Chin. Ann. of Math.
11B:3(1990), 330—341.

THE REGULARITY OF SOLUTIONS FOR NONLINEAR DEGENERATE ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract

This paper is devoted to the study of regularity of solutions for Dirichlet probelm of nonlinear degenerate elliptic equations in two dimensional case. A sufficient condition for their solutions in $C^{3+\alpha}(\bar{\Omega})$ to be certainly in $C^\infty(\bar{\Omega})$ is given.

§ 1. Introduction

Recently, the study of regularity of solutions for nonlinear equations has attracted much attention of many mathematicians. There are two ways to deal with such problems. One is a direct way which, as Nirenberg in [5, 1] carried out for elliptic equations, is based on the schauder's estimates for linear equation. Another one is micro-local analysis, which, as done in [2, 6], is based on the techniques of the harmonic analysis and pseudodifferential operators. It seems that the former is more suitable for elliptic equations and the latter is more powerful for hyperbolic equations. The present paper is devoted to the reqularity of solutions to boundary value problems of degenerated elliptic nonlinear equations. There are several papers concerning such problems, [7, 8] for the interior regularity of solutions to nonlinear operators satisfying Hörmander's square-sum condition, and [3] for the regularity up to the boundary.

Our result is as follows. Let $F(x, u, u_i, u_{ij})=F(x, \partial^\alpha u)$ be a smooth function of its arguments. Denote $u_i=\partial u/\partial x_i$, $u_{ij}=\partial^2 u/\partial x_i \partial x_j$ and $F_i=\partial F/\partial u_i$, $F_{ij}=\partial F/\partial u_{ij}$. Suppose that $\Omega \subset R^2$ is a smooth bounded domain.

Theorem. *Let $u(x) \in C^{3+\alpha}(\bar{\Omega})$ with $\alpha>0$ be a solution of Problem:*

$$F(x, \partial^\alpha u(x))=0 \text{ in } \Omega \tag{1.1}$$

with

$$u(x)=g(x) \text{ on } \partial\Omega. \tag{1.2}$$

Assume that F is elliptic in Ω for the solution $u(x)$ and

$$\det\,(F_{ij}(x, \partial^\alpha u(x)))=0 \text{ and } d(\det\,(F_{ij}(x, \partial^\alpha u(x))))\neq 0, \text{ on } \partial\Omega, \tag{1.3}$$

Manuscript received September 10, 1987.

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and that $\partial\Omega$ is noncharacteristics for the linearized operator of F in $u(x)$. Then $u\in C^\infty(\bar{\Omega})$ if g is smooth. Moreover, the result is still true if $u(x)\in C^{2+\alpha}(\bar{\Omega})$ with $\alpha>0$ and (1.1) is a quasilinear equation.

§ 2. Several Lemmas

According to[2], we shall use the standard decomposition of spectrum on R^1. Set

$$C_j=\{\xi\in R^1|2^jk^{-1}<|\xi|<2^{j+1}k\}\quad (j=0,\ \cdots),$$

where k is a constant>1. There are two functions $\psi(|\xi|),\ \varphi(|\xi|)\in C_c^\infty(R^1)$ satisying

$$\text{supp}\ \psi\subset(-1,\ 1)\ \text{and supp}\ \varphi\subset C_0 \tag{2.1}$$

and

$$\psi(|\xi|)+\sum_{j=0}^{\infty}\varphi(2^{-j}|\xi|)=1 \text{ for all } \xi\in R^1. \tag{2.2}$$

Define, for any $u\in C_c(R^1)$,

$$u_j=\varphi(2^{-j}D)u=F^{-1}(\varphi(2^{-j}|\xi|)\hat{u}(\xi)). \tag{2.3}$$

We shall list the well known properties for u without proof. For details see [1].

Proposition 2.1. *If $u\in C_c^\alpha(R^1)$, $\alpha\in R^1_+\backslash Z$, then*

$$|u_j|\leqslant C2^{-j\alpha}, \tag{2.4}$$

where constant C is independent of j. Conversely, if (2.4) is valid for any j, then $u\in C^\alpha$ and $|u|_\alpha\leqslant C_1C$ for another constant C_1. Some behavior of the Airy function $A_i(s)=\lambda(s)$, which is a solution of

$$\lambda''(s)-s\lambda(s)=0,\ s>0 \text{ and } \lambda(0)=1, \tag{2.5}$$

should be mentioned for the needs of later discussion. The integral expression in [4]

$$\lambda(s)=Cs^{1/2}\int_0^\infty\exp(-2s^{3/2}\cosh t/3)\cosh\ (t/3)\ dt,\ t>0$$
$$=s^{1/2}K_{1/3}(2s^{3/2}/3) \tag{2.6}$$

is very useful. From (2.6) and a slight computation, it follows that

$$0<\lambda(s)\leqslant C_n(1+s)^{-n},\ \text{for any } n\in Z^+ \tag{2.7}$$

and

$$(\lambda(as)/\lambda(bs))\leqslant(a/b)^{1/2}\exp\left(-\frac{2}{3}(a^{3/2}-b^{3/2})s^{3/2}\right)\ \text{if } a\geqslant b>0. \tag{2.8}$$

Lemma 2.2. *Let a positive function $f(t)\in C^2(R^1_+)$ approach to zero as $t\to+\infty$ and let $f''(t)>0$ on the zero points of $f'(t)$. Then $f'(t)<0$ for all $t>0$.*

Proof If the assertion is not true, it follows that $f'(t^*)=0$ for some $t^*>0$. By the assumption of this lemma, we have $f(t)>f(t^*)>0$ if t is in some interval $(t^*,\ t^*+\delta)$. In view of the fact that $f(t)\to0$ as $t\to+\infty$, one can conclude that $f(t)$ attains its maxima over the interval $(t^*,\ +\infty)$ at some point t^{**} where $f'(t^{**})=0$ and $f''(t^{**})\leqslant0$. This contradicts the assumption of the present lemma. The proof is completed.

Using a similar argument, we also have

Corollary 2.2 *Let a positive function $f(t)\in C^2(R^1_+)$ approach to $+\infty$ as $t\to+\infty$ and let $f''(t)<0$ if $f'(t)=0$. Then $f'(t)>0$ if $t>0$.*

Lemma 2.3. *The following functions:*

(1) $\lambda(s)$,

(2) $-\lambda'(s)$,

(3) $\lambda(as)/\lambda(bs)$ *with* $a\geqslant b>0$,

(4) $\lambda'(s)/\lambda(s)$

are monotone-decreasing for all $s\in R^1_+$.

Proof We only need to prove that the functions mentioned in this lemma satisfy the conditions of Lemma 2.1. From (2.7) and (2.5) it follows that $\lambda''(s)$ is always positive and $\lambda(s)\to 0$ as $s\to+\infty$. So $\lambda'(s)$ is monotone. An application of Lemma 2.2 to $\lambda(s)$ gives $\lambda'<0$. (2.8) implies $h=\lambda(as)/\lambda(bs)$ approaches to zero as $s\to+\infty$ if $a>b>0$. Assume, now, that

$$h_s=h[a\lambda'(as)/\lambda(as)-b\lambda'(bs)/\lambda(bs)]=0 \text{ at some point } s. \tag{2.9}$$

Therefore, at this point

$$h_{ss}=hs(a^3-b^3)>0. \tag{2.10}$$

In getting (2.10), we have used (2.5). This proves the monotonicity of h. Let us consider the last one. From (2.6) it follows that

$$-\lambda'(s)/\lambda(s)\geqslant s^{1/2}-\frac{1}{2}s^{-1/2}, \tag{2.11}$$

which means $-\lambda'(s)/\lambda(s)\to+\infty$, if $s\to+\infty$. Assume, similarly, that

$$(-\lambda'(s)/\lambda(s))'=-s+(\lambda'(s)/\lambda(s))^2=0 \text{ at some points.}$$

Then at these points

$$(-\lambda'(s)/\lambda(s))''=-1<0.$$

Using Corollary 2.2 we can obtain the assertion expected. This proves the present lemma.

Lemma 2.4 *There exists a positive constant ρ_0 such that $s^{-\rho_0}((\lambda(as)\lambda(bs)/\lambda^2(cs))'$ is negative and monotone-increasing for all $s\in R^1_+$ if a and b are bigger then c. Here ρ_0 is independent of a, b and c.*

Proof Setting $h=\lambda(as)\lambda(bs)/\lambda^2(cs)$, we have

$$h_s=h[(a\lambda'(as)/\lambda(as))+(b\lambda'(bs)/\lambda(bs))-2(c\lambda'(cs)/\lambda(cs))]=hw$$

and

$$(s^{-\rho_0}h_s)'=hs^{-(\rho_0+2)}[s^2w^2+H(as)+H(bs)-2H(cs)], \tag{2.12}$$

where

$$H(\sigma)=\sigma^3-\sigma^2(\lambda^1(\sigma)/\lambda(\sigma))^2-\rho_0\sigma\lambda'(\sigma)/\lambda(\sigma).$$

The remainder of the proof is to prove that there exists a constant ρ_0 such that $H(\sigma)$ is monotone-increasing in R^1_+. Indeed, with $g=-\lambda'(\sigma)/\lambda(\sigma)\sigma^{1/2}$, we find

$$H'(\sigma)=\sigma^2[3-2g^2+2\sigma^{3/2}g(1-g^2)+\rho_0(g\sigma^{3/2})'\sigma^{-2}]. \tag{2.13}$$

Asymptotic behavior of $\lambda'(\sigma)$ and $\lambda(\sigma)$ should be studied. Let us first consider the integral with $\zeta>0$,

$$\int_0^\infty \exp(-2\zeta\cosh t)\cosh(t/3)\,dt=\frac{1}{2}\int_0^\infty \exp(-\zeta(u+u^{-1}))u^{-2/3}du$$
$$=\frac{1}{2}\exp(-2\zeta)\int_0^\infty \exp(-\zeta(u+u^{-1}-2))u^{-2/3}du.$$

By means of the method of stational phase we can get its asymptotic expansion for big ζ of the form

$$=\frac{1}{2}\exp(-2\zeta)\sqrt{(\pi/\zeta)}\,[1+(u^{-2/3}u_w)''/4\zeta|_{w=0}+O(\zeta^{-2})] \tag{2.14}$$

where $w=(u-1)/\sqrt{u}$. Similarly,

$$\int_0^\infty \exp(-2\zeta\cosh t)\cosh(t/3)\cosh t\,dt$$
$$=\frac{1}{2}\exp(-2\zeta)\sqrt{(\pi/\zeta)}\,\{1+[u^{-2/3}u_w(w^2+2)]''/8\zeta|_{w=0}|+O(\zeta^{-2})\}. \tag{2.15}$$

Combining (2.14) (2.15) and (2.6) we have

$$g(s)=(1+4^{-1}s^{-3/2}+O(s^{-3}). \tag{2.16}$$

Inserting (2.16) into (2.13) yields

$$H'(\sigma)\geqslant\sigma^{1/2}\Big[\Big(\frac{3}{2}\rho_0-C\Big)+O(\sigma^{-2})\Big]$$

for some constant C. This implies that $H(\sigma)$ is monotone-increasing if $\rho_0\geqslant C$ and $\sigma>\sigma_0$ for some sufficiently large constant σ_0. Lemma 2.3 (4) shows that $(g\sigma^{3/2})'$ over $[0, \sigma^0]$ has a lower bound away from zero. So far we have derived the monotonicity in R^1_+ of $s^{-\rho_0}h$ if ρ_0 is chosen big enough. This completes the proof.

Consider the following problem

$$\begin{cases} u''-yu=-1, \ y>0, \\ u=0 \text{ as } y=0, \ \lim u=0 \text{ as } y\to+\infty. \end{cases} \tag{2.17}$$

It is not difficult to get its solution of the from

$$u=\mu(y)=\int_0^y dt\int_t^\infty (\lambda(y)\lambda(\sigma)/\lambda^2(t))\,d\sigma. \tag{2.18}$$

Moreover, we have

Lemma 2.5. *These inequalities*

$$|\mu(y)|\leqslant Cy^{-1},\ [\mu'(y)| \quad \text{and} \quad |\mu''(y)|\leqslant C^{-}y^2 \ \text{for all } y\geqslant 0 \tag{2.19}$$

are valid.

Proof Obviously, it only needs to prove (2.19) is valid for all $y\geqslant 2$. Let us now verify the first part of (2.19). (2.18) may be split into two parts as follows

$$\mu(y)=\lambda(y)\int_0^{y/2}+\int_{y/2}^y \lambda^{-2}(t)\,dt\int_t^\infty \lambda(\sigma)\,d\sigma=\mu_1(y)+\mu_2(y).$$

Applying (2.7), (2.8) to the second integral we can get

$$\mu_2(y)\leqslant yC^{-1}\int_{y/2}^y \exp(-2(y^{3/2}-t^{3/2})/3)\,dt^{3/2}\leqslant C'y^{-1}. \tag{2.20}$$

In the other hand

$$\mu_1(y)=\lambda(y)\int_0^1+\int_1^{y/2}\lambda^{-2}(t)dt\int_t^\infty\lambda(\sigma)d\sigma$$
$$\leqslant C\lambda(y)+\int_1^{y/2}\exp(-2(y^{3/2}-t^{3/2})/3)dt^{3/2}$$
$$\leqslant C_n y^{-n}+\exp(-y^{3/2}/3). \tag{2.21}$$

(2.20), (2.21) give $\mu(y)\leqslant Cy^{-1}$ if $y\geqslant 2$. Now we study its derivative. Since

$$\mu'(y)=\int_y^\infty(\lambda(\sigma)/\lambda(y))d\sigma+\lambda'(y)\mu(y)/\lambda(y),$$

from a similar argument it follows that the first term is controlled by $Cy^{-1/2}$. The sec ond term, with the aid of (2.16), approaches to zero if $y\to+\infty$. So, $\mu'(y)$ is the solution of

$$\begin{cases}u''-yu=\mu(y), & y>0,\\ u=0 \text{ as } y=0, & \lim u=0 \text{ as } y\to+\infty.\end{cases} \tag{2.22}$$

(2.22) has a unique solution which can be expressed in the form

$$\mu'(y)=\lambda(y)\mu'(0)+\lambda(y)\int_0^y\lambda^{-2}(t)dt\int_t^\infty\lambda(\sigma)\mu(\sigma)d\sigma.$$

Repeating the same argument as in (2.20)and(2.21) and using the resulted estimates for $\mu(y)$, we can immediately derive the second part of (2.19). This lemma is proved.

§ 3. A Priori Estimates for Linear Equation

This section is devoted to a priori estimates for solution of lineor degenerated elliptic equation. In order to do so we now consider two families of integral operators. The first one is defined as follows. For any $j=0, 1, \cdots$

$$B_jg(x, y)=\int_{-\infty}^\infty g_j(z)dz\int_{-\infty}^\infty\lambda(|\xi|^{2/3}y)X(2^{-j}|\xi|)\exp i(x-z)\xi d\xi, \tag{3.1}$$

where $g_j=\varphi(2^{-j}D)g$ and $X(t)$ is the characteristic function of set C_0, namely, $X(t)=1$ if t is in C_0 and $X(t)=0$ outside of C_0. Obviously, B_j is a mapping of $C_c^\infty(R^1)$ into $C(R^1_+, C^\infty(R^1))$. Furthermore we have

Lemma 3.1. *For any g in $C_c^\infty(R^1)$, the inequalities*

$$|yB_jg|\leqslant C|g|_\alpha 2^{-j(\alpha+2/3)}, \tag{3.2}$$
$$|\partial^2B_jg/\partial^2y|\leqslant C|g|_\alpha 2^{-j(\alpha-4/3)}, \tag{3.3}$$
$$|y^{1/2}\partial B_jg/\partial y|\leqslant C|g|_\alpha 2^{-j(\alpha-1/3)}, \tag{3.4}$$
$$|B_jg|\leqslant C|g|_\alpha 2^{-j\alpha}, \tag{3.5}$$
$$|\partial B_jg/\partial y|\leqslant C|g|_\alpha 2^{-j(\alpha-2/3)} \tag{3.6}$$

hold, where $2<\alpha$ in R^1/z and constant C is independent of g und j.

Proof From the fact $X(0)=0$ and the integration by parts, it follows that with $p_j^+=k2^{j+1}$, $p_j^-=k^{-1}2^j$ and $p_j=2^j$,

$$B_j g = \frac{4}{3}\int_{|x-z|<2^{-j+}}\int_{|x-z|>2^{-j}}(g_j(z)-g_j(x))\,dz$$
$$\times\int_{p_j^-}^{p_j^+}\lambda^1(y\xi^{2/3})\xi^{-1/3}\,\frac{y\sin(x-z)\xi}{(x-z)}\,d\xi$$
$$\pm 2\int_{-\infty}^{\infty}(g_j(z)-g_j(x))\,\frac{\sin(x-z)p_j^{\pm}}{(x-z)}\,\lambda(y(p_j^{\pm})^{2/3})\,dz$$
$$=I_1+I_2+I_3. \tag{3.7}$$

From the properties of g_j and Proposition 2.1, it is easily seen that for some $0<\beta<\alpha$

$$|I_1|\leqslant C|g_j|_\beta 2^{-j\beta}|\lambda(y(p_j^+)^{2/3})-\lambda(y(p_j^-)^{2/3})|\leqslant C'|g|_\alpha y^{-\rho}2^{-j(\alpha+2\rho/3)} \tag{3.8}$$

for any nonnegative ρ. In getting (3.8) we have used Lemma 2.3(2) and estimate $\lambda(s)\leqslant C_\rho s^{-\rho}$, $\rho\geqslant 0$. An application of Abel and Dirichlet test to the integral in ξ of I_2 gives at once

$$|I_2|\leqslant C|\lambda'(y(p_j^-)^{2/3})|(p_j)^{-1/3}y|g_j|\int_{|x-z|>2^{-j}}|(x-z)|^{-2}dz$$
$$\leqslant C'|g|_\alpha y^{-\rho}p_j^{-(\alpha+2\rho/3)},\quad \rho\geqslant 0. \tag{3.9}$$

Here the property $|\lambda'(s)|\leqslant C_\rho s^{-\rho}$, $\rho\geqslant 0$ has been utilized too.

Let us consider I_3. Obviously

$$|I_3|=2\lambda(y(p_j^{\pm})^{2/3}\left|\left[\int_{-\infty}^{\infty} g_j(z)\frac{\sin(x-z)}{(x-z)}p_j^{\pm}dz-\pi g_j(x)\right]\right|. \tag{3.10}$$

It remains to evaluate the integral in (3.10). By Parseval formula, the integral in (3.10) equals

$$\int_{-\infty}^{\infty}\hat{g}(\xi)\varphi(p_j\xi)F(\xi/p_i^{\pm})\exp(ix\xi)\,d\xi, \tag{3.11}$$

where

$$F(\xi)=\int_{-\infty}^{\infty}\exp(-iz\xi)(\sin z/z)\,dz$$
$$=\frac{\pi}{2}|\operatorname{sgn}(\xi+1)-\operatorname{sgn}(\xi-1)]. \tag{3.12}$$

Note that $\varphi(p_j\xi^{2/3})\ F(\xi/p_j^{\pm})=\frac{\pi}{2}(1\pm 1)\varphi(p_j\xi^{2/3})$. Therefore, inserting it into (3.11) and later into (3.10) we can get

$$|I_3|\leqslant C\lambda(y(p_j^{\pm})^{2/3})|g_j|.$$

So far (3.2) and (3.5) have been proved if we take $\rho=1$ and $\rho=0$.

Note that $B_j g$ satisfies the following equation

$$(B_j g)_{yy}+y(B_j g)_{xx}=0.\quad y>0, \tag{3.13}$$

Thus (3.3) is the trivial consequence of (3.2).

At last, we proceed to prove (3.4) and (3.6). Firstly, we have

$$\partial B_j/\partial y=\frac{4}{3}\int_{-\infty}^{\infty}(g_j(z)-g_j(x))\,dz\int_{p_j}^{p_j^+}(\sin(x-z)\xi/(x-z))(K_1+K_2)\,d\xi$$
$$\pm 2\lambda'(y(p_j^{\pm})^{2/3})(p_j^{\pm})^{2/3}\int_{-\infty}^{\infty}(g_j(z)-g_j(x))(\sin(x-z)p_j^{\pm}/(x-z))\,dz$$
$$=I_4+I_5+I_6, \tag{3.14}$$

where

$$K_1=\lambda'(y\xi^{2/3})\xi^{-1/3} \quad \text{and} \quad K_2=\lambda(y\xi^{2/3})\xi y^2\frac{2}{3}.$$

Comparing I_4 with I_1 and I_2, without difficulty we can similarly get the following estimates

$$|I_4|\leqslant Cy^{-\rho}|g|_\alpha p_j^{-(\alpha+2(\rho-1)/3)},\quad \rho\geqslant 0. \tag{3.15}$$

Comparing I_6 with I_3, and by means of the same argument as in (3.11) and (3.12), we can deduce that I_6 is controlled by the right hand side of (3.15).

Consider now I_5 which may be split into two parts

$$I_5=\frac{4}{3}\int_{|x-z|<2^{-j}}+\int_{|x-z|\geqslant 2^{-j}}(g_j(z)-g_j(x))dz\int_{p_j^-}^{p_j^+}\lambda(y\xi^{2/3})\xi y^2\frac{\sin(x-z)\xi}{(x-z)}d\xi$$
$$=I_{51}+I_{52}.$$

By means of the argument similar to that in getting (3.8), it is easily seen that

$$|I_{51}|\leqslant C|g|_\alpha y^{2-\rho_1}p_j^{-((\alpha-2)+2\rho_1/3)},\quad \rho_1\geqslant 0$$

or

$$\leqslant C|g|_\alpha y^{-\rho}p_j^{-(\alpha+2(\rho-1)/3)},\quad \rho\geqslant -2. \tag{3.16}$$

In order to estsmate I_{52}, we need to evaluate

$$\left|\int_{p_j^-}^{p_j^+}\lambda(y\xi^{2/3})\xi y^2\sin(x-z)\xi d\xi\right|$$

which, from the Dirichet and Abel test, is bounded above from

$$Cy^2\lambda(y(p_j^-)^{2/3})\max_{p_j^-<\xi<p_j^+}\left|\int_{p_j^-}^{\xi}\xi\sin(x-z)\xi d\xi\right|$$
$$\leqslant C'y^2|x-z|^{-1}\lambda(y(p_j^-)^{2/3})p_j.$$

From this, we can get

$$|I_{52}|\leqslant C|g|_\alpha y^{-\rho}p_j^{-(\alpha+2(\rho-1)/3)},\quad \rho\geqslant -2. \tag{3.17}$$

The proof of Lemma 3.1 is completed if we take $\rho=0$ or $1/2$ in (3.15) (3.16) and (3.17).

Now we turn to the second family of integral operators. Define, for each $j=0, 1, \cdots$,

$$T_jf(x, y)=2\int_0^y dt\int_t^\infty d\sigma\int_{-\infty}^{\infty}f_j(z, \sigma)dz\int T_j(\xi, \sigma, t, y)\cos(x-z)\xi d\xi, \tag{3.18}$$

where

$$T_j=\lambda(\sigma\xi^{2/3})\lambda(y\xi^{2/3})/\lambda^2(t\xi^{2/3})$$

and

$$f_j=\varphi(2^{-j}D)f.$$

Evidently, (3.18) defines a mapping of $C_c^\infty(\bar{R}_+^2)$ into $C(R_+^1, C^\infty(R^1))$. Indeed we have

Lemma 3.2. *For any $f\in C_c^\infty(\bar{R}_+^2)$, the inequalities*

$$|yT_jf|\leqslant C|f|_\alpha 2^{-j(\alpha+2)}. \tag{3.19}$$

$$|\partial^2T_jf/\partial^2y|\leqslant C|f|_\alpha 2^{-j\alpha}. \tag{3.20}$$

$$|y^{1/2}\partial T_j f/\partial y| \leqslant C|f|_\alpha 2^{-j(\alpha+1)}, \tag{3.21}$$

$$|T_j f| \leqslant C|f|_\alpha 2^{-j(\alpha+4/3)}, \tag{3.22}$$

$$|\partial T_j f/\partial y| \leqslant C|f|_\alpha 2^{-j(\alpha+2/3)}, \tag{3.23}$$

where $0<\alpha\in R^1\backslash Z$ *and* C *is independent of* f *and* j, *are valid.*

Here and later the norm $|f|_\alpha$ *refers to the* α*–Hölder norm only with respect to* x, *namely*

$$|f|_\alpha = \sum_{j=0}^{[\alpha]} \sup_{R_+^2} |\partial_x^j f| + \sup_{R_+^2, h>0} |(\partial_x^{[\alpha]} f(x+h, y) - \partial_x^{[\alpha]} f(x, y)/h^{\alpha-[\alpha]}|.$$

Proof The idea of the proof we shall be following is similar to what we used in proving Lemma 3.1. Integration by parts, splitting the result, yields

$$\begin{aligned} T_j f &= 2\int_0^y dt \int_t^\infty d\sigma \left(\int_{|x-z|<2^{-j}} + \int_{|x-z|\geq 2^{-j}}\right) \tilde{T}_j (f_j(z, \sigma) - f_j(x, \sigma))dz \\ &\quad \pm 2\int_0^y dt \int_t^\infty d\sigma \int_{-\infty}^\infty (f_j(z, \sigma) - f_j(x, \sigma)) \frac{\sin(x-z)p_j^\pm}{(x-z)} T_j(p_j^\pm, \sigma, t, y)dz \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{3.24}$$

where

$$\tilde{T} = \int_{p_j^-}^{p_j^+} \frac{\partial}{\partial\xi} T_j(\xi, \sigma, t, y) \sin(x-z)\xi/(x-z)d\xi.$$

In getting (3.24) we have used the property $X(0)=0$. Repeating the same argument as in proving (3.11), (3.12), we find

$$|I_3| \leqslant C|f|_\alpha p_j^{-\alpha} \int_0^y dt \int_t^\infty T_j(p_j^\pm, \sigma, t, y)d\sigma = C|f|_\alpha p_j^{-\alpha} \mu(y(p_j^\pm)^{2/3})(p_j^\pm)^{-4/3}.$$

From (2.18), (2.19) and $\mu(0)=0$ the inequality,

$$|I_3| \leqslant C|f|_\alpha y^{-\rho} p_j^{-(\alpha+4/3+2\rho/3)} \text{ for all } \rho \text{ in } [-1, 1], \tag{3.25}$$

comes at once. Lemma 2.4 guarantees

$$\int_{p_j^-}^{p_j^+} \left|\frac{\partial}{\partial\xi} T_j(\xi, \sigma, t, y)\right| d\xi \leqslant -C \frac{\partial}{\partial\xi} T_j(p_j^-, \sigma, t, y)p_j. \tag{3.26}$$

Inserting (3.26) into I_1 and in the same way as in (3.8) we find

$$|I_1| \leqslant C|f|_\alpha p_j^{-(\alpha-1)}|(\mu(\xi^{2/3}y)\xi^{-4/3})|_{\xi=p_j^-}.$$

In view of (2.19) it is not difficult to get

$$|I_1| \leqslant C|f|_\alpha y^{-\rho} p_j^{-(\alpha+4/3+2\rho/3)} \text{ for all } \rho \text{ in } [-1, 1]. \tag{3.27}$$

Now we shall estimate I_2. An application of Dirichlet and Mbel test to the integral in ξ of I_2 provides

$$\begin{aligned} &\left|\int_{p_j^-}^{p_j^+} \frac{\partial}{\partial\xi} T_j(\xi, \sigma, t, y)\cos(x-z)\xi d\xi\right| \\ &\leqslant -C \frac{\partial}{\partial\xi} T_j(p_j^-, \sigma, t, y)(p_j^-)^{-\rho_0} \max_{p_j^-<\xi<p_j^+} \int_{p_j^-}^{\xi} \xi^{\rho_0} \cos(x-z)\xi d\xi \\ &\leqslant -C \frac{\partial}{\partial\xi} T_j(p_j^-, \sigma, t, y)|x-z|^{-1}. \end{aligned} \tag{3.28}$$

With the aid of (3.28), by means of the similar argument to (3.9), I_2 is controlled by the right hand side of (3.27) too. So far we have proved (3.19) and (3.22) if

$\rho=1$ or $\rho=0$ in (3.25) and (3.27).

Indeed, from (3.25) and (3.27) we can get more. The inequality

$$|y^{-1/2}T_jf|\leqslant C|f|_\alpha 2^{-j(\alpha+1)} \tag{3.29}$$

comes at once from (3.25) and (3.27) with $\rho=-1/2$.

(3.20) is the direct consequence of (3.19) and the fact that T_jf is the solution of

$$(T_jf)_{yy}+y(T_jf)_{xx}=f_j,\ y>0.$$

(3.23) is the immediate consequence of (3.20) and (3.22) with the aid of the well known interpolation inequality of the form

$$|u_y^2|\leqslant C\sup|u_{yy}|\sup|u| \tag{3.30}$$

for any function u which has a piecewise continous derivative of second order in R^1 and a finite $\sup|u|$. Along the same line of proving (3.30), with a slight modification, we can obtain

$$|y|u_y^2\leqslant C(\sup|u_{yy}|\sup|yu|+\sup|u/y^{1/2}|) \tag{3.31}$$

for the same function mentioned above with the additional condition $u(0)=0$. The details of proof of (3.31) is omitted. Applying (3.31) to T_jf after it is extended to the below plane and noting (3.19) and (3.20) weo and educe (3.21) without difficulty.

We now proceed to study a priori estimates for boundary value problem

$$\begin{cases}Lu=a_{11}yu_{xx}+2a_{12}yu_{xy}+a_{22}u_{yy}+a_1u_x+a_2u_y+au=f,\\ u=g(x),\ y=0,\end{cases} \tag{3.22}$$

where

$$\begin{cases}a_i\cdot a_{ij}\in C^\alpha(\bar{R}_+^2),\\ a_{11},\ a_{22}>0 \text{ if } y\geqslant 0 \text{ and } y^2a_{12}^2-ya_{11}a_{22}<0 \text{ if } y>0.\end{cases} \tag{3.33}$$

Here we use $\dot{C}^\alpha(\bar{R}_+^2)$ to denote the set of all functions a in $C(\bar{R}_+^2)$ satisfying

$$|a|_\alpha<+\infty.$$

Theorem 3.1. *Let (3.33) be fulfiled. Then for any $u\in C_c^\infty(\bar{R}_+^2)$ with $Lu=f$ and $u(x,\ 0)=g$, the inequality,*

$$\begin{aligned}\mathscr{F}(u)&=|yu_{xx}|_\alpha+|y^{1/2}u_{xy}|_\alpha+|u_{yy}|_\alpha+|u_x|_{\alpha+1/3}+|u_y|_{\alpha+1/3}\\ &\leqslant C[|g|_{\alpha+2}+|f|_\alpha+|u|]\end{aligned} \tag{3.34}$$

holds for some constant C independent of u except the diameter of its support.

Proof Denote by $O_\delta(x,\ y)$ the circle with the centre $(x,\ y)$ and radius δ which is a constant to be determined. Suppose that $\{O_\delta(x_l,\ 0),\ l=1,\ 2,\ \cdots,\ l_0\}$ is an open covering of the intersection of suppu and $\{y=0\}$. Furthermore, $\{O_\delta(x_l,\ 0),\ l=1,\ 2,\ \cdots l_0\}$ and an open set N consist of an open covering of suppu. Let $\sigma_l(l=0,\ 1,\ \cdots\ l_0)$ be a partition of unity subordinate to this open covering. With $v_l=u\sigma_l$ we find

$$\begin{cases}Lv_l=f_l+L_lu=\tilde{f}_l,\ y>0,\\ v_l=g_l,\ y=0.\end{cases} \tag{3.35}$$

Moreover $|\tilde{f}_l|$ and $|g_l|$ can be controlled by the sum of the right hand side of (3.34) and $\gamma(\varepsilon)|u|+\varepsilon\mathscr{F}(u)$ for any $\varepsilon>0$ and some positive function $\gamma(\varepsilon)$ of ε. Therefore it suffices to prove (3.34) for each v_l, since (3.34) is a coercive inequality for (3.32). We shall, for convenience, put down v, f, g instead of v_l, $\tilde{f}_l$, g_l and assume $x_l=0$, $l\neq 0$.

If $l=0$, (3.34) is valid because (3.32) is elliptic in suppu. Thus it remains to prove the case of $l\neq 0$. The method of the proof we shall use is standard. Let us first consider the problem of frozen coefficients

$$\begin{cases} Lv = ya_{11}(0)v_{xx}+a_{22}(0)v_{yy}=f-Lv+ya_{11}(0)v_{xx}+a_{22}(0)v_{yy}=f^*. \\ v=g(x), \end{cases} \tag{3.36}$$

Without loss of generality we may assume $a_{11}(0)=a_{22}(0)=1$. With $v_j=\varphi(2^{-j}D)$ it is easy to see

$$v_j=B_jg+T_jf^*.$$

Because f^* and g are of compact support, one can approximate them respectively by sequences in $C_c^\infty(\bar{R}_+^2)$ and in $C_c^\infty(R^1)$. So Lemmas 3.1 and 3.2 are applicable for B_jg and T_jf^*. They provide

$$|yv_j|=|y(B_jg)|+|y(T_jf^*)|\leqslant C[|g|_{2+\alpha}+|f^*|_\alpha]2^{-j(\alpha+2)}, \tag{3.37}$$

where C is independent of j, f^* and g. From (3.36) it follows that with $\alpha_1=\min(\alpha, 1/2)$ we have that the right hand side of (3.37)

$$\leqslant C[|g|_{\alpha+2}+|f|_\alpha+\mathscr{F}(v)(\varepsilon+\delta^{\alpha_1})+\gamma(\varepsilon)|v|]2^{-j(\alpha+2)} \tag{3.37'}$$

for another constant C independent of j, f^* and g. Proposition 2.1 shows

$$|yv|_{\alpha+2}\leqslant C[|g|_{\alpha+2}+|f|_\alpha+\mathscr{F}(v)(\delta^{\alpha_1}+\varepsilon)+|v|\gamma(\varepsilon)]. \tag{3.38}$$

Similarly, we can prove from the right hand side of (3.38). that $|y^{1/2}\partial^2v/\partial x\partial y|$, $|\partial^2v/\partial^2y|$, $\cdots$ are bounded above. Summing up all estimates and choosing sufficiently small ε and δ we can deduce (3.34). This completes the proof of Theorem 3.1.

§ 4. The Proof of the Main Theorem

Based on a priori estimate resulted in § 3. We can proceed to discuss the regularity of solutions to boundary value problem for the nonlinear degenerated elliptic equation. Let $u\in C^{3+\alpha}(\bar{\Omega})$ with $\alpha>0$ satisfy

$$\begin{cases} F(x, u, u_i, u_{ij})=F(x, \partial^\alpha u)=0, \text{ in } \Omega, \\ u=g(x), \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

Without loss of generality, we may assume that $\Omega=R_+^2=\{(x, y)|y>0\}$ and

$$\alpha+i/3\bar{\in}Z \text{ for any integer } i. \tag{4.2}$$

Obviously, it suffices to locally prove the main theorem. Consider the boundary value problem for the quotient $\Delta u/h=(u(x+h, y)-u(x, y))/h$ of the difference of u

$$\begin{aligned} L(\Delta u/h) &= F_{ij}(\Delta u/h)_{ij}+\bar{F}_i(\Delta u/h)_i+\bar{F}_u(\Delta u/h) \\ &= -\bar{F}_x+\Sigma(F_{ij}-\bar{F}_{ij})(\Delta u/h)_{ij},\ y>0 \end{aligned} \tag{4.3}$$

and

$$(\Delta u/h) = (\Delta g/h),\ y=0, \tag{4.4}$$

where

$$F_{ij} = F_{ij}(x,\ y,\ \partial^\alpha u(x,\ y)),\ \bar{F}_{ij}$$
$$= \int_0^1 F_{ij}(x+\lambda h,\ y,\ \lambda\partial^\alpha u(x+h,\ y) + (1-\lambda)\partial^\alpha u(x,\ y))\,d\lambda,$$

etc. Setting $W = (\Delta\partial_x u/h)$ and differentiating both sides of (4.3) with respect x, we have

$$LW = \partial f/\partial x - \dot{L}(\Delta u/h) \in \dot{C}^\alpha. \tag{4.5}$$

Here $\dot{L}$ stands for the operator after differentiating the coefficients of L. The ellipticity of F in R^2_+ provides $F_{11}F_{22} - F^2_{12} > 0$ in R^2_+. The assumption of this theorem about the degeneracy on $\partial\Omega$ implies

$$F_{11}F_{22} - F^2_{12} = 0 \text{ and } F_{22} > 0,\ y=0, \tag{4.6}$$

since (1.3) is coordinate-free. In order to reduce (4.5) to the form like that studied in Theorem 3.1, a change of independent variables should be introduced. Because the coefficients of (4.5) are only in $C^{1+\alpha}$, in doing so we have to do carefully. Suppose the change of the independent variables is of the form

$$\bar{x} = \sigma(x,\ y) \in C^{2+\alpha},\ \bar{y} = y \tag{4.7}$$

and

$$\sigma(x,\ 0) = x,\ \sigma_y(x,\ 0) = -(F_{12}/F_{22})(x,\ 0). \tag{4.8}$$

The function satisfying (4.7) and (4.8) can be constructed by solving

$$(\Delta - I)\tilde{\sigma} = 0,\ y>0 \text{ and } \partial\tilde{\sigma}/\partial y = -(F_{12}/F_{22}) \text{ if } y=0$$

and then taking $\sigma = \tilde{\sigma} - \tilde{\sigma}(x,\ 0) + x$. Under the new coordinate system (4.5) may be rewritten as follows

$$\begin{cases} LW = a_{11}\bar{y}W_{\bar{x}\bar{x}} + 2a_{12}\bar{y}W_{\bar{x}\bar{y}} + a_{22}W_{\bar{y}\bar{y}} + \cdots = \partial f/\partial x - \dot{L}(\Delta u/h), \\ W \in C^\infty \text{ if } \bar{y}=0, \end{cases} \tag{4.5'}$$

where $a_{ij} \in \dot{C}^\alpha$ and a_{11}, a_{22} are positive. This comes from the reason that $u \in C^{3+\alpha}$, and the coefficients of the equation, under new coordinates

$$A_{11} = F_{ij}\sigma_i\sigma_j = -A_{12} = F_{12}\sigma_x + F_{22}\sigma_y = 0, \text{ and } A_{22} = F_{22} > 0,\ y=0. \tag{4.9}$$

The second part of the assumption (1.3) may be written in the form

$$\frac{\partial}{\partial\bar{y}}(\det(F_{ij})) = \lim_{\bar{y}\to 0}(a_{11}a_{22} - a^2_{12}\bar{y}) = a_{11}a_{22} > 0,$$

which implies $a_{11} > 0$ near $y=0$. Without loss of generality, we assume that W is of compact support and $\sigma_x(x,\ y) \neq 0$ if necessary, multiplying it by a cutoff function. Applying Theorem 3.1 to W one can obtain, according to the notation in (3.34),

$$\mathscr{F}(W) \leqslant C, \text{ independent of } h, \tag{4.10}$$

From (4.10) and letting $h\to 0$, we see that $\partial u_{xx}/\partial\bar{x}$ and $\partial u_{xx}/\partial\bar{y}$ are in $\dot{C}^{\alpha+1/3}(\bar{R}^2_+)$. Thus u_{xxx} and u_{xxy} are in $\dot{C}^{\alpha+1/3}(\bar{R}^2_+)$ in $(x,\ y)$ since $\sigma_x \neq 0$ in supp W. After letting

$h \to 0$ in (4.3) and using the latter part of (4.6) we can solve $u_{yyx} = \Phi_1(x, y, u, \partial^\alpha u, u_{xxx}, u_{xxy})$, where Φ_1 is a smooth function of its argument. So u_{xyy} is in $C^{\alpha+1/3}$ in (x, y) as well as in $(\bar{x}, \bar{y})$. In view of the fact that $F_{22} > 0$, by means of the theorem of implicit function, one can solve

$$u_{yy} = \Phi_2(x, y, u, \partial u, u_{xx}, u_{xy}). \tag{4.11}$$

Φ_2 is also a smooth function. This shows $u_{yyy} \in \dot{C}^{\alpha+1/3}$ in (x, y) as well as in $(\bar{x}, \bar{y})$. An application of the regularity theorem of standard elliptic problem provides $\sigma \in C^{2+\alpha+1/3}$. So the right hand side of (4.5) is in $\dot{C}^{\alpha+1/3}$ in $(\bar{x}, \bar{y})$. Repeating the same argument as done previously, we have $\partial^\beta u \in \dot{C}^{\alpha+i/3}$ for all $|\beta| = 3$ and $i \in Z^+$, which implies $\partial^\beta u \in C(\bar{R}^1_+, C^\infty(R^1))$ for all $|\beta| = 3$. Combining this with (4.11) we have completed the proof of the present theorem.

Remark. There is no difficulty in repeating the discussion carried out here for quasilinear equation if $u \in C^{2+\alpha}(\bar{\Omega})$.

The author would like to thank Professor Abdus Salam, and Internatsonal Atomic Energy Agency UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. The author is also grateful to Professor James Eells for his encouragement.

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