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ON THE LORENTZ CONJECTURES UNDER THE L_1-NORM

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Abstract

Let $f(x)\in C[-1, 1]$, $p_n^*(x)$ be the best approximation polynomial of degree n to $f(x)$. G. Lorentz conjectured that if for all n, $p_{2n}^*(x)=p_{2n+1}^*(x)$, then f is even; and if $p_{2n+1}^*(x)=p_{2n+2}^*(x)$, $p_0^*(x)\equiv 0$, then f is odd.

In this paper, it is proved that, under the L_1-norm, the Lorentz conjecture is valid conditionally, i. e. if (i) $(1-x^2)f(x)$ can be extended to an absolutely convergent Tchebyshev series; (ii) for every n, $f(x)-p_{2n+1}^*(x)$ has exactly $2n+2$ zeros (or, in the second situation, $f(x)-p_{2n+2}^*(x)$ has exactly $2n+3$ zeros), then Lorentz conjecture is valid.

Let f be a function of $C[-1, 1]$, π_n be the set of polynomials of degree n, and $p_n^*(f, x)$ the best approximation to f in π_n. The follwing conjectures were proposed by G. G. Lorentz[1]:

Conjecture 1. *Suppose that $f\in C[-1, 1]$. If for all $n\geqslant 0$*

$$p_{2n}^*(f, x)=p_{2n+1}^*(f, x), \tag{1}$$

then f is even.

Conjecture 2. *Suppoes that $f\in C[-1, 1]$. If for all $n\geqslant 0$*

$$p_{2n+1}^*(f, x)=p_{2n+2}^*(f, x),$$

then f is odd.

In practice, there is something wrong with Conjecture 2. And E. Saff and R. Varga[2] added a condition to it, that is

Conjecture 2′. *Suppose that $f\in C[-1, 1]$, and $f(0)=0$. If for all $n\geqslant 0$*

$$p_{2n+1}^*(f, x)=p_{2n+2}^*(f, x),$$

then f is odd.

We think it might be more reasonable if we modify the conjecture as follows:

Conjecture 2″. *Suppose that $f\in C[-1, 1]$. If for all $n\geqslant 0$*

$$p_{2n+1}^*(f, x)=p_{2n+2}^*(f, x) \quad \text{and} \quad p_0^*(f, x)\equiv 0, \tag{2}$$

then f is odd.

Though Conjectures 1 and 2 were published in many conferences and papers, the

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answers are very few. The following result was given by E. Saff and R. Varga[2].

If the function $F(t)$ has an analytic extension $F(z)$ which is an entire function of exponential type τ with $0<\tau<\pi/2$, i. e.

$$\limsup_{r\to\infty}\frac{\ln M_F(r)}{r}=\tau,\quad M_F(r)=\max\{|F(z)|,\ |z|=r\},$$

then Conjecture 1 (or 2′) is valid. Here, for Conjecture 1, $F(t)$ is defined by $f(x)+f(-x)=F(x^2)$, and for Conjecture 2′, by $f(x)-f(-x)=xF(x^2)$.

It is not difficult to get that Conjectures 1 and 2″ are true under the L_2-norm.

In this paper we prove that, under the L_1-norm, the Lorentz conjectures 1 and 2″ are valid conditionally.

Suppose that $f(x)\in C[-1, 1]$, and $p_n^*(x)\in\pi_n$ is the best approximation to $f(x)$ under the L_1-norm. We denote the error function by $e_n(x)=f(x)-p_n^*(x)$.

Besides, a Tchebyshev series means the sum from Tchebyshev polynomials, $\sum\limits_{n=0}^{\infty}a_nT_n(x)$, where $T_n(x)$ is the Tchebyshev polynomial: $T_n(x)=\cos(n\arccos x)$.

Theorem 1. *Suppose that $f(x)\in C[-1, 1]$. If*

(i) *$(1-x^2)f(x)$ can be extended to an absolutely convergent Tchebyshev series, i. e.*

$$(1-x^2)f(x)=\sum_{n=0}^{\infty}a_nT_n(x),\quad x\in[-1, 1] \tag{3}$$

and

$$\sum_{n=0}^{\infty}|a_n|<\infty; \tag{4}$$

(ii) *for every $n=0, 1, \cdots$, the error function $e_{2n+1}(x)$ has exactly $2n+2$ zeros; then the Lorentz conjecture 1 is valid under the L_1-norm.*

Theorem 2. *Suppose that $f(x)\in C[-1, 1]$. If*

(i) *$(1-x^2f)(x)$ can be extended to an absolutely convergent Tchebyshev series;*

(ii) *for every $n=0, 1, \cdots$, the error function $e_{2n+2}(x)$ has exactly $2n+3$ zeros; then the Lorentz conjecture 2″ is valid under the L_1-norm.*

The Proof of the Theorem

Here we only give the proof of Theorem 1. The proof of Theorem 2 is similar.

Lemma 1[3]. *Let $f(x)\in C[-1, 1]$, $p_n^*(x)\in\pi_n$ be the best approximation to $f(x)$ under the L_1-norm. If the error function $e_n(x)$ has exactly $n+1$ zeros $\xi_1, \xi_2, \cdots, \xi_{n+1}$, then*

$$\xi_i=\cos\frac{i\pi}{n+2},\quad i=1, 2, \cdots, n+1. \tag{4}$$

Now we begin to prove Theorem 1.

Suppose that $f(x)\in C[-1, 1]$, $p_{2n}^*(f, x)=p_{2n+1}^*(f, x)$. Denote

$$p_{2n}^*(f, x)=b_0+b_1x+\cdots+b_{2n}x^{2n}.$$

By Lemma 1, for $\xi_i=\cos\frac{i\pi}{2n+3}$ we have

$$b_0+b_1\xi_i+\cdots+b_{2n}\xi_i^{2n}=f(\xi_i),\ i=1,\ 2,\ \cdots,\ 2n+2.$$

This can be regarded as a linear system with $2n+1$ unknows and $2n+2$ equations. Since this system has a non-trivial solution, the determinant

$$\begin{vmatrix} 1 & \xi_1 & \cdots & \xi_1^{2n} & f(\xi_1) \\ 1 & \xi_2 & \cdots & \xi_2^{2n} & f(\xi_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \xi_{2n+2} & \cdots & \xi_{2n+2}^{2n} & f(\xi_{2n+2}) \end{vmatrix}=0.$$

This means that the divided difference of $f(x)$

$$[\xi_1,\ \xi_2,\ \cdots,\ \xi_{2n+2}]f=0. \tag{6}$$

Now we assert that if $\eta_i=\cos\frac{i\pi}{m+1}$ $(i=0,\ 1,\ \cdots,\ m+1)$, then

$$[\eta_1,\ \eta_2,\ \cdots,\ \eta_m]f=\frac{2^m}{m+1}\sum_{i=1}^{m}(-1)^{i+1}(1-\eta_i^2)f(\eta_i). \tag{7}$$

In fact, from the definition of the divided diffrence we know that

$$[\eta_1,\ \eta_2,\ \cdots,\ \eta_m]f=\sum_{i=1}^{m}\frac{f(\eta_i)}{\omega'(\eta_i)},$$

where $\omega(x)=\prod_{j=0}^{m}(x-\eta_j)$. Note that $\{\eta_i\}$ are all the m zeros of the polynomial $T'_{m+1}(x)$, so we have

$$\omega(x)=\frac{1}{2^m(m+1)}\ T'_{m+1}(x).$$

Because the Tchebyshev polynomial $T_m(x)$ satisfies the differential equation[4]

$$(1-x^2)T''_m(x)-xT'_m(x)+m^2T_m(x)=0,$$

it follows that

$$\omega'(x)=\frac{xT'_{m+1}(x)-(m+1)^2T_{m+1}(x)}{2^m(m+1)(1-x^2)}.$$

Finally we get

$$[\eta_1,\ \eta_2,\ \cdots,\ \eta_m]f=\frac{2^m}{m+1}\sum_{i=1}^{m}(-1)^{i+1}(1-\eta_i^2)f(\eta_i),$$

and (7) is obtained.

When $(1-x)^2f(x)$ can be extended to an absolutely Tchebyshev series, i.e.

$$\bar{f}(x)=(1-x^2)f(x)=\sum_{j=0}^{\infty}a_jT_j(x),$$

if we denote $\sum_{i=0}^{m+1}{}''\ u_i=\frac{1}{2}\ (u_0+u_{m+1})+\sum_{i=1}^{m}u_i$, from (7) it follows that

$$[\eta_1,\ \eta_2,\ \cdots,\ \eta_m]f=\frac{2^m}{m+1}\sum_{i=0}^{m+1}{}''(-1)^{i+1}\bar{f}(\eta_i)=-\frac{2^m}{m+1}\sum_{i=0}^{m+1}{}''(-1)^i\sum_{j=0}^{\infty}a_jT_j(\eta_i)$$

$$=-\frac{2^m}{m+1}\sum_{j=0}^{\infty}a_j\sum_{i=0}^{m+1}{}''\,T_{m+1}(\eta_i)T_j(\eta_i). \tag{8}$$

According to [4],

$$\frac{2^m}{m+1}\sum_{i=0}^{m+1}{}''T_{m+1}(\eta_i)T_j(\eta_i)=\begin{cases}2^m, & j=(2k+1)(m+1),\ k=0,\ 1,\ \cdots,\\ 0, & \text{otherwise.}\end{cases}$$

Hence we get

$$[\eta_1,\ \eta_2,\cdots,\ \eta_m]f=-2^m\sum_{k=0}^{\infty}a_{(2k+1)(m+1)}. \tag{9}$$

Let $L_m(\bar{f})=\sum_{k=0}^{\infty}a_{(2k+1)m}$, from (6) it follows that

$$L_{2n+3}(\bar{f})=0,\ n=0,\ 1,\ \cdots. \tag{10}$$

And it is not difficult to know that $L_1(\bar{f})=0$.

Let $\mu(i)$ be the Möbius function. From (4) and (10) we have

$$0=\sum_{i=0}^{\infty}\mu(2i+1)L_{(2n+1)(2i+1)}(\bar{f})=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\mu(2i+1)a_{(2n+1)(2k+1)(2i+1)}$$

$$=\sum_{m=0}^{\infty}a_{(2n+1)(2m+1)}\sum_{(2i+1)\mid(2m+1)}\mu(2i+1)=a_{2n+1}$$

for all $n=0,\ 1,\ \cdots$.

This means that

$$(1-x^2)f(x)=\sum_{n=0}^{\infty}a_{2n}T_{2n}(x).$$

Hence $f(x)$ is even. Theorem 1 is proved.

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