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CUMULATING CENTRAL POLYNOMIALS AND IDENTITIES FOR $M_n(F_p)$

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Abstract

Analogue of Formanek Central Polynomial, the auther constructs a cumulating centrdl polynomial for $M_n(F_p)$ and a cumulating identity for $M_n(F_p)$.

§1. Introduction

Let F_p denote a field of characteristic $p\neq 0$; F, a field of characteristic 0 throughout; $\mathfrak{M}_n(A)$ denote the set of identities satisfied by $M_n(A)$, where A is a commutative ring. We have the map

$$\phi: \mathfrak{M}_n(\mathbb{Z}) \to \mathfrak{M}_n(F_p),$$
$$f(x_1, x_2, \cdots, x_t) \to f(x_1, x_2, \cdots, x_t) \pmod p.$$

An important question (see [1, 2]) in the theory of PI-algebra says that: Does $\phi(\mathfrak{M}_n(\mathbb{Z}))$ generate $\mathfrak{M}_n(F_p)$ as a vector space over F_p? To answer this question it is important to find as many elements of the set $\mathfrak{M}_n(F_p)-\phi(\mathfrak{M}_n\mathbb{Z}))$ as possible.

All polynomial identities and central polynomials for $M_n(F)$, or for $M_n(F_p)$, which the author has met before, are based on the concept of the "alternating" property of polynomials, but for F_p, $M_n(F_p)$ possesses the most fundamental property that for any $a\in M_n(F_p)$, $p\cdot a=0$, if a polynomial $f(x_1, \cdots, x_t)$ can "cumulate" the elements of $M_n(F_p)$ to p times, i. e. for any $a_1, \cdots, a_t\in M_n(F_p)$ $f(a_1, \cdots, a_t)=p\cdot a$ for some suitable $a\in M_n(F_p)$, certainly $f(x_1, \cdots, x_t)$ is a polynomial identity for $M_n(F_p)$. The polynomials which are precisely of cumulating type are the symmetric ones:

$$S_t(x_1, \cdots, x_t)=\sum_{\pi\in \mathrm{sym}(t)} X_{\pi 1}X_{\pi 2}\cdots X_{\pi t}. \qquad (1)$$

According to this consideration, the author has independently established the following[3]

Theorem 1. *If* $t\geqslant pn$, *then* (1) *is a polynomial identity for* $M_n(F_p)$; *if* $t<pn$, *then* (1) *is not an identity for* $M_n(F_p)$.

(Later the author knew from Professor Edward Formanek by communication that the Russian mathematician A. E. Zalesski had obtained the same results). In

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this paper we shall establish a cumulating central polynomial and another cumulating identity for $M_n(F_p)$ by using Formanek's method[4] and Jacobson's argument[5,6].

§2. A Cumulating Central Polynomial for $M_n(F_p)$

Let $x_1, x_2, \cdots, x_{n+1}$ be commuting variables over F_p. Denote

$$p(i, j)=x_i^{p-1}+x_i^{p-2}x_j+x_i^{p-3}x_j^2+\cdots+x_j^{p-1}$$

and

$$g(x_1,x_2, \cdots, x_{n+1})=\prod_{2\leq i\leq n} p(1, i)p(n+1, i)\cdot \prod_{2\leq k<j\leq n} p^2(k, j).$$

Use the Formanek map[4]

$$\rho: x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n+1}^{\alpha_{n+1}} \to X^{\alpha_1}Y_1X^{\alpha_2}Y_2\cdots Y_nX^{\alpha_{n+1}}.$$

We transform $g(x_1, x_2, \cdots, x_{n+1})$ into $G(X, Y_1, Y_2, \cdots, Y_n)$ with coefficients unaltered.

Theorem 2. $G(X, Y_1, \cdots, Y_n)+G(X, Y_2, Y_3, \cdots, Y_n, Y_1)+\cdots+G(X, Y_n, Y_1, \cdots, Y_{n-1})$ *is a central polynomial for* $M_n(F_p)$, *but not for* $M_n(F)$.

Proof of Theorem 2 $G(X, Y_1, \cdots, Y_n)$ is linear in Y_i, so we can assume $Y_s=e_{i_sj_s}$, $s=1, 2, \cdots, n$, being matrix units, and $x=\operatorname{diag}\{x_1, x_2, \cdots, x_n\}$. We have

$$X^{\alpha_1}e_{i_1j_1}X^{\alpha_2}e_{i_2j_2}\cdots X^{\alpha_n}e_{i_nj_n}X^{\alpha_{n+1}}=x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_n}^{\alpha_n}x_{j_n}^{\alpha_{n+1}}e_{i_1j_n}\neq 0$$

if and only if

$$e_{i_1j_1}e_{i_2j_2}\cdots e_{i_nj_n}\cdots e_{i_nj_n} \text{ is a path, i. e. } j_k=i_{k+1} \text{ for } k=1, 2, \cdots, n-1. \tag{2}$$

So when (2) holds, $G(X, e_{i_1j_1}, \cdots, e_{i_nj_n})=g(x_{i_1}, x_{i_2}, \cdots, x_{i_n}, x_{j_n})e_{i_1j_n}$. Note that if $i=j$, $P(i, j)=px_i^{p-1}=0$, so $g(x_{i_1}, x_{i_2}, \cdots, x_{i_n}, x_{j_n})\neq 0$ if and only if

$$i_1, i_2, \cdots, i_n \text{ is a permutation of } 1, 2, \cdots, n \text{ and } j_n=i_1. \tag{3}$$

This implies:

$$G(X, e_{i_1j_1}\cdots e_{i_nj_n})=\begin{cases}\prod_{1\leq i<j\leq n} p^2(i, j)e_{i_1i_1}, & \text{if and only if (2) and (3) hold,}\\ 0, & \text{otherwise.}\end{cases}$$

So

$$G(X, e_{i_1j_1}, \cdots, e_{i_nj_n})+\cdots+G(X, e_{i_nj_n}, e_{i_1j_1}, \cdots, e_{i_{n-1}j_{n-1}})$$
$$=\begin{cases}\prod_{1\leq i<j\leq n} p^2(i, j)E, & \text{if and only if (2) and (3) hold,}\\ 0, & \text{otherwise.}\end{cases}$$

This ends our proof.

§3. Another Cumulating Identity for $M_n(F_p)$

Using Formanek's method[4], we can easily establish an identity for $M_n(F)$ (also for $M_n(F_p)$). Let

$$\rho^*: x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n+1}^{\alpha_{n+1}} \to X^{\alpha_1}Y_1X^{\alpha_2}Y_2\cdots X^{\alpha_{n-1}}Y_{n-1}X^{\alpha_n}Y_nX^{\alpha_{n+1}}Y_{n+1}.$$

Let $f_1(x_1, x_2, \cdots, x_{n+1}) = \prod_{1\leq i<j\leq n+1} (x_i - x_j)$ be polynomial of commutative variables. Then the image $G^*(X, Y_1, \cdots, Y_{n+1})$ of $f_1(x_1, \cdots, x_{n+1})$ under ρ^* is an identity for $M_n(F)$ (proof is trivial, similar to that of Theorem 2).

Analogue of this example, let $f_2(x_1, \cdots, x_{n+1}) = \prod_{1\leq i<j\leq n+1} p(i, j)$, then the image $G^{**} = G^{**}(x, y_1, \cdots, y_n)$ of $f_2(x_1, \cdots, x_{n+1})$ under ρ^* is an identity for $M_n(F_p)$ (obviously, G^{**} is not an identity for $M_n(F)$). To see this we note that G^{**} is linear in $Y_1, \cdots, Y_n$, so we assume $Y_s = e_{i_s j_s}$, $s=1, 2, \cdots, n$ and $x = \mathrm{diag}\,\{x_1, x_2, \cdots, x_n\}$. So $G^{**}(x, e_{i_1 j_1}, \cdots, e_{i_n j_n}) = f_2(x_{i_1} x_{i_2}, \cdots, x_{i_n} x_{j_n}) e_{i_1 j_n}$ iff (2) holds, otherwise $G^{**}(x, e_{i_1 j_1}, \cdots, e_{i_n j_n}) = 0$. But $x_{i_1}, x_{i_2}, \cdots, x_{i_n}, x_{j_n} \in \{x_1, \cdots, x_n\}$. This forces, at least two of $x_{i_1}, x_{i_2}, \cdots, x_{i_n}, x_{j_n}$ are the same. In this case $f_1(x_{i_1}, x_{i_2}, \cdots, x_{i_n}, x_{j_n}) = 0$. So G^{**} is an identity of $M_n(F_p)$ and of cumulating type, surely not a P. I for $M_n(F)$. So we have Theorem 3. G^* is an identity for $M_n(F)$ and also for $M_n(F_p)$. G^{**} is an identity for $M_n(F_p)$ only.

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