). \tag{2.36}$$

Then, for all $k\geq 1$, by Lemma 2.2 we can prove that there exist two positive contsants T and b_0, such that for suitably small E $(\leq E_0)$ if u^0 and u^1 satisfy (2.2) and the suitable compatibily condition, we have

$$u^{(k)}\in X(T,\ E),\quad k\geq 1. \tag{2.37}$$

Next, from (2.35) we know for $k\geq 2$,

$$\begin{cases} (u^{(k)}-u^{(k-1)})_{tt}-\sum_{i,j=1}^n (a_{ij}(u^{(k-1)},\ Du^{(k-1)})(u^{(k)}-u^{(k-1)})_{x_ix_j} \\ \quad =b(u^{(k-1)},\ Du^{(k-1)})-b(u^{(k-2)},\ Du^{(k-2)}) \\ \quad +\sum_{i,j=1}^n (a_{ij}(u^{(k-1)},\ Du^{(k-1)})-a_{ij}(u^{(k-2)},\ Du^{(k-2)}))u_{x_ix_j}^{(k-2)},\ (t,\ x)\in Q_T, \\ (u^{(k)}-u^{(k-1)})(0,\ x)=0,\ (u^{(k)}-u^{(k-1)})_t(0,\ x)=0, \\ (u^{(k)}-u^{(k-1)})|_{\Sigma_T}=0. \end{cases} \tag{2.38}$$

Applying Lemma 2.2 with $m-1$ instead of m and noticing (2.37), we obtain

$$\sum_{|a|+b\leq m-1}\|D_x^a\partial_t^b(u^{(k)}-u^{(k-1)})(t)\|_{L^2}^2$$

$$\leq C_1e^{cT}(1+T)E^4\max_{t\in[0,T]}\sum_{|a|+b\leq m-1}\|D_x^a\partial_t^b(u^{(k-1)}-u^{(k-2)})\|_{L^2}^2, \tag{2.39}$$

where C_1 is a positive constant depending only on E_0. So, if T and E are suitably small, we have

$$\sum_{|a|+b\leqslant m-1}\|D_x^a\partial_t^b(u^{(k)}-u^{(k-1)})(t)\|_{L^2}^2$$

$$\leqslant\frac{1}{2}\max_{t\in[0,T]}\sum_{|a|+b\leqslant m-1}\|D_x^a\partial_t^b(u^{(k-1)}-u^{(k-2)})(t)\|_{L^2}^2. \tag{2.40}$$

Horeover, by using (2.38), we see that there exists a $u(t, x)\in\prod_{k=0}^{m}C^k(0, T, H^{m-k-1})$ such that

$$u^{(k)}\to u,\ \text{strongly in}\ \prod_{k=0}^{m}C^k([0, T], H^{m-k-1}(\Omega)). \tag{2.41}$$

On the other hand, from (2.39), by weak compactness and (2.41), we see that for every fixed $t\in[0, T]$ there exists a subsequence $\{m_j(t)\}$ such that

$$\partial_t^k u^{(m_j(t))}\to\partial_t^k u,\ \text{weakly in}\ H^{m-k}(\Omega),\ 0\leqslant k\leqslant m. \tag{2.42}$$

Thus, we obtbin a solution of (1.1)-(1.3) satisfying

$$u\in\prod_{k=0}^{m-1}W^{k,\infty}(0, T, H^{m-k}(\Omega)\cap H_0^1(\Omega))\cap W^{m,\infty}(0, T, L^2(\Omega)) \tag{2.43}$$

and

$$\sup_{t\in[0,T]}\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(t)\|_{L^2}^2\leqslant E. \tag{2.44}$$

Finally, using the method similar to what used in the last step of proof of Lemma 2.2, we can get $u\in X(T, E)$. The proof of Theorem 2.1 is complete.

§ 3. The Energy Estimate

In this section, we establish a fine energy estimate for the local solutions of (1.1)-(1.3).

Lemma 3.1. *Suppose that* $a_{ij}(y)$ $(i, j=1, 2, \cdots n)$ *and* $b(y)$ *satisfy* (1.4)—(1.6), *and* $u^0\in H^m(\Omega)$, *and* $u^1\in H^{m-1}(\Omega)$ *satisfy the suitable compactibility condition where* $m\geqslant 2[n/2]+3$ *is an integer. Then, there exists a positive constant* $E_1(\leqslant E^*\leqslant E_0$, *where* E^* *is determided by Theorem* 2.1) *such that if* $u\in X(T, E_1)$ *is the solution of* (1.1)-(1.3), *the following energy estimate holds*

$$\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(t)\|_{L^2}\leqslant C_m\Big\{\|u^0\|_{H^m}+\|u^1\|_{H^{m+1}}+\|u(t)\|_{L^2}$$

$$+\int_0^T\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(s)\|_{L^2}(\|u_S(s)\|_{L^\infty}+\|u(s)\|_{L^\infty})ds\Big\}, \tag{3.1}$$

where C_m *is a positive constant independent of* T.

Proof Let $u\in X(T, E)$ $(E\leqslant E_0$ is determined later) be the solution of (1.1)-(1.3). Set

$$u_\delta=\phi_\delta * u, \tag{3.2}$$

where $\phi_\delta *$ is Friedrich's mollifier with respect to t. We easily see

$$u_\delta\in\prod_{k=0}^{m+1}C^k([0, T], H^{m+1-k}(\Omega)\cap X(T, E) \tag{3.3}$$

and u_δ satisfies

$$u_{\delta tt}-\sum_{i,j=1}^{n} a_{ij}(u,\ Du)u_{\delta x_i x_j}=h_\delta+R_\delta+b(u_\delta,\ Du_\delta),\quad \delta\leqslant t\leqslant T-\delta, \tag{3.4}$$

where

$$h_\delta=\phi_\delta * b(u,\ Du)-b(u_\delta,\ Du_\delta), \tag{3.5}$$

$$R_\delta=\sum_{i,j=1}^{n}(\phi_\delta * a_{ij}(u,\ Du)u_{x_i x_j}-a_{ij}(u,\ Du)u_{\delta x_i x_j}). \tag{3.6}$$

Using the usual energy integration method and noticing (1.4)-(1.6), $u\in X(T,\ E)$ and $m\geqslant 2[n/2]+3$, we have

$$\sum_{k=1}^{m}(\|\partial_t^k u_\delta(t)\|_{L^2}+\|\partial_t^{k-1}\nabla u_\delta(t)\|_{L^2})\leqslant C\Big\{\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u_\delta(\varepsilon)\|_{L^2}$$
$$+\int_\varepsilon^T\sum_{|a|+b\leqslant r}\|D_x^a\partial_t^b u_\delta(s)\|_{L^2}(\|u_\delta(s)\|_{L^\infty}+\|u_{\delta_s}(s)\|_{L^\infty})ds$$
$$+\int_\varepsilon^T\sum_{k=0}^{m}\|\partial_t^k h_\delta(s)\|_{L^2}+\|\partial_t^k g_\delta(s)\|_{L^2}ds\Big\},\quad \varepsilon\leqslant t\leqslant T-\varepsilon, \tag{3.7}$$

where $\varepsilon<\delta$ is a small positive constant. Through a reasoning process similar to what used in section 2, first let ε be fixed and $\delta\to 0$, then let $\varepsilon\to 0$ and notice

$$\|\partial_t^k u(0)\|_{H^{m-k}}\leqslant C(\|u^0\|_{H^m}+\|u^1\|_{H^{m-1}}),\ 0\leqslant k\leqslant m, \tag{3.8}$$

we obtain

$$\sum_{k=1}^{m}\|\partial_t^k u(t)\|_{L^2}+\sum_{k=1}^{m-1}\|\nabla\partial_t^k u(t)\|_{L^2}\leqslant C\Big\{\|u^0\|_{H^m}+\|u^1\|_{H^{m-1}}$$
$$+\int_0^T(\|u_s(s)\|_{L^\infty}+\|u(s)\|_{L^\infty})\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(s)\|_{L^2}\ ds\Big\}. \tag{3.9}$$

Next, by using a method similar to what used in section 2, e. g., by applying the regularity results of elliptic operator and induction, we get

$$\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(t)\|_{L^2}\leqslant C_0\Big\{\|u^0\|_{H^m}+\|u^1\|_{H^{m-1}}+\|u(t)\|_{L^2}$$
$$+E\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(t)\|_{L^2}+\int_0^T\sum_{|a|+b\leqslant m}\|D_x^a\partial_t^b u(s)\|_{L^2}$$
$$(\|u(s)\|_{L^\infty}+\|u_s(s)\|_{L^\infty})ds\Big\}, \tag{3.10}$$

where C_0 is a positive constant independent of T.

Thus, taking $E_1=\min\ (E^*,\ 1/(2D_0))$, we get (3.1). This completes the proof of Lemma 3.1.

§ 4. The Decay Estimate

In this section we shall discuss the time decay estimate of the solution for the following problem

$$\begin{cases} u_{tt}-\Delta u=f, \quad (t,x)\in Q, & (4.1)\\ u(0,x)=u^0(x), \quad u_t(0,x)=u^1(x), & (4.2)\\ u|_\Sigma=0, & (4.3)\end{cases}$$

where Ω is the same as in problem (1.1)-(1.3), u^0, u^1 and f are sufficiently smooth and satisfy the suitable compatibility condition.

For the sake of convenience, we introduce some notations. Let $G=R^n$ or $G\subset R^n$ be an open set, $g(t,x)$ and $h(x)$ be functions defined on $R^+\times G$ and G, respectively. For all integers $L\geqslant 0$, positive numbers $p\geqslant 1$ and $k\geqslant 0$, we denote

$$[g, k, p, L, G](t)=\sup_{s\in(0,t)}(1+s)^k\sum_{|a|+b\leqslant L}\|D_x^a\partial_t^b g(s)\|_{L^p(G)}, \tag{4.4}$$

$$B(h, p, L)=\sum_{|a|\leqslant L}\|D_x^a h\|_{L^p(\Omega)}, \tag{4.5}$$

$$B^*(h, p, L)=\sum_{|a|\leqslant L}\|D_x^a h\|_{L^p(R^n)}, \tag{4.6}$$

and

$$\Omega_r=\{x|x\in\Omega, |x|\leqslant r\}. \tag{4.7}$$

Lemma 4.1. *Let Ω be non-trapping and $p>2$ be a number satisfying $(n-1)(1-2/p)/2>1$. Then, for each integer $L\geqslant 0$, the solution of (4.1)-(4.3) satisfies the estimates*

$$[u, (n-1)(1-2/p)/2, p, L, \Omega](t)\leqslant C\{B(u^0, q, L+2n+1) + B(u^1, q, L+2n)+[f, (n-1)(1-2/p)/2, q, L+2n, \Omega](t)\}, \tag{4.8}$$

$$[u, 0, 2, 0, \Omega](t)\leqslant C\{B(u^0, q, 2n+1)+B(u^1, q, 2n) + [f, (n-1)(1-2/p)/2, q, 2n, \Omega](t)\}, \tag{4.9}$$

where

$$1/q=2(1/2+1/n)/p+1-2/p. \tag{4.10}$$

In order to prove Lemma 4.1, we first state and prove several lemmas.

Lemma 4.2. *Suppose that $p>2$ is a number satisfying*

$$(n-1)(1-2/p)/2>1 \tag{4.11}$$

and u is the solution of the following Cauchy problem

$$\begin{cases} u_{tt}-\Delta u=f, \quad (t,x)\in R^+\times R^n, & (4.12)\\ u(0,x)=u^0(x), \quad u(0,x)=u^1(x), \quad x\in R^n. & (4.13)\end{cases}$$

Then, for each integer $L\geqslant 0$, we have

$$[u, (n-1)(1-2/p)/2, p, L, R^n](t) \leqslant C\{B^*(u^0, q, L+n)+B^*(u^1, q, L+n-1) + [f, (n-1)(1-2/p)/2, q, L+N-1, R^n](t)\}, \tag{4.14}$$

$$[u, 0, 2, 0, R^n](t)\leqslant C\{B^*(u^0, 2, 0)+B^*(u^1, a, 0) + [f, (n-1)(1-2/p)/2, a, 0, R^n](t)\}, \tag{4.15}$$

where

$$1/a=1/2+1/n \quad 1/q=2(1/2+1/n)/p+1-2/p. \tag{4.16}$$

Proof From S. Klainerman's[5] and W. A. Strauss's[6] results, we know if

$u^0=0$ and $f=0$ in (4.12), then

$$[u,\ (n-1)/2,\ \infty,\ L,\ R^n](t)\leqslant CB^*(u^1,\ 1,\ L+n-1), \tag{4.17}$$

$$[u,\ 0,\ 2,\ 0,\ R^n](t)\leqslant CB^*(u^1,\ a,\ 0). \tag{4.18}$$

By using Rietz-Thorin interpolation theorem[1] and noticing (4.11), we get (if $u^0=0$ and $f=0$)

$$[u,\ (n-1)(1-2/p)/2,\ p,\ L,\ R^n](t)\leqslant CB^*(u^1,\ q,\ L+n-1). \tag{4.19}$$

Applying this result and (4.18) to problem (4.12)-(4.13), we obtain (4.14) and (4.15) immediately.

Lemma 4.3. *Let $n\geqslant 3$ and $u(t,\ x)$ be the solution of the following IBV problem in an exterior domain*

$$\begin{cases} u_{tt}-\Delta u=0,\quad (t,\ x)\in Q, & (4.20)\\ u(0,\ x)=u^0(x),\ u_t(0,\ x)=u^1(x),\ x\in\Omega, & (4.21)\\ u|_\Sigma=0, & (4.22)\end{cases}$$

where Ω is the same as in problem (1.1)-(1.3) and is not-trapping. We assume that there exists a positive constant R such that

$$\operatorname{supp} u^0,\ \operatorname{supp} u^1\subset\Omega_R. \tag{4.23}$$

Then, there exists a positive constant C depending only on n, R and Ω, such that the following local energy decay estimate holds

$$[u,\ n/2,\ 2,\ 1,\ \Omega_R](t)\leqslant C\{B(u^0,\ 2,\ 1)+B(u^1,\ 2,\ 0)\}. \tag{4.24}$$

Proof When n is odd, refer to [9] for the proof. When n is even, refer to [8] for the proof. For the estimate of $\|u(t)\|_{L^2(\Omega)}$ we can use Poicaré's inequality.

Lemma 4.4. *Let $\Omega\in R^n$ be an exterior domain of a compact set with smooth boundary $\partial\Omega$, and $u(x)\in H^1(\Omega)$ be a solution of*

$$\begin{cases} \Delta u=f,\quad x\in\Omega,\\ u_{\partial\Omega}=0.\end{cases} \tag{4.25}$$

Let r_3 and r_4 be two positive constants satisfying $r_4<r_3$ and $\partial\Omega\subset\{x\in R^n,\ |x|<r_4\}$. Then, for each integer $L\geqslant 0$, if $f\in H^L(\Omega)$, we have $u\in H^{L+2}(\Omega)$, and

$$\|u\|_{H^{L+2}(\Omega_4)}\leqslant C(\|f\|_{H^L(\Omega_{r_3})}+\|u\|_{L^2(\Omega_3)}), \tag{4.26}$$

with C depending on L, n, r_3, r_4 and Ω, where $\Omega_4=\Omega_{r_4}$, $\Omega_3=\Omega_{r_3}$.

For the proof of this lemma, refer to [4].

Now, we prove Lemma 4.1. The method of proof follows [11].

Proof of Lemma 4.1 We extend u^0, u^1 and $f(t,\ \cdot)$ from Ω to R^n in the same regularity class. Note that relevant Sobolev norms of extensions of u^0, u^1 and $f(t,\ \cdot)$ can be bounded by corresponding Sobolev norms multiplied by a constant independent of u^0, u^1, $f(t,\ \cdot)$ and t. We also denote these extended functions by original notations.

Let u_1 be the solution of

$$\begin{cases} u_{tt}-\Delta u=f, & (t, x)\in R^+\times R^n, \\ u(0, x)=u^0(x), & u_t(0, x)=u^1(x). \end{cases} \tag{4.27}$$

Then, we fix a constant $d>0$ such that $\partial\Omega\subset\{x\in R^n, |x|<d\}$. Choose two functions $v(x)$ and $w(x)$ satisfying $v(x), w(x)\in C^\infty(R^n)$ and

$$v(x)=\begin{cases} 1 & |x|>d+2, \\ 0 & |x|>d+1, \end{cases} \quad w(x)=\begin{cases} 1 & |x|>d+3/4, \\ 0 & |x|<d+1/4. \end{cases} \tag{4.28}$$

Define

$$u_2(t, x)=v(x)u_1(t, x), \tag{4.29}$$

$$u_3(t, x)=u(t, x)-u_2(t, x), \tag{4.30}$$

$$u_4(t, x)=w(x)u_3(t, x). \tag{4.31}$$

From Lemma 4.2 and the properties of extension we get

$$[u_1, (n-1)(1-2/p)/2, p, L, R^n](t)\leqslant A(L+n)(t) \tag{4.32}$$

and

$$[u_2, (n-1)(1-2/p)/2, p, L, R^n](t)\leqslant A(L+n)(t), \tag{4.33}$$

with $A(L+n)(t)$ defined by

$$A(L+n)(t)=C\{B(u^0, q, L+u)+B(u^1, q, L+n-1) \\ +[f, (n-1)(1-2/p)/2, q, L+n-1, \Omega](t)\}. \tag{4.34}$$

From (4.30), we easily see that u_3 satisfies

$$\begin{cases} u_{tt}-\Delta u=(1-v)f+g, \ (t, x)\in Q, & (4.35) \\ u(0, x)=(1-v)u^0, \ u_t(0, x)=(1-v)u^1, \ x\in\Omega, & (4.36) \\ u|_\Sigma=0, & (4.37) \end{cases}$$

where

$$g(t, x)=u_1\Delta v+2\nabla u_1\nabla v. \tag{4.38}$$

From (4.32) and (4.38), it is easy to verify

$$\operatorname{supp} g\subset[0, +\infty)\times\{x\in R^n, d+1<|x|<d+2\}, \tag{4.39}$$

$$[g, (n-1)(1-2/p)/2, q, L, \Omega](t)\leqslant A(L+n+1)(t). \tag{4.40}$$

Let

$$h=(1-v)f+g. \tag{4.41}$$

For each integer $k>0$, we have

$$\begin{cases} \partial_t^{k+2}u_3-\Delta\partial_t^k u_3=\partial_t^k h, \ (t, x)\in Q, & (4.42) \\ \partial_t^k u_3(0, x)=w_k, \ \partial_t^{k+1}u_3(0, x)=w_{k+1}, & (4.43) \\ \partial_t^k a_3|_\Sigma=0, \end{cases}$$

where

$$w_k=\begin{cases} \Delta^s u^0+\sum_{j=0}^{s-1}\partial_t^{2j}\Delta^{s-j-1}h(0, x), \ k=2s, \\ \Delta^s u^1+\sum_{j=0}^{s-1}\partial_t^{2j+1}\Delta^{s-j-1}h(0, x), \ k=2s+1, \end{cases}$$

$$w_{k+1}=\begin{cases}\Delta^s u^1+\sum_{j=0}^{s-1}\partial_t^{2j+1}\Delta^{s-j-1}h(0,x), & k=2s,\\ \Delta^{s+1}u^0+\sum_{j=0}^{s}\partial_t^{2j}\Delta^{s-j}h(0,x), & k=2s+1.\end{cases} \tag{4.45}$$

By Lemma 4.3 and (4.40), we obtain

$$\left[\partial_t^k u_3, \frac{1}{2}(n-1)(1-2/p),\ 2,\ 1,\ \Omega_{d+2}\right](t)\leqslant A(k+n+1)(t). \tag{4.46}$$

Now, we assert that, for each integer $L\geqslant 0$, the following estimate is valid

$$\sum_{|\alpha|\leqslant J}\|D_x^\alpha\partial_t^{L-J}u_3(t)\|_{L^2(\Omega_{d+2-J/L})}\leqslant C(1+t)^{(n-1)(1-2/p)/2}A(L+n+1)(t). \tag{4.47}$$

In fact, it follows from (4.6) that (4.47) holds for $J=0$ and $J=1$. Now, we suppose (4.47) holds for all integers J with $0\leqslant J\leqslant L-1$. We want to prove (4.47) also holds with J replaced by $J+1$. It is obvious tha $\partial_t^{L-J-1}u_3(t,x)$ satisfies

$$\begin{cases}\Delta\partial_t^{L-J-1}u_3=\partial_t^{L-J+1}u_3-\partial_t^{L-J-1}h, & x\in\Omega,\\ (\partial_t^{L-J-1}u_3)|_{\partial\Omega}=0.\end{cases} \tag{4.48}$$

From Lemma 4.4 and (4.46)-(4.47), we get

$$\begin{aligned}\sum_{|\alpha|\leqslant J+1}\|D_x^\alpha\partial_t^{L-J-1}u_3(t)\|_{L^2(\Omega_{d+2-(J+1)/L})}&\leqslant C\{\|\partial_t^{L-J-1}u_3(t)\|_{L^2(\Omega_{d+2-J/L})}\\&+\sum_{|\alpha|\leqslant J-1}\|D_x^\alpha\partial_t^{L-J+1}u_3(t)\|_{L^2(\Omega_{d+2-(J+1)/L})}+\sum_{|\alpha|\leqslant J}\|D_x^\alpha\partial_t^{L-J-1}h(t)\|_{L^2(\Omega_{d+2-J/L})}\\&\leqslant C(1+t)^{-(n-1)(1-2/p)/2}A(L+n+1)(t).\end{aligned} \tag{4.49}$$

Thereby, an induction argument gives

$$[u_3,\ (n-1)(1-2/p)/2,\ 2,\ L,\ \Omega_{d+1}](t)\leqslant A(L+n+1)(t). \tag{4.50}$$

By Sobolev imbedding theorem, (4.50) implies

$$[u_3,\ (n-1)(1-2/p)/2,\ p,\ L,\ \Omega_{d+1}](t)\leqslant A(L+n+[n/2]+2)(t). \tag{4.51}$$

Finally, we shall evaluate $u_4(t,x)$. It implies the estimate of $u_3(t,x)$ on $|x|>d+1$. Because $u_4(t,x)$ satisfies

$$\begin{cases}u_{tt}-\Delta u=K(t,x), & (t,x)\in R^+\times R^n,\\ u(0,x)=u^0w(1-v),\ u_t(0,x)=u^1w(1-v), & x\in\Omega,\end{cases} \tag{4.52}$$

with $K(t,x)$ defined by

$$K(t,x)=-u_3\Delta w-2\nabla u_3\nabla w+wh, \tag{4.53}$$

from Lemma 4.2 and noticing the fact that supp Δw and supp ∇w $\{x\in R^n, |x|<d+1\}$, we know

$$\begin{aligned}&[u_4,\ (n-1)(1-2/p)/2,\ p,\ L+1,\ \Omega](t)\\&\leqslant C[u_4,\ (n-1)(1-2/p)/2,\ p,\ L+1,\ R^n](t)\leqslant CA(L+n+2)(t).\end{aligned} \tag{4.54}$$

From (4.51) and (4.54), we have

$$[u_3,\ (n-1)(1-2/p)/2,\ p,\ L,\ \Omega](t)\leqslant A(L+n+[n/2]+2)(t). \tag{4.55}$$

From (4.38) and (4.55), we get (4.8) immediately.

For (4.9), by noticing $q<\alpha$ and using Lemma 4.2, we have

$$\begin{aligned}[u_2,\ 0,\ 2,\ 0,\ \Omega](t)&\leqslant[u_1,\ 0,\ 2,\ 0,\ R^n](t)\leqslant C\{B(u^0,\ 2,\ 0)+B(u^1,\ \alpha,\ 0)\\&+[f,\ (n-1)(1-2/p)/2,\ \alpha,\ 0,\ \Omega](t)\}\leqslant A([n/2]n+1).\end{aligned} \tag{4.56}$$

Moreover, we have

$$[u_3,\ 0,\ 2,\ 0,\ \Omega](t) \leqslant [u_3,\ 0,\ 2,\ 0,\ \Omega](t) + [u_4,\ 0,\ 2,\ 0,\ R^n](t) \leqslant A(2n+1)(t). \tag{4.57}$$

In (4.56), we have used Sobolev imbedding theorem. Combining (4.56) with (4.57), we get (4.9). Thus, the proof of Lemma 4.1 is complete.

§ 5. A Priori Estimate and the Proof of Main Theorem

Lemma 5.1. *In problem (1.1)-(1.3) we assume that n satisfies (1.10), the domain Ω is non-trapping, $a_{ij}(u, Du)$ $(i, j=1, 2, \cdots n)$ and $b(u, Du)$ satisfy (1.4)-(1.6), and $u^0 \in H^m(\Omega) \cap W^{m,r}(\Omega)$ and $u^1 \in H^{m-1}(\Omega) \cap W^{m-1,r}(\Omega)$ satisfy the suitable compatibility condition, where $m \geqslant 4n+6$ is an integer and r is determined by (1.14). Then, there exist two positive constants E_2 $(E_2 \leqslant E_1 \leqslant E^* \leqslant E_0$, where E_1 is defined in Lemma 3.1) and C'_m, such that if $u \in X(T, E_2)$ is the solution of (1.1)-(1.3) satisfying (3.1), we have*

$$\sum_{|a|+b<m} \|D_x^a \partial_t^b u(t)\|_{L^2} \leqslant C'_m\{B(u^0, r, m) + B(u^0, r, m-1) + B(u^0, 2, m) + B(u^1, 2, m-1)\}. \tag{5.1}$$

Proof We only prove Lemma 5.1 for $k=1$ which is an important case of nonlinear terms and critical in our proof. The proof of case $k \geqslant 2$ is similar to $k=1$. We consider the following instead of (1.1)

$$u_{tt} - \Delta u = F(u, Du, D_x^2 u), \tag{5.2}$$

where

$$E(u, Du, D_x^2 u) = \sum_{i,j=1}^{n} (a_{ij}(u, Du) - \delta_{ij}) u_{x_i x_j} + b(u, Du). \tag{5.3}$$

From (1.6) and $u \in X(T, E_2)$ $(E_2 \leqslant E_0)$, we know

$$F(u, Du, D_x^2 u) = O((|u| + |Du| + |D_x^2 u|)^2), \tag{5.4}$$

and

$$[F,\ (n-1)(1-2/p)/2,\ r,\ 4n+4,\ \Omega](t) \leqslant C[u,\ (n-1)(1-2/p)/2,\ p,\ 2n+4,\ \Omega](t)[u,\ 0,\ 2,\ 4n+6,\ \Omega](t), \tag{5.5}$$

where

$$p = 4(1-1/n),\ r = (4n-4)/(3n-2). \tag{5.6}$$

Set

$$M(t) = [u,\ (n-1)(1-2/p)/2,\ p,\ 2n+4,\ \Omega](t) \quad (= [u,\ (n-2)/4,\ 4(1-1/n),\ 2n+4,\ \Omega](t)). \tag{5.7}$$

By Lemma 4.1 $(q=r=(4n-4)/(3n-2),\ p=4(1-1/n))$ and (5.5), we have

$$M(t) \leqslant C\{B(u^0, r, 4n+5) + B(u^1, r, 4n+4) + EM(t)\}. \tag{5.8}$$

Taking $E=E_2$ sufficiently small such that if $u \in X(T, E_2)$, we have

$$M(t) \leqslant C_m\{B(u^0, r, m) + B(u^1, r, m-1)\}, \tag{5.9}$$

where C_m is independent of t and E_2 is dependent on m.

By Sobolev imbedding theorem, we get

$$\|u_t(t)\|_{L^\infty}+\|u(t)\|_{L^\infty}\leqslant C(1+t)^{-(n-2)/4}(B(u^0, r, m)+B(u^1, r, m-1)). \quad (5.10)$$

Similarly, from (4.9), (5.5) and (5.9), we get

$$[u, 0, 2, 0, \Omega](t)\leqslant C(B(u^0, r, m)+B(u^1, r, m-1)). \quad (5.11)$$

From (3.1), (5.10) and (5.11), by using Gronwall's inequality and noticing $(n-2)/4>1$ if $n\geqslant 7$ (see (1.10)), we obtain (5.1). The proof of Lemma 5.1 is complete.

Proof of Main Theorem Take

$$\beta=\min(b_0E_2, b_0E_2/C'_m), \quad (5.12)$$

where E_2 and C'_m are determined by Lemma 5.1 and b_0 is by Theorem 2.1. By applying Theorem 2.1 and Lemma 5.1 repeatedly, we find that if u^0 and u^1 satisfy (1.11), problem (1.1)-(1.3) has a unique global solution $u(t, x)$, $u\in X(\infty, E)$ with $E\leqslant E_2$. The decay estimate (1.13) follows from (5.10). The proof of Main Theorem is complete.

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