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A NECESSARY AND SUFFICIENT CONDITION ON THE EXISTENCE AND UNIQUENESS OF 2π-PERIODIC SOLUTION OF DUFFING EQUATION

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Abstract

This paper gives a necessary and sufficient condition on the existence and uniqueness of 2π-periodic solution of Duffing equation

$$d^2x/dt^2+g(x)=P(t) \quad (=P(t+2\pi)).$$

1. In this paper, we study the existence and uniqueness problem of 2π-periodic solution for Duffing equation

$$d^2x/dt^2+g(x)=P(t), \tag{*}$$

where $g(x)\in C^1(R, R)$, and there exists u_0 such that $g'(u_0)>0$, $P(t)\in C(R, R)$, $P(t)=P(t+2\pi)$, for all $t\in R^1$.

D. E. Leach[1] proved the existence and uniqueness of 2π-periodic solution of (*) under the Lond condition:

$$m^2<\lambda\leqslant g'(x)\leqslant\mu<(m+1)^2 \quad (g(0)=0), \tag{1.1}$$

with a given integer $m\geqslant 0$ and two such constants λ and μ. R. Reissing[2] proved the existence of periodic solution of (*) under a weaker condition:

$$m^2<\lambda\leqslant g(x)/x\leqslant\mu<(m+1)^2 \quad |x|\geqslant C>0. \tag{1.2}$$

Tung-Ren Ding[3] resolved the existence problem of (*) under a weakened version of (1.1)

$$m^2\leqslant g'(x)\leqslant(m+1)^2, \tag{1.3}$$

and in this paper, by using a theorem due to T. R. Ding, we improve the result of D. E. Leach[1].

2. We need the following hypotheses:

(H_1) $m^2\leqslant g'(x)\leqslant(m+1)^2$,

(H_2) $H(g)=\min\{\sup_{x\in R}|g(x)-m^2x|, \sup_{x\in R}|g(x)-(m+1)^2x|\}=+\infty$,

(H_3) $\operatorname{int} A=\phi$, where $A=\{x\,|\,g'(x)=m^2, (m+1)^2\}$,

where $m\geqslant 0$ is an integer.

The main purpose of this paper is to show

Theorem. *For every 2π-periodic function $P(t)$, (*) has one and only one 2π-*

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periodic solution if and only if (H_1)-(H_3) *hold.*

To prove the theorem, we need two lemmas.

Lemma 1. *For any x, y, $x \neq y$, we assume that*

$$m^2 < (g(x)-g(y))/(x-y) < (m+1)^2,$$

where $m \geqslant 0$ is a given integer. Then (*) *has at most one 2π-periodic solution.*

Proof Aussuming both $x(t)$, $y(t)$ are the 2π-periodic solution of (*) and $x(t) \neq y(t)$, for some $t \in R^1$, we get

$$x''(t) - y''(t) + g(x(t)) - g(y(t)) = 0. \tag{2.1}$$

Let

$$x(t) - y(t) = r(t)\cos\theta(t), \quad x'(t) - y'(t) = r(t)\sin\theta(t). \tag{2.2}$$

We can prove easily that $r(t)$ is a 2π-periodic function, $r(t) \neq 0$ and

$$\cos\theta(2\pi) = \cos\theta(0), \quad \sin\theta(2\pi) = \sin\theta(0). \tag{2.3}$$

By applying (2.2) to (2.1), we get

$$-d\theta/dt = \sin^2\theta(t) + \cos\theta(t)\,(g(x(t)) - g(y(t)))/r(t)$$
$$= \begin{cases} \sin^2\theta(t) + \cos^2\theta(t)\,(g(x(t)) - g(y(t)))/(x(t) - y(t)), & x \neq y, \\ 1, & x = y. \end{cases}$$

It is obvious that $d\theta/dt < 0$. Let T_i denote the time in which $\theta(t)$ decreases from θ_0 to $\theta_0 - 2(m+i)$, $(i = 0, 1)$. Then we have

$$T_0 = \int_0^{T_0} dt = \int_{\theta_0 - 2m\pi}^{\theta_0} \frac{d\theta}{\sin^2\theta + \cos\theta\,(g(x) - g(y))/r}$$
$$< \int_{\theta_0 - 2m\pi}^{\theta_0} \frac{d\theta}{\sin^2\theta + m^2\cos^2\theta} = 2\pi.$$

In the similar way, we have $T_1 > 2\pi$. So we get

$$2m\pi < \theta(2\pi) - \theta(0) < 2(m+1)\pi.$$

This is a contradiction with (2.3).

Lemma 2. *Let $f(x) \in C(R, R)$. Suppose that there exist x_0, x_1, $x_0 < x_1$, such that $f(x_1) > 0 (<0)$, $f(x_0) < (>0)$ and* int $B = \phi$, *where $B = \{x \mid f(x) = 0\}$. Then there exists $y \in [x_0, x_1]$ such that $f(y) = 0$ and for every neighbourhood $\delta(y)$ of y, there are y_1 and $y_2 \in \delta(y)$ satisfying that $f(y_1) \cdot f(y_2) < 0$.*

Proof Assume the result is not true, so for every point $y \in [x_0, x_1]$, we can get an open neighbourhood $\delta(y)$ of y, such that any two points $z_1, z_2 \in \delta(y)$ satisfy $f(z_1) \cdot f(z_2) \geqslant 0$. By using Heine-Borel Theorem, we can get $y_1 (= x_0) < y_2 < \cdots\cdots < y_{n-1} < y_n (= x_1)$ such that $\bigcup_{i=1}^{n} \delta(y_i) \supset [x_0, x_1]$ and $\delta(y_i) \cap \delta(y_{i+1}) \neq \varnothing$. From int $B = \varnothing$, we can get $l_i \in \delta(y_i) \cap \delta(y_{i+1})$ such that $f(l_i) \neq 0$. By the fact l_i, $l_{i+1} \in \delta(y_{i+1})$, we have

$$f(l_i) \cdot f(l_{i+1}) > 0 \quad (i = 1, \cdots\cdots, n-1).$$

So we have

$$f(x_0) \cdot f(x_1) > 0.$$

This is a contradiction.

3. The proof of the theorem.

1) The proof of "sufficent" part.

By using the result of [3], under the conditionse (H_1), (H_2), we can prove the existence of 2π-periodic solution of $(*)$.

Under conditions (H_1), (H_3), for any x, y, $x\neq y$, we can prove easily that

$$m^2<(g(x)-g(y))/(x-y)<(m+1)^2.$$

Using Lemma 1, we get the uniqueness of 2π-periodic solution of $(*)$.

2) The proof of "necessary" part.

At first, we prove that $g(x)$ satisfies (H_3).

Assume that it is not true. Let $x_0\in$ int A, so there is a neighbourhood $\delta(x_0)$ of x_0 such that for every $x\in\delta(x_0)$, $g'(x)=m^2$. We consider equation

$$x''+g(x)=g(x_0). \tag{3.1}$$

We know that $x=x_0+L\cos mt$ $(L\ll 1)$ are solutions of (3.1). This is a contradiction with uniqueness.

Now we want to prove $g'(x)\geqslant 0$ $(x\in R^1)$.

Assume it is not true. Then there exists y_0 such that $g'(y_0)<0$. Without loss of generality, we assume $y_0<u_0$. From $g'(u_0)>0$, we can get two points z_1, $z_2\in[y_0, u_0]$ such that $g(z_1)=g(z_2)$. We consider equation

$$x''+g(x)=g(z_1). \tag{3.2}$$

It is obvious that both $x=z_1$, $x=z_2$ are 2π-periodic solutions of (3.2). This contradicts with the uniqueness.

Now we prove that (H_1) holds.

Let $I=\text{im}(g'(x))$. Suppose that $n^2\in\text{int } I$, n is an integer. From $g'(x)\geqslant 0$ we know $n^2\geqslant 1$. Considering the function $f(x)=g'(x)-n^2$, we can see that $f(x)$ satisfies the condition of Lemma 2. So we can get a point x_0 and two sequences y_i, w_i such that w_i, $y_i\to x_0(i\to+\infty)$ with $f(w_i)>0$, $f(y_i)<0$. Thus $g'(y_i)<n^2$, $g'(w_i)>n^2$. Let us consider equation

$$x''+g(x)=g(x_0). \tag{3.3}$$

(3.3) has a center at point $(x_0, 0)$, and all orbits are closed orbits around the center. Let Γ denote the orbit which passes through the point $(x_0+1, 0)$ and whose period is $\tau(\Gamma)$; from the uniqueness, we know $\tau(\Gamma)\neq 2\pi/n$. Without loss of generality, we assume $\tau(\Gamma)-2\pi/n\geqslant\delta>0$. Now we consider the family of equations

$$x''+g(x)=g(w_i),\ i=1, 2\cdots\cdots. \tag{3.4$_i$}$$

Let Γ_i denote the orbit of $(3.4)_i$ which passes through $(x_0+1, 0)$ and whose period is $\tau(\Gamma_i)$. From $w_i\to x_0$, we have $\lim\limits_{i\to+\infty}\tau(\Gamma_i)=\tau(\Gamma)$. so if k is large enough, we have

$$\tau(\Gamma_k)>2\pi/n. \tag{3.5}$$

The period of any orbit near center $(w_k, 0)$ is about

$$2/(g'(w_k))^{1/2}<2\pi/n \quad (k\gg 1). \tag{3.6}$$

From (3.5) and (3.6), we know that, between center $(w_k, 0)$ and Γ_k, there exists a closed orbit H_k whose period is $2\pi/n$. We obtain two 2π-periodic solutions of $(3.4)_k$: $x=w_k$ and H_k for any $k\gg 1$. This contradicts the uniqueness too. So $n^2\in$ int I. Then (H_1) holds.

At last we prove (H_2) holds.

Assume it is not true, i. e., $H(g)=M<+\infty$. Without loss of generality, we assume that $H(g)=\sup_{x\in R}|g(x)-m^2x|$. Let $h(x)=g(x)-m^2x$. Then we consider equation

$$x''+g(x)=3M\cos mt. \tag{3.7}$$

Let $x=\varphi(t)$ be a 2π-periodic solution of (3.7). So $\varphi(t)$ is a 2π-periodic solution of equation

$$x''+m^2x=3M\cos mt-h(\varphi(t)). \tag{3.8}$$

From (3.8), we get

$$\int_0^{2\pi}\cos mt(3M\cos mt-h(\varphi(t))dt=0. \tag{3.9}$$

From (3.9), we obtain

$$3M\pi=\int_0^{2\pi}3M\cos^2 mtdt=\int_0^{2\pi}\cos mt(h(\varphi(t)))dt$$
$$\leqslant\int_0^{2\pi}|\cos mt(h(\varphi(t)))|dt\leqslant M\int_0^{2\pi}dt=2M\pi.$$

But (H_3) implies that $H(g)=M\neq 0$. Hence we obtain a contradiction.

The proof of the theorem is thus completed.

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