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ON LINEAR AND NONLINEAR RIEMANN-HILBERT PROBLEMS FOR REGULAR FUNCTION WITH VALUES IN A CLIFFORD ALGEBRA

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Abstract

This paper deals with the boundary value problems for regular function with values in a Clifford algebra:

$$\bar{\partial} W=0, \quad x\in \mathbf{R}^n\setminus\Gamma,$$
$$W^+(x)=G(x)W^-(x)+\lambda f(x, W^+(x), W^-(x)), \; x\in\Gamma;\; W^-(\infty)=0,$$

where Γ is a Liapunov surface in $\mathbf{R}^n$, the differential operator $\bar{\partial}=\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}e_2+\cdots+\frac{\partial}{\partial x_n}e_n$, $W(x)=\sum_A e_A W_A(x)$ are unknown functions with values in a Clifford algebra $\mathscr{A}_n$. Under some hypotheses, it is proved that the linear baundary value problem (where $\lambda f(x, W^+(x), W^-(x))\equiv g(x)$) has a unique solution and the nonlinear boundary value problem has at least one solution.

§ 1. Introduction

Let V_n be an n-dimensional real vector space with orthonormal basis $e_1=1$, e_2, $\cdots$, e_n. Let $\mathscr{A}_n$ be a Clifford algebra over V_n. Then an arbitrary element of the basis for $\mathscr{A}_n$ may be written as $e_A=e_{\alpha_1}e_{\alpha_2}\cdots e_{\alpha_h}$, where $A=\{\alpha_1, \alpha_2, \cdots, \alpha_h\}\subset\{1, 2, \cdots, n\}$ and $1\leqslant\alpha_1<\alpha_2<\cdots<\alpha_h\leqslant n$. The elements $\{e_j\}_{j=2}^n$ satisfy the relation $e_ie_j+e_je_i=-2\delta_{ij}$, where δ_{ij} is the Kroneker delta. Each of the elements in $\mathscr{A}_n$ may be written as $a=\sum_A a_A e_A$, where a_A are real numbers. We define that $|a|^2=\sum_A|a_A|^2$, then we have $|a+b|\leqslant|a|+|b|$, $|ab|\leqslant 2^{n-1}|a||b|$. It is clear that this algebra is incommutative.

Let x denote a point in R^n and $x=x_1e_1+x_2e_2+\cdots+x_ne_n$. We define $\bar{x}=x_1e_1-x_2e_2-\cdots-x_ne_n$, then $x\bar{x}=\bar{x}x=|x|^2$. If there exists y such that $xy=yx=1$, then x is said to be invertible and its inverse is written as $y=x^{-1}$. Obviously, for $x\neq0$, we have $x^{-1}=\frac{\bar{x}}{|x|^2}$.

Let D be an open connected set in R^n. The set of C^r-functions in D with values in $\mathscr{A}_n$ is denoted by $F_D^{(r)}=\{f\,|\,f: D\to\mathscr{A}_n,\ f(x)=\sum_A f_A(x)e_A,\ f_A(x)\in C^r(D)\}$. We

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define also the defferential operators $\bar{\partial}=e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}+\cdots+e_n\frac{\partial}{\partial x_n}$ and $\partial=e_1\frac{\partial}{\partial x_1}-e_2\frac{\partial}{\partial x_2}\cdots\cdots-e_n\frac{\partial}{\partial x_n}$. For $f\in F_D^{(r)}$ $(r\geqslant 1)$, $\bar{\partial}f=\sum_{\alpha,A}e_\alpha e_A\frac{\partial f_A}{\partial x_\alpha}$ and $f\bar{\partial}=\sum_{\alpha,A}e_A e_\alpha\frac{\partial f_A}{\partial x_\alpha}$. Note that the formal product of the operators $\bar{\partial}$ and ∂ is $\bar{\partial}\partial=\partial\bar{\partial}=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_n^2}=\Delta$, the Laplacian operator. We define that f is left regular in D if $\bar{\partial}f=0$ in D, in what follows, f is simply called a regular function. For $n=2$, the regular function is just a holomorphic function in the plane.

Up to now, quite a lot of function theory on regular function with values in $\mathscr{A}_n$ has been established*[1-9] However, the various boundary value problems have not been fully investigated. This paper deals with the Riemann problem** for regular function with values in $\mathscr{A}_n$. Let D^+ denote a simply connective bounded domain in R^n with boundary Γ which is a Liapunov surface. Let D^- denote the complementary space of $D^++\Gamma$. This problem is to find out a function $w(x)$, sectionally regular in the domain D^+ and D^-, whore boundary values $w^+(t)$ and $w^-(t)$ at each point of the boundary Γ satisfy the following relation

$$w^+(t)=G(t)w(t)+g(t),\quad t\in\Gamma \tag{1}$$

and the function $w(x)$ vanishes at the infinity. Here $G(t)$ and $g(t)$ are given Hölder continuous functions with values in $\mathscr{A}_n$ on Γ, i. e. they satisfy the following ine-qualities

$$|g(t)-g(\tilde{t})|=(\sum_A|g_A(t)-g_A(\tilde{t})|^2)^{1/2}\leqslant k|t-\tilde{t}|^\alpha,\quad 0<\alpha<1.$$

$$|G(t)-G(\tilde{t})|=(\sum_A|G_A(t)-G_A(\tilde{t})|^2)^{1/2}\leqslant k'|t-\tilde{t}|^\alpha,\quad t,\ \tilde{t}\in\Gamma, t\neq\tilde{t}.$$

where k and k' are positive constants which do not depend on t and $\tilde{t}$.

§ 2. Linear Riemann Problem

Let us define the Cauchy integral over Clifford algebra

$$w(x)=\frac{1}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}\,d\sigma_t\omega(t), \tag{2}$$

where $\omega_n=2\pi^{n/2}/\Gamma(n/2)$ is the area of a unit sphere in R^n, and $d\sigma_t=(e_1\cos(n,e_1)+e_2\cos(n,e_2)+\cdots+e_n\cos(n,e_n))ds_t$, in which ds_t is a differential of the area, and n denotes the exterior normal direction at the point t on Γ. Obviously, the integral (2) is a sectionally regular function in the domains D^+ and D^-. If $\omega(t)$ is a Hölder continuous function on Γ, then, for $x\in\Gamma$, the integral (2) is finite in the sense of Cauchy's principal values. In what follows, we refer the integral (2) to Cauchy's

*) As $n=3$, see [10].

**) As $n=3$, see [11]

principal value integral for $x\in\Gamma$. We have the following Plemelj formula[4].

Lemma 1 *If $w(t)$ is a Hölder continuous function on Γ, then both $w^+(x)$ and $w^-(x)$ are Hölder continuous functions on Γ, and the following equalities hold*

$$w^+(x)=\frac{1}{2}w(x)+\frac{1}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}d\sigma_t w(t),\quad x\in\Gamma, \tag{3}$$

$$w^-(x)=-\frac{1}{2}w(x)+\frac{1}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}d\sigma_t w(t),\ x\in\Gamma. \tag{4}$$

At first, we consider a special linear Riemann problem.

Theorem 1. *Let $G(t)$ be a Clifford constant G belonging to the center of $\mathscr{A}_n$ and it has an inverse G^{-1}. Then there exists a unique solution to the Riemann problem*

$$\bar\partial w(x)=0,\ x\in R^n\backslash\Gamma,$$

$$w^+(t)=Gw^-(t)+g(t),\ t\in\Gamma;\ w^-(\infty)=0,$$

and the solution may be represented by the following formula

$$w(x)=\frac{X(x)}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}d\sigma_t G^{-1}g(t), \tag{5}$$

where

$$X(x)=\begin{cases}G, & x\in D^+,\\ 1, & x\in D^-.\end{cases}$$

Proof Obviously, the function $w(x)$ determined by (5) is sectionally regular in D^+ and D^-, and it vanishes at infinity. From (3) and (4), we have

$$\begin{aligned}w^+(x)-Gw^-(x)&=\frac{X^+(x)}{2}G^{-1}g(x)+\frac{G}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}d\sigma_t G^{-1}g(t)\\&\quad -G\left[-\frac{X^-(x)}{2}G^{-1}g(x)+\frac{1}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}d\sigma_t G^{-1}g(t)\right]\\&=g(x),\ x\in\Gamma.\end{aligned}$$

Therefore, the function $w(x)$ is a solution to this problem. Conversely, if there is another solution $\tilde{w}(x)$ to this problem, then the function $w^*(x)=X^{-1}(x)(w(x)-\tilde{w}(x))$ is regular in $R^n\backslash\Gamma$ and $w^*(x)$ is continuous on Γ and vanishes at infinity. According to Liouville theorem[2], we have $w^*(x)=0$, i. e. $w(x)=\tilde{w}(x)$. Thus the theorem is proved.

Let $\mathscr{H}(\Gamma,\alpha)$ denote the set of Hölder continuous functions with values in $\mathscr{A}_n$ on Γ (the Hölder exponent is α, $0<\alpha<1$). We define the norm in $\mathscr{H}(\Gamma,\alpha)$ as

$$\|f\|_\alpha=C(f,\Gamma)+H(f,\Gamma,\alpha),$$

where

$$C(f,\Gamma)=\max_{t\in\Gamma}|f(t)|,\ H(f,\Gamma,\alpha)=\sup_{\substack{t\neq x\\ t,x\in\Gamma}}\frac{|f(t)-f(x)|}{|t-x|^\alpha},\ 0<\alpha<1.$$

It is easy to prove that $\mathscr{H}(\Gamma,\alpha)$ is a Banach space. And it is also not difficult to prove that $f(x)+g(x)$, $f(x)g(x)\in\mathscr{H}(\Gamma,\alpha)$ for $f(x)$, $g(x)\in\mathscr{H}(\Gamma,\alpha)$, and the following inequalities hold

$$\|f+g\|_\alpha\leqslant\|f\|_\alpha+\|g\|_\alpha,\ \|fg\|_\alpha\leqslant 2^{n-1}\|f\|_\alpha\|g\|_\alpha. \tag{6}$$

Here Γ is a Liapunov surface (See [4]). So there exist a number $d>0$ and a constant δ, $0<\delta\leqslant 1$. For any point $x\in\Gamma$, we construct a sphere K with centre at the point x of radius d in R^n, Γ' denotes the part of Γ lying inside the sphere K. We consider a rectangular coordinate system with origin x, and the direction of the positive x_n axes is taken to be the exterior normal direction at x to Γ. Then the surface Γ' may be represented in the form $\xi_n=\xi_n(\xi_1, \xi_2, \cdots, \xi_{n-1})$. We refer n to the exterior normal direction at ξ on Γ'. Then, for any point ξ on Γ', there exists a constant $\tilde{c}$ which does not depend on ξ such that

$$\begin{aligned}&\cos(n, x_n)\geqslant 1/2,\ 1-\cos(n, x_n)\leqslant\tilde{c}\cdot\rho^{2\delta},\\ &|\xi_n|\leqslant\tilde{c}\cdot\rho^{\delta+1},\ |\cos(n, x_k)|\leqslant\tilde{c}\cdot\rho\delta,\ k=1, 2, \cdots, n-1,\end{aligned}\tag{7}$$

where ρ is the length of projection of $|x_\xi|$ onto the plane $x_n=0$.

Lemma 2. *The integral operator K*

$$(K\omega)(x)=\frac{2}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}d\sigma_t\omega(t),\ x\in\Gamma$$

is a bounded linear operator mapping from the function spaee $\mathscr{H}(\Gamma, \alpha)$ into itself, i. e. for any $\omega(t)\in\mathscr{H}(\Gamma, \alpha)$, there exsits a positive constant c which does not depend on ω such that

$$\|K\omega\|_\alpha\leqslant C\cdot\|\omega\|_\alpha.$$

Proof Since the Cauchy principal value integral $\frac{2}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}d\sigma_t=1$, for $x\in\Gamma$, we have

$$\begin{aligned}|(K\omega)(x)|&=\left|\frac{2}{w_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-\bar{x}|^n}d\sigma_t(\omega(t)-\omega(x))+\omega(x)\right|\\ &\leqslant c_1'H(\omega, \Gamma, \alpha)\int_\Gamma\frac{1}{|t-x|^{n-1-\alpha}}ds_t+|\omega(x)|\\ &\leqslant c_1H(\omega, \Gamma, \alpha)+c(\omega, \Gamma),\end{aligned}\tag{8}$$

where c_1 is a constant independent of ω. Now we estimate $H(K\omega, \Gamma, \alpha)$. For any $x, \tilde{x}\in\Gamma$, denote $\eta=|x-\tilde{x}|$, we suppose that $4\eta\leqslant d$ at first. $\Gamma'_{2\eta}$ denotes the part of Γ lying inside a sphere of radius 2η with centre at the point x. Denote $\Gamma_{2\eta}=\Gamma-\Gamma'_{2\eta}$. Then we have

$$\begin{aligned}|(K\omega)(x)-(K\omega)(\tilde{x})|\leqslant&\left|\frac{2}{\omega_n}\int_{\Gamma'_{2\eta}}\frac{\bar{t}-\bar{x}}{|t-x|^n}d\sigma_t(\omega(t)-\omega(x))\right|\\ &+\left|\frac{2}{\omega_n}\int_{\Gamma'_{2\eta}}\frac{\bar{t}-\bar{\tilde{x}}}{|t-\tilde{x}|^n}d\sigma_t(\omega(t)-\omega(\tilde{x}))\right|\\ &+\left|\frac{2}{\omega_n}\int_{\Gamma_{2\eta}}\frac{\bar{t}-\bar{x}}{|t-x|_n}d\sigma_t(\omega(t)-\omega(x))\right.\\ &\left.-\frac{2}{\omega_n}\int_{\Gamma_{2\eta}}\frac{\bar{t}-\bar{\tilde{x}}}{|t-\tilde{x}|^n}d\sigma_t(\omega(t)-\omega(\tilde{x}))\right|\\ &+|\omega(x)-\omega(\tilde{x})|\\ =&J_1+J_2+J_3+|\omega(x)-\omega(\tilde{x})|.\end{aligned}$$

From (7), we have

$$J_1 \leqslant c_2' \cdot H(\omega, \Gamma, \alpha) \cdot \int_{\Gamma_{2\eta}'} \frac{ds_t}{|t-x|^{n-1-\alpha}}$$

$$= c_2' \cdot H(\omega, \Gamma, \alpha) \int_{\Sigma_{2\eta}'} \frac{1}{|t-x|^{n-1-\alpha}} \frac{d\xi_1 \cdots d\xi_{n-1}}{\cos(n, x_n)} \leqslant c_2'' \cdot H(\omega, \Gamma, \alpha) \int_0^{2\eta} \frac{\rho^{n-2}}{\rho^{n-1-\alpha}} d\sigma$$

$$= c_2 \cdot H(\omega, \Gamma, \alpha) \cdot \eta^\alpha = c_2 \cdot H(\omega, \Gamma, \alpha) \cdot |x-\tilde{x}|^\alpha,$$

where $\Sigma_{2\eta}'$ is the projective domain of $\Gamma_{2\eta}'$ onto the tangent plane at x, and c_2 is a constant which does not depend on x and $\tilde{x}$. Similarly we may estimate J_2, where we consider the integral region which is the part of Γ lying inside the sphere of radius 4η with centre at the point $\tilde{x}$ instead of the original one. Then we may also have $J_2 \leqslant c_3 H(\omega, \Gamma, \alpha) |x-\tilde{x}|^\alpha$, where c_3 does not depend on x and $\tilde{x}$. Now we estimate J_3.

$$J_3 = \left| \frac{2}{\omega_n} \int_{\Gamma_{2\eta}} \left(\frac{\bar{t}-\bar{x}}{|t-x|^n} - \frac{\bar{t}-\bar{\tilde{x}}}{|t-\tilde{x}|^n} \right) d\sigma_t (\omega(t) - \omega(\tilde{x})) \right.$$

$$\left. - \frac{2}{\omega_n} \int_{\Gamma_{2\eta}} \frac{\bar{t}-\bar{x}}{|t-x|^n} d\sigma_t |\omega(x) - \omega(\tilde{x})) \right|$$

$$\leqslant \left| \frac{2}{\omega_n} \int_{\Gamma_{2\eta}} \left(\frac{\bar{t}-\bar{x}}{|t-x|^n} - \frac{\bar{t}-\bar{\tilde{x}}}{|t-\tilde{x}|^n} \right) d\sigma_t (\omega(t) - \omega(\tilde{x})) \right|$$

$$+ \left| \frac{2}{\omega_n} \int_{\Gamma_{2\eta}} \frac{\bar{t}-\bar{x}}{|t-x|^n} d\sigma_t (\omega(x) - \omega(\tilde{x})) \right|.$$

It is clear that the second term of the right hand side of the above formula is not greater than $c_4' H(\omega, \Gamma, \alpha) |x-\tilde{x}|^\alpha$, where c_4' is a positive constant which does not depend on x and $\tilde{x}$. By [2], we have

$$\left| \frac{\bar{t}-\bar{x}}{|t-x|^n} - \frac{\bar{t}-\bar{\tilde{x}}}{|t-\tilde{x}|^n} \right| \leqslant \frac{\sum_{k=0}^{n-2} |t-x|^{n-2-k} |t-\tilde{x}|^k}{|t-x|^{n-1} \cdot |t-\tilde{x}|^{n-1}} \cdot |x-\tilde{x}|$$

$$= \left(\sum_{k=0}^{n-2} |t-x|^{-1-k} |t-\tilde{x}|^{k-n+1} \right) |x-\tilde{x}|.$$

Since $\frac{2}{3} \leqslant \left| \frac{t-x}{t-\tilde{x}} \right| \leqslant 2$ for $t \in \Gamma_{2\eta}$, we have $|t-\tilde{x}| \geqslant \frac{1}{2} |t-x|$. Then

$$J_3 \leqslant c_4'' \cdot H(\omega, \Gamma, \alpha) \int_{\Gamma_{2\eta}} |t-x|^{-n+\alpha} ds_t \cdot |x-\tilde{x}| + c_4' \cdot H(\omega, \Gamma, \alpha) |x-\tilde{x}|^\alpha$$

$$\leqslant c_4 H(\omega, \Gamma, \alpha) \cdot |x-\tilde{x}|^\alpha,$$

where c_4 is a constant which does not depend on x and $\tilde{x}$. Thus, for $4|x-\tilde{x}| < d$, we have

$$|(K\omega)(x) - (K\omega)(\tilde{x})| \leqslant c' \cdot H(\omega, \Gamma, \alpha) \cdot |x-x|^\alpha,$$

where c' is a constant which does not depend on x and $\tilde{x}$. Obviously, for $4|x-\tilde{x}| \geqslant d$, there exsists an estimation similar to the above one. From (8) and the above estimation, the lemma is proved.

Theorem 2. *Suppose that $G(x)$ and $g(x)$ are given Hölder continuous function on Γ with values in $\mathscr{A}_n$, and $G(x)$ satisfies the following condition:*

$$\beta = 2^{n-2} \cdot \|1 - G(x)\|_\alpha \cdot (c+1) < 1, \tag{9}$$

where c is a positive constant mentioned in Iemma 2. Then there exsits a unique solution to the linear Riemann problem

$$\bar{\partial} w(x)=0,\quad x\in R^n\backslash\Gamma, \tag{10}$$

$$w^+(t)=G(t)w^-(t)+g(t),\quad t\in\Gamma;\ w^-(\infty)=0. \tag{11}$$

Proof The solution to this problem may be written in the form

$$w(x)=\frac{1}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}\,d\sigma_t\omega(t).$$

where $\omega(t)$ is a Hölder continuous function to be determined on Γ. Then, from Lemma 1, the Riemann problem (10)-(11) can be reduced to an equivalent singular integral equation for $\omega(t)$,

$$\omega(x)=\frac{1-G(x)}{2}\Big[\omega(x)-\frac{2}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}\,d\sigma_t\omega(t)\Big]+g(x),\quad x\in\Gamma. \tag{12}$$

Let A denote an integral operator defined by the right hand side of (12), i. e.

$$(A\omega)(x)=\frac{1-G(x)}{2}[\omega(x)-(K\omega)(x)]+g(x).$$

From Lemma 2 and (6) and (9), the integral operator A is a contraction operator mapping the Banach space $\mathscr{H}(\Gamma,\alpha)$ into itself, therefore there is a unique fixed point for the operator A. Thus there exists a unique solution to (12) and the theorem is proved.

§ 3. Nonlinear Riemenn Problem

Now we consider the nonlinear Riemann problem

$$\begin{aligned}&\bar{\partial} w(x)=0,\ x\in R^n\backslash\Gamma,\\ &w^+(x)=G(x)w^-(x)+\lambda f(x,w^+(x),w^-(x)),\ x\in\Gamma;\ w^-(\infty)=0,\end{aligned} \tag{13}$$

where λ is a real parameter, $G(x)$ is a given Hölder continuous function with values in $\mathscr{A}_n$ determined on Γ, and $f(x,w^{(1)},w^{(2)})$ is a given function with values in $\mathscr{A}_n$ determined on $\Gamma\times\mathscr{A}_n\times\mathscr{A}_n$ and $f(x,0,0)=0$. For any Clifford numbers $w^{(1)}$ and $w^{(2)}$, the function $f(x,w^{(1)},w^{(2)})$ is a Hölder continuous function for $x\in\Gamma$, and for any point $x\in\Gamma$, the function $f(x,w^{(1)},w^{(2)})$ satisfies the Lipschitz condition with respect to the last two variables, i. e.

$$\begin{aligned}&|f(x,w^{(1)},w^{(2)})-f(\tilde{x},\tilde{w}^{(1)},\tilde{w}^{(2)})|\\ &\leqslant l_0|x-\tilde{x}|^\alpha+l_1|w^{(1)}-\tilde{w}^{(1)}|+l_2|w^{(2)}-\tilde{w}^{(2)}|,\ 0<\alpha<1,\end{aligned} \tag{14}$$

where l_0, l_1 and l_2 are positive constants which do not depend on x, $\tilde{x}$, $w^{(1)}$, $\tilde{w}^{(1)}$, $w^{(2)}$ and $\tilde{w}^{(2)}$.

We can represent the solution in the form $w(x)=\frac{1}{\omega_n}\int_\Gamma\frac{\bar{t}-\bar{x}}{|t-x|^n}\,d\sigma_t\omega(t)$, where $\omega(t)$ is a Hölder continuous function with values in $\mathscr{A}_n$ to be determined on Γ. Then the nonlinear Riemann problem may be reduced to an equivalent nonlinear singular integral equation

$$\omega(x)=\frac{1-G(x)}{2}\Big[\omega(x)-\frac{2}{\omega_n}\int_\Gamma\frac{\bar t-\bar x}{|t-x|^n}d\sigma_t\omega(t)\Big]$$
$$+\lambda f\left(x,\ \frac{\omega(x)}{2}+\frac{1}{\omega_n}\int_\Gamma\frac{\bar t-\bar x}{|t-x|^n}d\sigma_t\omega(t),\right.$$
$$\left.-\frac{\omega(x)}{2}+\frac{1}{\omega_n}\int_\Gamma\frac{\bar t-\bar x}{|t-x|^n}d\sigma_t\omega(t)\right),\ x\in\Gamma \tag{15}$$

Let Bw denote an integral operator defined by the right hand side of (15), i. e.

$$B\omega=\frac{1-G(x)}{2}[\omega(x)-(K\omega)(x)]$$
$$+\lambda f\left(x,\ \frac{\omega(x)}{2}+\frac{1}{\omega_n}\int_\Gamma\frac{\bar t-\bar x}{|t-x|^n}d\sigma_t\omega(t),\right.$$
$$\left.-\frac{\omega(x)}{2}+\frac{1}{\omega_n}\int_\Gamma\frac{\bar t-\bar x}{|t-x|^n}d\sigma_t\omega(t)\right).$$

Now we consider the operator B in the continuous function space $C(\Gamma)$. In the function space $C(\Gamma)$, the norm is defined as

$$\|\omega(x)\|=C(\omega,\Gamma)=\max_{x\in\Gamma}|\omega(x)|,$$

and $C(\Gamma)$ is a Banach space. Let M denote a subset in $C(\Gamma)$,

$$M=\{\omega(x)\,|\,(\omega x)\in\mathscr{H}(\Gamma,\alpha),\ \|\omega\|_\alpha\leqslant l\},,$$

where l is a constant, and then M is a convex closed set in $C(\Gamma)$. Suppose that the function $G(x)$ satisfies the condition (9). From (9), (14) and (7), it is not difficult to find out that for any $\omega\in M$, the following inequality holds

$$\|B\omega\|_\alpha\leqslant\beta\|\omega\|_\alpha+|\lambda|(\beta'\|\omega\|_\alpha+l_0), \tag{16}$$

where β is a positive constant mentioned in Theorem 2, and $\beta'=\frac{1+c}{2}(l_1+l_2)$. If $|\lambda|$ is so small that

$$|\lambda|<\frac{l(1-\beta)}{l_0+\beta' l}, \tag{17}$$

then the operator B maps the set M into itself. Now we proceed to prove that the operator B is a continuous mapping. Suppose that the sequence of functions $\{\omega^{(n)}(x)\}$ is uniformly convergent to a function $\omega(x)$, where each $\omega^{(n)}(x)\in M$. We have to prove that the sequence of functions $\{K\omega^{(n)}\}$ is also uniformly convergent to the function $K\omega$. For any $x\in\Gamma$, $\Gamma'_{2\eta}$ denotes the part of Γ lying inside a sphere of radius 2η with centre at the point x, $\Gamma_{2\eta}=\Gamma-\Gamma'_{2\eta}$. Then

$$|(K\omega^{(n)})(x)-(K\omega)(x)|\leqslant\left|\frac{2}{\omega_n}\int_{\Gamma'_{2\eta}}\frac{\bar t-\bar x}{|t-x|^n}d\sigma_t(\omega^{(n)}(t)-\omega(t))\right|$$
$$+\left|\frac{2}{\omega_n}\int_{\Gamma_{2\eta}}\frac{\bar t-\bar x}{|t-x|^n}d\sigma_t(\omega^{(n)}(t)-\omega(t))\right|=J_4+J_5.$$

Suppose $2\eta<d$. From (7) we have

$$J_4=\left|\frac{2}{\omega_n}\int_{\Gamma_{2\eta}^{i}}\frac{\bar{t}-\bar{x}}{|t-x|^n}\,d\sigma_t(\omega^{(n)}(t)-\omega^{(n)}(x))-\frac{2}{\omega_n}\int_{\Gamma_{2\eta}^{i}}\frac{\bar{t}-\bar{x}}{|t-x|^n}\,d\sigma_t(\omega(t)-\omega(x))\right.$$
$$\left.+\frac{2}{\omega_n}\int_{\Gamma_{2\eta}^{i}}\frac{\bar{t}-\bar{x}}{|t-x|^n}\,d\sigma_t(\omega^{(n)}(x)-\omega(x))\right|$$
$$\leqslant\left|\frac{2}{\omega_n}\int_{\Gamma_{2\eta}^{i}}\frac{\bar{t}-\bar{x}}{|t-x|^n}\,d\sigma_t(\omega^{(n)}(t)-\omega^{(n)}(x))\right|$$
$$+\left|\frac{2}{\omega_n}\int_{\Gamma_{2\eta}^{i}}\frac{\bar{t}-\bar{x}}{|t-x|^n}\,d\sigma_t\,(\omega(t)-\omega(x))\right|$$
$$+2^{n-1}\left|\frac{2}{\omega_n}\int_{\Gamma_{2\eta}^{i}}\frac{\bar{t}-\bar{x}}{|t-x|^n}\,d\sigma_t\right|\cdot|\omega^{(n)}(x)-\omega(x)|$$
$$\leqslant c_5'\cdot l\cdot\int_0^{2\eta}\frac{1}{\rho^{n-1-\alpha}}\,\rho^{n-2}d\rho+c_5''\cdot l\cdot\int_0^{2\eta}\frac{\rho^{1+\delta}}{\rho^n}\,\rho^{n-2}d\rho=c_6'\cdot\eta^{\alpha}+c_6''\cdot\eta^{\delta},$$

where c_6' and c_6'' are constants which do not depend on $\omega^{(n)}$, ω, x and η. Therefore we can let η be so small that $J_4<\varepsilon/2$. Obviously, $J_5\leqslant c_7\cdot\|\omega^{(n)}-\omega\|$, where c_7 is a constant which does not depend on $\omega^{(n)}$, ω and x, but depenas on η. For the above η, according to the uniformly convergence of the sequence of functions $\{\omega^{(n)}(t)\}$, there is a number N so that for any $n>N$ we have $J_5<\varepsilon/2$. Thus, for $n>N$, $|(K\omega^{(n)}(x)-(K\omega)(x)|<\varepsilon$. So the operator B is a continuous mapping from M into itself. According to Arzela–Ascoli theorem, the set M is a compact set in the space $C(\Gamma)$. Thus the continuous operator B maps the convex closed subset M of $C(\Gamma)$ into itself, and $B(M)$ is a compact set in the space $C(\Gamma)$. Based on Schauder's fixed point principle, there at least exists a Hölder continuous solution to the nonlinear singular integral equation (15). Thus we have proved the following theorem.

Theorem 3. *Suppose that the function $f(x, w^{(1)}, w^{(2)})$ satisfies the condition* (14), *and $G(t)$ satisfies the condition* (9). *If λ satisfies* (17), *then the nonlinear Riemann problem* (13) *has at least one solution.*

We can also prove that there exists a unique solution to the above nonlinear Riemann problem, if some other suitable conditions are imposed on the function $f(x, w^{(1)}, w^{(2)})$.

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