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LOCAL LIKELIHOOD METHOD IN THE PROBLEMS RELATED TO CHANGE POINTS

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Abstract

In this paper, the so-called local likelihood method is suggested for solving the change point problems when the data are distributed as multivariate normal. The detection procedures proposed not only provide strongly consistent estimates for the number and locations of the change points, but also simplify significantly the computation.

§1. Introduction

Recently, considerable attention has been devoted to change point problems. These problems originally arise in various fields such as economics and industrial quality control. They are also related to the problem of edge detection (see Mazumdar, Sinha and Li (1985)).

When the underlying distribution of the data is normal, the change points are characterized by those of the mean function and/or covariance matrix function. In Section 2, we list five cases we are interested in.

Some authors studied the change points of the mean function or regression function when the data are distributed as normal. The related literature is quite extensive. Most of these authors only considered the cases when there is at most one change point or the number of the change points is known. Here we quote Hinkley (1970), Sen and Srivastava (1973, 1975), Vostrikova (1981) and Schulze (1982).

Generally, the number of change points may be unknown. For this case, Krishnaiah, Miao and Zhao (1986a, 1986b) investigated the detection of the number and locations of change points for the mean functions using information theoretic criteria. However, the computations of these methods are too much when there are many change points. Later, Yin (1986) proposed a nonparametric method so as to save the computation.

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In this paper, we use so-called local likelihood to detect the number and locations of the change points when the data are distributed as multivariate normal. One of the conspicuous merits of this method is that it has much less computation in applications. Besides, the detection pro-cedures are strongly consistent.

The problems are proposed in Section 2 and the various cases are treated in Sections 3—6 respectively.

§ 2. The Problems Considered

Let $\underset{\sim}{X}(t)$ be an independent p-dimensional process on $(0, 1]$ such that

$$\underset{\sim}{X}(t)=\underset{\sim}{\mu}(t)+\underset{\sim}{V}(t),\ 0<t\leqslant 1, \tag{2.1}$$

where $\underset{\sim}{V}(t)$ is an independent p-variate normal process with means $\underset{\sim}{0}$ and covariances $\Lambda(t)>0$, $\underset{\sim}{\mu}(t)$ is a $p\times 1$ real vector function. The problems to be considered can be formulated as the following five forms.

Case 1. $\underset{\sim}{\mu}(t)$ is a left-continuous step function with the jump points $t_1<\cdots<t_q$, which are called the change points of the mean function of $\underset{\sim}{X}(t)$, and $\Lambda(t)=\Lambda>0$ for $t\in(0, 1]$, where Λ is an unknown matrix.

Case 2. $\underset{\sim}{\mu}(t)$ is a left-continuous step function with the change points mentioned above. Write $t_0=0<t_1<\cdots<t_q<t_{q+1}=1$. We assume that $\Lambda(t)=\Lambda_j>0$ for $t\in(t_{j-1}, t_j]$, $j=1, 2, \cdots, q+1$, and $\Lambda_1, \cdots, \Lambda_{q+1}$ are all unknown.

Case 3. $\Lambda(t)>0$ is a left-continuous step function with the jump points $t_1<\cdots<t_q$, which are called the change points of the covariance function of $\underset{\sim}{X}(t)$, and $\underset{\sim}{\mu}(t)=\underset{\sim}{\mu}$ for $t\in(0, 1]$, where $\underset{\sim}{\mu}$ is unknown.

Case 4. $p=1$, $\underset{\sim}{\mu}(t)$ is a left-continuous step function with the jump points $t_1<\cdots<t_q$, and $\Lambda(t)=\lambda>0$ for $t\in(0, 1]$. Write $0=t_0<t_1<\cdots<t_q<t_{q+1}=1$, we assume that $\underset{\sim}{\mu}(t)=\mu_j$ for $t\in(t_{j-1}, t_j]$ and $\mu_1>\mu_2>\cdots>\mu_q>\mu_{q+1}$.

Case 5. $p=1$, $\mu(t)$ satisfies the conditions in the Case 4, and $\Lambda(t)=\lambda_j>0$ for $t\in(t_{j-1}, t_j]$.

In the above cases, we assume that the number q of the change points is unknown. We want to determine q and the locations of $t_1, t_2, \cdots, t_q$, based on the data $\underset{\sim}{X}(i/N)$, $i=1, 2, \cdots, N$.

§ 3. The Case 1

Put $\underset{\sim}{X}_i=\underset{\sim}{X}(i/N)$. Take a positive integer $m<N$. For $k=m, m+1, \cdots, N-m$, consider $\underset{\sim}{X}_{k-m+1}, \underset{\sim}{X}_{k-m+2}, \cdots, \underset{\sim}{X}_k, \underset{\sim}{X}_{k+1}, \cdots, \underset{\sim}{X}_{k+m}$. First we assume that $\underset{\sim}{X}_i$, $i=k-m+1, \cdots, k+m$, are independent p-variate normal variables such that

$$E\underset{\sim}{X}_{k-m+1}=\cdots=E\underset{\sim}{X}_k=\underset{\sim}{\mu}^{(1)},\quad E\underset{\sim}{X}_{k+1}=\cdots=E\underset{\sim}{X}_{k+m}=\underset{\sim}{\mu}^{(2)}, \tag{3.1}$$

and

$$\operatorname{Var}(\underset{\sim}{X}_{k-m+1})=\cdots=\operatorname{Var}(\underset{\sim}{X}_{k+m})=\Lambda>0, \tag{3.2}$$

where $\underset{\sim}{\mu}^{(1)}$, $\underset{\sim}{\mu}^{(2)}$ and Λ are all unknown. Put

$$\begin{gathered}\overline{\underset{\sim}{X}}_{(1,k)}=\frac{1}{m}\Sigma_{(1,k)}\underset{\sim}{X}_i,\quad \overline{\underset{\sim}{X}}_{(2,k)}=\frac{1}{m}\Sigma_{(2,k)}\underset{\sim}{X}_i,\\ \overline{\underset{\sim}{X}}_{(k)}=\frac{1}{2m}\Sigma_{(k)}\underset{\sim}{X}_i,\end{gathered} \tag{3.3}$$

where $\Sigma_{(1,k)}=\sum_{i=k-m+1}^{k}$, $\Sigma_{(2,k)}=\sum_{i=k+1}^{k+m}$, $\Sigma_{(k)}=\sum_{i=k-m+1}^{k+m}$. Then, the logarithm of the likelihood ration test (LRT) statistic for testing the null hypothesis H_k: $\underset{\sim}{\mu}^{(1)}=\underset{\sim}{\mu}^{(2)}$ against the alternative K_k: $\underset{\sim}{\mu}^{(1)}\neq\underset{\sim}{\mu}^{(2)}$ is given by

$$G_N(k)=m\log|A_{k,N}|-m\log|B_{k,N}|, \tag{3.4}$$

and $A_{k,N}=\frac{1}{2m}[\Sigma_{(1,k)}(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(1,k)})(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(1,k)})'+\Sigma_{(2,k)}(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(2,k)})(\underset{\sim}{X}_i-\underset{\sim}{X}_{i(,k)})']$,

$$B_{k,N}=\frac{1}{2m}\Sigma_{(k)}(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(k)})(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(k)})'. \tag{3.5}$$

Take $m=m_N$ and positive number C_N such that

$$N\gg m\gg C_N\gg\log N. \tag{3.6}$$

Hereafter, $\alpha_N\gg\beta_N$ means $\lim_{N\to\infty}\alpha_N/\beta_N=\infty$.

Define

$$D_N=\{k:\ k=m,\ m+1,\ \cdots,\ N-m,\ -G_N(k)>C_N\}. \tag{3.7}$$

Assume that $k_{1,N}=\min\{k:\ k\in D_N\}$. Define

$$D_{1,N}=\{k:\ k\in D_N,\ k-k_{1,N}<3m\}. \tag{3.8}$$

Assume that $k_{2,N}=\min\{k:\ k\in D_N-D_{1,N}\}$. Then we define

$$D_{2,N}=\{k:\ k\in D_N-D_{1,N},\ k-k_{2,N}<3m\}. \tag{3.9}$$

Continuing this procedure, we obtain

$$D_N=D_{1,N}+D_{2,N}+\cdots+D_{\hat{q},N}, \tag{3.10}$$

and every $D_{j,N}$ is not empty. Put

$$\hat{t}_j=\frac{1}{2N}\{k_{j,N}+\max(k:\ k\in D_{j,N})\},\quad j=1,\ \cdots,\ \hat{q}, \tag{3.11}$$

then we can use $(\hat{q},\ \hat{t}_1,\ \cdots,\ \hat{t}_q)$ as an estimate of $(q,\ t_1,\ \cdots,\ t_q)$. We have the following

Theorem 3.1. *Under the conditions of the Case 1, $(\hat{q},\ \hat{t}_1,\ \cdots,\ \hat{t}_{\hat{q}})$ is a strongly consistent estimate of $(q,\ t_1,\ \cdots,\ t_q)$.*

We need the following

Lemma 3.1. *Let $x_1, x_2, \cdots, x_n$ be i.i.d. random variables such that $Ex_1=0$ and $\phi(t)=E\{\exp(tx_1)\}$ is finite in a neighborhood at $t=0$. Then for any given constant $s>0$,*

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}x_i\right|\geqslant s\right\}<C_1e^{-Cn},$$

where C and C_1 are positive constants.

This result is well-known and the proof is omitted.

Proof of Theorem 3.1. It is easily seen that there exist integers $k_1^0, \cdots, k_q^0$, $0<k_1^0<\cdots<k_q^0<N$ such that

$$\left|\frac{k_j^0}{N}-t_j\right|\leqslant\frac{1}{N}, \quad \text{for } j=1, 2, \cdots, q, \tag{3.12}$$

and

$$\underset{\sim}{\mu}(i/N)=\underset{\sim}{\mu}_j, \ i=k_{j-1}^0+1, \cdots, k_j^0, \ j=1, \cdots, q+1, \tag{3.13}$$

where $k_0^0=0$, $k_{q+1}^0=N$, and $\underset{\sim}{\mu}_1\neq\underset{\sim}{\mu}_2, \cdots, \underset{\sim}{\mu}_q\neq\underset{\sim}{\mu}_{q+1}$.

Fix $j, j=1, 2, \cdots, q$, and put $k=k_j^0$. Since $m\ll N$, (3.1) and (3.2) hold for N large, where $\underset{\sim}{\mu}^{(1)}=\underset{\sim}{\mu}_j$, $\underset{\sim}{\mu}^{(2)}=\underset{\sim}{\mu}_{j+1}$. Using Lemma 3.1 and Borel-Cantelli's lemma, we have

$$\lim_{N\to\infty} A_{k,N}=A, \quad \text{a.s.} \tag{3.14}$$

and

$$\begin{aligned}\lim_{N\to\infty} B_{k,N}&=\lim_{N\to\infty}\Big\{\frac{1}{2m}\Sigma_{(1,k)}(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(1,k)})(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(1,k)})'\\&\quad+\frac{1}{2m}\Sigma_{(2,k)}(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(2,k)})(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(2,k)})'\\&\quad+\frac{1}{2}(\overline{\underset{\sim}{X}}_{(1,k)}\overline{\underset{\sim}{X}}'_{(1,k)}+\overline{\underset{\sim}{X}}_{(2,k)}\overline{\underset{\sim}{X}}'_{(2,k)})-\overline{\underset{\sim}{X}}_{(k)}\overline{\underset{\sim}{X}}'_{(k)}\Big\}\\&=A+\frac{1}{2}(\underset{\sim}{\mu}_j\underset{\sim}{\mu}'_j+\underset{\sim}{\mu}_{j+1}\underset{\sim}{\mu}'_{j+1})-\frac{1}{4}(\underset{\sim}{\mu}_j+\underset{\sim}{\mu}_{j+1})(\underset{\sim}{\mu}_j+\underset{\sim}{\mu}_{j+1})'\\&=A+\frac{1}{4}(\underset{\sim}{\mu}_j-\underset{\sim}{\mu}_{j+1})(\underset{\sim}{\mu}_j-\underset{\sim}{\mu}_{j+1})', \quad \text{a. s.}\end{aligned} \tag{3.15}$$

By (3.4), (3.14), (3.15) and $\underset{\sim}{\mu}_j\neq\underset{\sim}{\mu}_{j+1}$, with probability one for N large,

$$-G_N(k_j^0)>\alpha_j m \text{ for some constant } \alpha_j>0. \tag{3.16}$$

By (3.16) and $m\gg C_N$, with probability one for N large,

$$k_j^0\in D_N, \ j=1, 2, \cdots, q. \tag{3.17}$$

By the definition of $D_{j,N}$'s, noting (3.12) and $m\ll N$, we see that with probability one for N large, $k_1^0, \cdots, k_q^0$ belong to different $D_{j,N}$'s, which implies

$$\hat{q}\geqslant q. \tag{3.18}$$

Take $\varepsilon_1\in(0, 1/2)$. Write

$$K_N=\{k: m\leqslant k\leqslant N-m, \ \left|\frac{k}{N}-t_j\right|\geqslant\frac{m}{N}(1+\varepsilon_1) \text{ for } j=1, 2, \cdots, q\}. \tag{3.19}$$

Assume that $k\in K_N$. By $m\ll N$, it is easily seen that (3.1) and (3.2) hold with $\underset{\sim}{\mu}^{(1)}=\underset{\sim}{\mu}^{(2)}=\underset{\sim}{\mu}_j$ for some $j=1, 2, \cdots, q+1$ and N large.

For a $p\times p$ matrix $A=(a_{ij})$, we write $\|A\|=(\sum_{i,j}|a_{ij}|^2)^{1/2}$ and use $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\cdots\geqslant\lambda_p(A)$ to denote its eigenvalues. Also, we write $\|\underset{\sim}{\alpha}\|^2=\sum_{i=1}^{p}|\alpha_i|^2$ for $\underset{\sim}{\alpha}=(\alpha_1, \cdots, \alpha_p)'$.

By (3.5),

$$A_{k,N}=\frac{1}{2m}\Sigma_{(k)}(\underset{\sim}{X}_i-\underset{\sim}{\mu}_j)(\underset{\sim}{X}_i-\underset{\sim}{\mu}_j)'-\frac{1}{2}\sum_{\nu=1}^{2}(\overline{\underset{\sim}{X}}_{(\nu,k)}-\underset{\sim}{\mu}_j)(\overline{\underset{\sim}{X}}_{(\nu,k)}-\underset{\sim}{\mu}_j)'$$
$$\triangleq\widetilde{A}_{k,N}-\widetilde{C}_{k,N}. \tag{3.20}$$

By Lemma 3.1, for any $\varepsilon>0$, there exist constants $C>0$ and $C_1>0$ such that

$$P\{\|\widetilde{A}_{k,N}-A\|\geqslant\varepsilon\}<C_1e^{-Cm},$$

and

$$P\{\|\overline{\underset{\sim}{X}}_{(\nu,k)}-\underset{\sim}{\mu}^{(\nu)}\|\geqslant\varepsilon\}<C_1e^{-Cm},$$

which imply that for any $\varepsilon>0$,

$$P\{\|A_{k,N}-A\|\geqslant\varepsilon\}<C_1e^{-Cm}. \tag{3.21}$$

Hereafter we will use C and C_1 for some new positive constants without statement.

It is well-known that for any $p\times p$ symmetric matrices A and B,

$$\operatorname{tr}AB\leqslant\sum_{i=1}^{p}\lambda_i(A)\lambda_i(B) \tag{3.22}$$

(von Neumann (1937)), which implies that

$$\sum_{i=1}^{p}(\lambda_i(A_{k,N})-\lambda_i(A))^2\leqslant\operatorname{tr}(A_{k,N}-A)^2. \tag{3.23}$$

By (3.21) and (3.23),

$$P\{\lambda_p(A_{k,N})\leqslant\frac{1}{2}\lambda_p(A)\}<C_1e^{-Cm}. \tag{3.24}$$

By (3.5),

$$B_{k,N}-A_{k,N}=\frac{1}{4}(\overline{\underset{\sim}{X}}_{(1,k)}-\overline{\underset{\sim}{X}}_{(2,k)})(\overline{\underset{\sim}{X}}_{(1,k)}-\overline{\underset{\sim}{X}}_{(2,k)})'\geqslant0. \tag{3.25}$$

By (3.4), (3.22) and (3.25),

$$\begin{aligned}-G_N(k)&=m\log|I_p+A_{k,N}^{-1/2}(B_{k,N}-A_{k,N})A_{k,N}^{-1/2}|\\&\leqslant m\operatorname{tr}A_{k,N}^{-1}(B_{k,N}-A_{k,N})\leqslant m\operatorname{tr}(B_{k,N}-A_{k,N})/\lambda_p(A_{k,N})\\&=\frac{m}{4}\|\overline{\underset{\sim}{X}}_{(1,k)}-\overline{\underset{\sim}{X}}_{(2,k)}\|^2/\lambda_p(A_{k,N}).\end{aligned} \tag{3.26}$$

Assume that $y_N\sim N(0,1)$. Then for $N\geqslant3$,

$$P(|y_N|>\sqrt{6\log N})\leqslant\frac{2}{\sqrt{2\pi(6\log N)}}\exp(-3\log N)<N^{-3}. \tag{3.27}$$

Since $\sqrt{m}\ (\overline{\underset{\sim}{X}}_{(\nu,k)})-\underset{\sim}{\mu}_j\sim N(0,A)$ for $\nu=1,2$, it is easily seen that if we take the constant C_2 large enough,

$$P\left\{\frac{m}{4}\|\overline{\underset{\sim}{X}}_{(1,k)}-\overline{\underset{\sim}{X}}_{(2,k)}\|^2\geqslant C_2\log N\right\}<2N^{-3}. \tag{3.28}$$

By (3.26), (3.24) and (3.28), noting $C_N\gg\log N$, we have

$$P\{-G_N(k)>C_N\}<C_1(e^{-Cm}+N^{-3}). \tag{3.29}$$

Note that C_1 and C are the same constants for all $k\in K_N$ and $\#(K_N)\leqslant N$. Thus

$$P\{-G_N(k)>C_N \text{ for some } k\in K_N\}\leqslant C_1(Ne^{-Cm}+N^{-2}), \tag{3.30}$$

which is a general term of a convergent series by $m\gg\log N$. Use Borel-Cantelli's lemma, with probability one, we assert that for large N

$$D_N\cap K_N=\varnothing. \tag{3.31}$$

By the definition of $D_{j,N}$'s and the fact

$$2m(1+\varepsilon_1)<3m,$$

with probability one for N large,

$$\hat{q}\leqslant q. \tag{3.32}$$

From (3.18), (3.32) and (3.31), it follows that with probability one for N large

$$\hat{q}=q, \tag{3.33}$$

and

$$\lim_{N\to\infty}\hat{t}_j=t_j \text{ a.s.}, \ j=1, 2, \cdots, q. \tag{3.34}$$

Theorem 3.1 is proved.

§ 4. The Case 2

Take $\underset{\sim}{X}_i=\underset{\sim}{X}(i/N)$, $i=k-m+1$, $k-m+2$, $\cdots$, $k+m$. For $k=m$, $m+1$, $\cdots$, $N-m$, suppose that (3.1) and the following hold:

$$\begin{aligned}\operatorname{Var}(\underset{\sim}{X}_{k-m+1})=\cdots=\operatorname{Var}(\underset{\sim}{X}_k)=\Lambda^{(1)}>0,\\ \operatorname{Var}(\underset{\sim}{X}_{k+1})=\cdots=\operatorname{Var}(\underset{\sim}{X}_{k+m})=\Lambda^{(2)}>0,\end{aligned} \tag{4.1}$$

where $\underset{\sim}{\mu}^{(1)}$, $\underset{\sim}{\mu}^{(2)}$, $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are all unknown. Continue to use the notations of (3.3). Then, the logarithm of the LRT statistic for testing the null hypothesis H_k: $\underset{\sim}{\mu}^{(1)}=\underset{\sim}{\mu}^{(2)}$ and $\Lambda^{(1)}=\Lambda^{(2)}$ against the alternative K_k: $\underset{\sim}{\mu}^{(1)}\neq\underset{\sim}{\mu}^{(2)}$, $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are arbitrary is given by

$$G_N(k)=\frac{m}{2}\log|A_{1,k,N}|+\frac{m}{2}\log|A_{2,k,N}|-m\log|B_{k,N}|, \tag{4.2}$$

where

$$A_{\nu,k,N}=\frac{1}{m}\Sigma_{(\nu,k)}(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(\nu,k)})(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(\nu,k)})', \ \nu=1, 2, \tag{4.3}$$

and $B_{k,N}$ is defined in (3.5).

Take integer $m=m_N$ and positive number C_N such that

$$N\gg m\gg C_N\geqslant\log^2 N. \tag{4.4}$$

Define D_N, $D_{j,N}$, $\hat{t}_j$, $j=1, \cdots, \hat{q}$ by (3.7)—(3.11). Then we have the following

Theorem 4.1. *Under the conditions of the Case 2, $(\hat{q}, \hat{t}_1, \cdots, \hat{t}_{\hat{q}})$ is a strongly consistent estimate of $(q, t_1, \cdots, t_q)$.*

To prove this theorem, we introduce the following

Lemma 4.1. *Assume that $y_N=\sum_{i=1}^{m}z_{iN}/\sqrt{m}$, where $m=m_N\to\infty$ as $N\to\infty$, and $z_{1N}, \cdots, z_{mN}$ are i.i.d. random variables such that the distribution of z_{1N} is independent of N, $Ez_{1N}=0$, and $\phi(t)=E\{\exp(tz_{1N})\}$ is finite for $|t|<T$ with T being a constant. Then, for any given constant $\varepsilon>0$ and $\lambda>0$,*

$$P\{|y_N|\geqslant\varepsilon\log N\}\leqslant C_1N^{-\lambda},$$

where $C_1>0$ is a constant.

Proof For N large, $\lambda/(\sqrt{m}\varepsilon)\leqslant T/2$. It is easily shown that $\phi''(t)$ is continuous for $t\in[0, T/2]$, and $M\triangleq\sup\{|\phi''(t)|: 0\leqslant t\leqslant T/2\}<\infty$. We have

$$P(y_N\geqslant\varepsilon\log N)\leqslant\exp(-\lambda\log N)E\left\{\exp\left[\frac{\lambda}{\sqrt{m}\varepsilon}\sum_{i=1}^{m}z_{iN}\right]\right\}=N^{-\lambda}\phi^m(\lambda/(\sqrt{m}\varepsilon)).$$

By Taylor's expansion, and $\phi'(0)=0$,

$$\phi(\lambda/(\sqrt{m}\varepsilon))=1+\frac{1}{2}\phi''(t_*)(\lambda/(\sqrt{m}\varepsilon))^2,$$

where $0\leqslant t_*\leqslant\lambda/(\sqrt{m}\varepsilon)\leqslant T/2$. Thus

$$P(y_N\geqslant\varepsilon\log N)\leqslant N^{-\lambda}(1+\lambda^2M/(2\varepsilon^2m))^m\leqslant N^{-\lambda}\exp\{\lambda^2M/(2\varepsilon^2)\}.$$

The same is true for $P(y_N\leqslant-\varepsilon\log N)$. Lemma 4.1 is proved.

Now we give a proof of Theorem 4.1.

Proof As before, there exist integers $k_1^0, \cdots, k_q^0$, $k_0^0=0<k_1^0<\cdots<k_q^0<k_{q+1}^0=N$ such that (3.12) and the following hold:

$$\underset{\sim}{\mu}(i/N)=\underset{\sim}{\mu}_j,\ A(i/N)=A_j,\ \text{for } i=k_{j-1}^0+1, \cdots, k_j^0,\ j=1, 2, \cdots, q+1, \tag{4.5}$$

where $\underset{\sim}{\mu}_1\neq\underset{\sim}{\mu}_2, \cdots, \underset{\sim}{\mu}_q\neq\underset{\sim}{\mu}_{q+1}$.

Fix $j\,(j=1, 2, \cdots, q)$, and put $k=k_j^0$. By $m\ll N$, (3.1) and (4.1) hold for N large, where $\underset{\sim}{\mu}^{(1)}=\underset{\sim}{\mu}_j$, $\underset{\sim}{\mu}^{(2)}=\underset{\sim}{\mu}_{j+1}$, $A^{(1)}=A_j$ and $A^{(2)}=A_{j+1}$. By Lemma 3.1 and Borel-Cantelli's lemma, we have

$$\begin{aligned}&\lim_{N\to\infty}A_{1,k,N}=A_j, &&\text{a. s.}\\ &\lim_{N\to\infty}A_{2,k,N}=A_{j+1}, &&\text{a. s.}\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}\lim_{N\to\infty}B_{k,N}&=\lim_{N\to\infty}\left\{\frac{1}{2}(A_{1,k,N}+A_{2,k,N})+\frac{1}{4}(\underset{\sim}{X}_{(1,k)}-\underset{\sim}{X}_{(2,k)})(\underset{\sim}{X}_{(1,k)}-\underset{\sim}{X}_{(2,k)})'\right\}\\ &=\frac{1}{2}(A_j+A_{j+1})+\frac{1}{4}(\underset{\sim}{\mu}_j-\underset{\sim}{\mu}_{j+1})(\underset{\sim}{\mu}_j-\underset{\sim}{\mu}_{j+1})', \quad \text{a. s.}\end{aligned}\tag{4.7}$$

Put $H=\frac{1}{4}(\underset{\sim}{\mu}_j-\underset{\sim}{\mu}_{j+1})(\underset{\sim}{\mu}_j-\underset{\sim}{\mu}_{j+1})'$ and $A=\frac{1}{2}(A_j+A_{j+1})$. By Jensen's inequality, we have

$$-\frac{1}{2}(\log|A_j|+\log|A_{j+1}|)+\log|A|\geqslant 0. \tag{4.8}$$

By (4.2) and (4.6)—(4.8), noting that $\underset{\sim}{\mu}_j\neq\underset{\sim}{\mu}_{j+1}$, we have

$$\begin{aligned}\lim_{N\to\infty}\left\{-\frac{1}{m}G_N(k)\right\}&=-\frac{1}{2}(\log|A_j|+\log|A_{j+1}|)+\log|A+H|\\ &\geqslant\log|A+H|-\log|A|=\log|I_p+A^{-1/2}HA^{-1/2}|>0, \quad \text{a. s.}\end{aligned}\tag{4.9}$$

Using the argument used in establishing (3.18), we see that with probability one for N large,

$$\hat{q}\geqslant q. \tag{4.10}$$

Take $\varepsilon_1\in(0, 1/2)$. Define K_N by (3.19) and take $k\in K_N$. By $m\ll N$, (3.1) and (4.1) hold with $\underset{\sim}{\mu}^{(1)}=\underset{\sim}{\mu}^{(2)}=\underset{\sim}{\mu}_j$ and $A^{(1)}=A^{(2)}=A_j$ for some $j=1, 2, \cdots, q+1$ and large N.

Put $\tilde{A}_{\nu,k,N}=\frac{1}{m}\Sigma_{(\nu,k)}(\underset{\sim}{X}_i-\underset{\sim}{\mu}_j)((\underset{\sim}{X}_i-\underset{\sim}{\mu}_j)'$, $\nu=1, 2$, and

$\tilde{B}_{k,N}=\frac{1}{2}(\tilde{A}_{1,k,N}+\tilde{A}_{2,k,N})=\frac{1}{2m}\Sigma_{(k)}(\underset{\sim}{X}_i-\underset{\sim}{\mu}_j)(\underset{\sim}{X}_i-\underset{\sim}{\mu}_j)'$. By Lemma 4.1, for any $\varepsilon>0$,

$$
\begin{aligned}
&P\{\|\tilde{A}_{\nu,k,N}-A_j\|\geqslant\varepsilon\log N/\sqrt{m}\}<C_1N^{-3},\ \nu=1,2,\\
&P\{\|\tilde{B}_{k,N}-A_j\|\geqslant\varepsilon\log N/\sqrt{m}\}<C_1N^{-3}.
\end{aligned}
\tag{4.11}
$$

Note that $\sqrt{m}(\overline{\underset{\sim}{X}}_{(\nu,k)}-\underset{\sim}{\mu}_j)\sim N(0, A_j)$, $\sqrt{2m}(\overline{\underset{\sim}{X}}_{(k)}-\underset{\sim}{\mu}_j)\sim N(0, A_j)$, $\nu=1, 2$. Using (3.27), we can get

$$
\begin{aligned}
&P\{\|\overline{\underset{\sim}{X}}_{(\nu,k)}-\underset{\sim}{\mu}_j)(\overline{\underset{\sim}{X}}_{(\nu,k)}-\underset{\sim}{\mu}_j)'\|\geqslant C_2\log N/m\}<C_1N^{-3},\ \nu=1,2,\\
&P\{\|(\overline{\underset{\sim}{X}}_{(k)}-\underset{\sim}{\mu}_j)(\overline{\underset{\sim}{X}}_{(k)}-\underset{\sim}{\mu}_j)'\|\geqslant C_2\log N/m\}<C_1N^{-3},
\end{aligned}
\tag{4.12}
$$

provided the constant C_2 is large. By (4.3) and (3.5),

$$
\begin{aligned}
&\tilde{A}_{\nu,k,N}-A_{\nu,k,N}=(\overline{\underset{\sim}{X}}_{(\nu,k)}-\underset{\sim}{\mu}_j)(\overline{\underset{\sim}{X}}_{(\nu,k)}-\underset{\sim}{\mu}_j)'\geqslant0,\ \nu=1,2,\\
&\tilde{B}_{k,N}-B_{k,N}=(\overline{\underset{\sim}{X}}_{(k)}-\underset{\sim}{\mu}_j)(\overline{\underset{\sim}{X}}_{(k)}-\underset{\sim}{\mu}_j)'\geqslant0.
\end{aligned}
\tag{4.13}
$$

By (4.11) and (4.12), for any $\varepsilon>0$,

$$
\begin{aligned}
&P\{\|A_{\nu,k,N}-A_j\|\geqslant\varepsilon\log N/\sqrt{m}\}<C_1N^{-3},\ \nu=1,2,\\
&P\{\|B_{k,N}-A_j\|\geqslant\varepsilon\log N/\sqrt{m}\}<C_1N^{-3}.
\end{aligned}
\tag{4.14}
$$

Using the argument similar to (3.22)—(3.26), we have

$$P\{\lambda_p(B_{k,N})\leqslant\frac{1}{2}\lambda_p(A_j)\}<C_1N^{-3},\tag{4.15}$$

and

$$
\begin{aligned}
0&\leqslant m\log|\tilde{B}_{k,N}|-m\log|B_{k,N}|=m\log|I+B_{k,N}^{-1/2}(\tilde{B}_{k,N}-B_{k,N})B_{k,N}^{-1/2}|\\
&\leqslant m\operatorname{tr}B_{k,N}^{-1}(\tilde{B}_{k,N}-B_{k,N})\leqslant m\|\overline{\underset{\sim}{X}}_{(k)}-\underset{\sim}{\mu}_j\|^2/\lambda_p(B_{k,N}).
\end{aligned}
\tag{4.16}
$$

By (4.12), (4.15) and (4.16), when the constant C_2 is large, we have

$$P\{|m\log|\tilde{B}_{k,N}|-m\log|B_{k,N}||\geqslant C_2\log N\}<C_1N^{-3}.\tag{4.17}$$

In the same way, for C_2 large, $\nu=1, 2$,

$$P\left\{\left|\frac{m}{2}\log|\tilde{A}_{\nu,k,N}|-\frac{m}{2}\log|A_{\nu,k,N}|\right|\geqslant C_2\log N\right\}<C_1N^{-3}.\tag{4.18}$$

Denote by $\hat{\lambda}_1^{(\nu)}\geqslant\hat{\lambda}_2^{(\nu)}\geqslant\cdots\geqslant\hat{\lambda}_p^{(\nu)}$ the eigenvalues of $\tilde{A}_{\nu,k,N}$ for $\nu=1, 2$, and denote by $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_p$ the eigenvalues of A_j. Put $\delta_i^{(\nu)}=(\hat{\lambda}_i^{(\nu)}-\lambda_i)/\lambda_i$, $i=1, \cdots, p$, $\nu=1, 2$. By (3.23) and (4.11), for any $\varepsilon>0$,

$$P\{|\delta_i^{(\nu)}|^2\geqslant\varepsilon\log^2N/m,\ \text{for some } i,\ \nu\}<C_1N^{-3}.\tag{4.19}$$

When $|x_i|\leqslant1/4$, $|y_i|\leqslant1/4$, $i=1, \cdots, p$, by Taylor's expansion, there exists a positive constant C_3 such that

$$\frac{m}{2}\sum_{i=1}^{p}\left(2\log\left(1+\frac{x_i+y_i}{2}\right)-\log((1+x_i)-\log(1+y_i)\right)\leqslant C_3m\sum_{i=1}^{p}(x_i^2+y_i^2).\tag{4.20}$$

Put

$$-\widetilde{G}_N(k)=m\log|\widetilde{B}_{k,N}\{-\frac{m}{2}\log|\widetilde{A}_{1,k,N}|-\frac{m}{2}\log|\widetilde{A}_{2,k,N}|$$

$$=\frac{m}{2}\sum_{i=1}^{p}\Big[2\log\frac{\hat{\lambda}_i^{(1)}+\hat{\lambda}_i^{(2)}}{2}-\log\hat{\lambda}_i^{(1)}-\log\hat{\lambda}_i^{(2)}\Big]$$

$$=\frac{m}{2}\sum_{i=1}^{p}\Big[2\log\Big(1+\frac{\lambda_i^{(1)}+\lambda_i^{(2)}}{2}\Big)-\log(1+\delta_i^{(1)})-\log(1+\delta_i^{(2)})\Big]. \quad (4.21)$$

Since $m\gg\log^2 N$, there exists a constant N_0 such that

$$\delta\log^2 N/m<1/4 \text{ for } N\geqslant N_0. \quad (4.22)$$

By (4.19) – (4.22), for any $\varepsilon>0$, and $N\geqslant N_0$,

$$P\Big\{-\widetilde{G}_N(k)\geqslant\frac{\varepsilon}{2}\log^2 N\Big\}\leqslant C_1N^{-3}+P\Big\{C_3m\sum_{i=1}^{p}(|\delta_i^{(1)}|^2+|\delta_i^{(2)}|^2)$$
$$\geqslant\frac{\varepsilon}{2}\log^2 N\Big\}\leqslant C_1N^{-3}+C_1N^{-3}. \quad (4.23)$$

By (4.17) and (4.18), for any $\varepsilon>0$,

$$P\Big\{|(-G_N(k))-(-\widetilde{G}_N(k))|\geqslant\frac{\varepsilon}{2}\log^2 N\Big\}<C_1N^{-3}. \quad (4.24)$$

By (4.23) and (4.24), for any $\varepsilon>0$ and $N\geqslant N_0$,

$$P\{-G_N(k)\geqslant\varepsilon\log^2 N\}<C_1N^{-3}. \quad (4.25)$$

Note that we can choose the constants C_1, N_0, etc., independent of $k\in K_N$. Since $\#(K_N)\leqslant N$, for $N\geqslant N_0$ we have

$$P\{-G_N(k)>\varepsilon\log^2 N, \text{ for some } k\in K_N\}<C_1N^{-2}, \quad (4.26)$$

which is the general term of a convergent series. By Borel-Cantelli's lemma, with probability one for N large,

$$D_N\cap K_N=\varnothing. \quad (4.27)$$

Using the argument similar to (3.21)—(3.24), we can prove that with probability one for N large, $\hat{q}\leqslant q$, $\hat{q}=q$, and

$$\lim_{N\to\infty}\hat{t}_j=t_j \quad \text{a.s.}, j=1, 2, \cdots, q.$$

The theorem is proved.

§ 5. The Case 3

Following the local likelihood principle, we can define

$$G_N(k)=\frac{m}{2}\log|A_{1,k,N}|+\frac{m}{2}\log|A_{2,k,N}|-m\log|B_{k,N}|, \quad (5.1)$$

where $k=m, m+1, \cdots, N-m$, and

$$A_{\nu,k,N}=\frac{1}{m}\Sigma_{(\nu,k)}(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(k)})(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(k)})', \quad \nu=1, 2,$$
$$B_{k,N}=\frac{1}{2m}\Sigma_{(k)}(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(k)})(\underset{\sim}{X}_i-\overline{\underset{\sim}{X}}_{(k)})'. \quad (5.2)$$

Take $m=m_N$ and C_N satisfying (4.4), and define D_N, $D_{j,N}$, $\hat{t}_j$, $j=1, \cdots, \hat{q}$ by

(3.7)—(3.11). Then we have

Theorem 5.1. *Under the conditions of the Case 3, $(\hat{q}, \hat{t}_1, \cdots, \hat{t}_{\hat{q}})$ is a strongly consistent estimate of $(q, t_1, \cdots, t_q)$.*

The proof is similar to that of Theorem 4.1. Here we omit this proof.

§ 6. The Cases 4 and 5

Put $X_i = X(i/N)$. Take a positive integer $m<N$. For $k=m, m+1, \cdots, N-m$, we assume that $X_{k-m+1}, \cdots, X_{k+m}$ are independent normal variables such that

$$EX_{k-m+1}=\cdots=EX_k=\mu^{(1)},\ EX_{k+1}=\cdots=EX_{k+m}=\mu^{(2)}, \tag{6.1}$$

and

$$\begin{aligned}\mathrm{Var}(X_{k-m+1})&=\cdots=\mathrm{Var}(X_k)=\lambda^{(1)},\\ \mathrm{Var}(X_{k+1})&=\cdots=\mathrm{Var}(X_{k+m})=\lambda^{(2)}.\end{aligned} \tag{6.2}$$

Similar to (3.3), we can define $\overline{X}_{(1,k)}$, $\overline{X}_{(2,k)}$ and $\overline{X}_{(k)}$.

First we consider the Case 4. In this case, we assume that $\lambda^{(1)}=\lambda^{(2)}=\lambda$, and $\mu^{(1)}$, $\mu^{(2)}$, λ are all unknown.

The logarithm of the LRT statistic for testing the null hypothesis H_k: $\mu^{(1)}=\mu^{(2)}$ against the alternative K_k: $\mu^{(1)}>\mu^{(2)}$ is given by

$$G_N(k)=I(\overline{X}_{(1,k)}>\overline{X}_{(2,k)})\, m\log(A_{k,N}/B_{k,N}), \tag{6.3}$$

where $I(A)$ denotes the indicator of a set A, and

$$\begin{aligned}A_{k,N}&=\frac{1}{2m}\left[\Sigma_{(1,k)}(X_i-\overline{X}_{(1,k)})^2+\Sigma_{(2,k)}(X_i-\overline{X}_{(2,k)})^2\right],\\ B_{k,N}&=\frac{1}{2m}\ \Sigma_{(k)}(X_i-\overline{X}_{(k)})^2.\end{aligned} \tag{6.4}$$

Take $m=m_N$ and C_N such that

$$N\gg m\gg C_N\gg\log N. \tag{6.5}$$

Define D_N, $D_{j,N}$, $\hat{t}_j$, $j=1, \cdots, \hat{q}$ by (3.7)—(3.11). Then we have the following

Theorem 6.1. *Under the conditions of the Case 4, $(\hat{q}, \hat{t}_1, \cdots, \hat{t}_{\hat{q}})$ is a strongly consistent estimate of $(q, t_1, \cdots, t_q)$.*

Now we consider the Case 5, and we assume that (6.1) and (6.2) hold, and $\mu^{(1)}$, $\lambda^{(1)}$, $\mu^{(2)}$, $\lambda^{(2)}$ are all unknown.

It can be proved that the logarithm of the LRT statistic for testing the null hypothesis H_k: $\mu^{(1)}=\mu^{(2)}$, $\lambda^{(1)}=\lambda^{(2)}$ against the alternative K_k: $\mu^{(1)}>\mu^{(2)}$ and $\lambda^{(1)}$, $\lambda^{(2)}$ are arbitrary is given by

$$\begin{aligned}G_N(k)=&I(\overline{X}_{(1,k)}>\overline{X}_{(2,k)})\left\{\frac{m}{2}\log A_{1,k,N}+\frac{m}{2}\log A_{2,k,N}-m\log B_{k,N}\right\}\\ &+I(\overline{X}_{(1,k)}\leqslant\overline{X}_{(2,k)})\left\{\frac{m}{2}\log\hat{\lambda}^{(1)}(k)+\frac{m}{2}\log\hat{\lambda}^{(2)}(k)-m\log B_{k,N}\right\},\end{aligned} \tag{6.6}$$

where

$$A_{1,k,N}=\frac{1}{m}\Sigma_{(1,k)}(X_i-\overline{X}_{(1,k)})^2,$$
$$A_{2,k,N}=\frac{1}{m}\Sigma_{(2,k)}(X_i-\overline{X}_{(2,k)})^2, \tag{6.7}$$
$$B_{k,N}=\frac{1}{2m}\Sigma_{(k)}(X_i-\overline{X}_{(k)})^2,$$

and $\lambda^{(1)}(k)$, $\lambda^{(2)}(k)$ satisfy the following conditions:

$$-\frac{1}{\hat{\lambda}^{(1)}(k)}+\frac{1}{(\hat{\lambda}^{(1)}(k))^2}A_{1,k,N}+\frac{(\overline{X}_{(1,k)}-\overline{X}_{(2,k)})^2}{(\hat{\lambda}^{(1)}(k)+\hat{\lambda}^{(2)}(k))^2}=0,$$
$$-\frac{1}{\hat{\lambda}^{(2)}(k)}+\frac{1}{(\hat{\lambda}^{(2)}(k))^2}A_{2,k,N}+\frac{(\overline{X}_{(1,k)}-\overline{X}_{(2,k)})^2}{(\hat{\lambda}^{(1)}(k)+\hat{\lambda}^{(2)}(k))^2}=0. \tag{6.8}$$

The proof of (6.6) is routine, and is omitted.

Take integer $m=m_N$ and positive number C_N such that

$$N\gg m\gg C_N\geqslant\log^2 N. \tag{6.9}$$

Define D_N, $D_{j,N}$, $\hat{t}_j$, $j=1,\cdots,\hat{q}$ by (3.7)—(3.11). Then we have the following:

Theorem 6.2. *Under the conditions of the Case* 5, $(\hat{q},\hat{t}_1,\cdots,\hat{t}_{\hat{q}})$ *is a strongly consistent estimate of* $(q,t_1,\cdots,t_q)$.

Since the proof of Theorem 6.1 is similar to that of Theorem 6.2, we only give the proof of Theorem 6.2.

Proof of Theorem 6.2. It is easily seen that there exists integers $k_1^0,\cdots,k_q^0$, $k_0^0=0<k_1^0<\cdots<k_q^0<k_{q+1}^0=N$ such that

$$\left|\frac{k_j^0}{N}-t_j\right|\leqslant 1/N,\quad \text{for } j=1,2,\cdots,q, \tag{6.10}$$

and

$$\mu(i/N)=\mu_j,\ A(i/N)=\lambda_j \text{ for } i=k_{j-1}^0+1,\cdots,k_j^0,\ j=1,\cdots,q+1, \tag{6.11}$$

where $\mu_1>\mu_2>\cdots>\mu_{q+1}$.

Fix j, $j=1,2,\cdots,q$, and put $k=k_j^0$. Since $m\ll N$, (6.1) and (6.2) hold for N large, where $\mu^{(1)}=\mu_j>\mu^{(2)}=\mu_{j+1}$, and $\lambda^{(1)}=\lambda_j$, $\lambda^{(2)}=\lambda_{j+1}$. It can be proved that with probability one for N large, $\overline{X}_{(1,k)}>\overline{X}_{(2,k)}$, and by (6.6)

$$-\frac{1}{m}G_N(k)=\log B_{k,N}-\frac{1}{2}\log A_{1,k,N}-\frac{1}{2}\log A_{2,k,N}. \tag{6.12}$$

By Lemma 3.1 and Borel-Cantelli's lemma, we have

$$\lim_{N\to\infty}A_{1,k,N}=\lambda_j,\qquad \text{a. s.}$$
$$\lim_{N\to\infty}A_{2,k,N}=\lambda_{j+1},\qquad \text{a. s.} \tag{6.13}$$

and

$$\lim_{N\to\infty}B_{k,N}=\frac{1}{2}(\lambda_j+\lambda_{j+1})+\frac{1}{4}(\mu_j-\mu_{j+1})^2,\qquad \text{a. s.} \tag{6.14}$$

(Refer to (4.7)). Along the line of proof from (4.8) to (4.10), we see that with probability one for N large,

$$\hat{q}\geqslant q. \tag{6.15}$$

Take $\varepsilon_1 \in (0, 1/2)$. Define K_N by (3.19) and take $k \in K_N$. By $m \ll N$, (6.1) and (6.2) hold with $\mu^{(1)} = \mu^{(2)} = \mu_j$ and $\lambda^{(1)} = \lambda^{(2)} = \lambda_j$ for some $j = 1, 2, \cdots, q+1$ and for large N.

Put $\hat{\lambda}^{(1)}(k) = x$, $\hat{\lambda}^{(2)}(k)/\hat{\lambda}^{(1)}(k) = z$, $A_{1,k,N} = a$, $A_{2,k,N} = b$ and $(\bar{X}_{(1,k)} - \bar{X}_{(2,k)})^2 = c$. The equation (6.8) can be rewritten as

$$-\frac{1}{x} + \frac{a}{x^2} + \frac{1}{x^2(1+z)^2} = 0, \tag{6.16}$$

$$-\frac{1}{xz} + \frac{b}{x^2 z^2} + \frac{1}{x^2(1+z)^2} = 0. \tag{6.17}$$

It is easily seen that with probability one we have $a>0$, $b>0$ and $c>0$. Multiplying both sides of (6.16) by x^2, we get

$$x = \frac{c}{(1+z)^2} + a. \tag{6.18}$$

Multiplying both sides of (6.17) by x^2, substituting (6.18) into it, and multiplying both sides of the obtained equation by $z^2(1+z)^2$, we get

$$az(1+z)^2 - b(1+z)^2 - cz^2 + cz = 0. \tag{6.19}$$

Equation (6.19) has at least one real solution which is neither 0 nor -1, and if $z \neq -1$ is real, $a + c/(1+z)^2 > 0$. Hence, by means of the above method of solution, we guarantee that the equation system (6.8) and the system of equations (6.18), (6.19) have the same real solution. Also, any real solution of (6.8) must be positive. We denote by $(\hat{\lambda}^{(1)}(k), \hat{\lambda}^{(2)}(k))$ such a solution of (6.8). By (6.8),

$$\begin{aligned} |\hat{\lambda}^{(1)}(k) - A_{1,k,N}| / (\hat{\lambda}^{(1)}(k))^2 &= (\bar{X}_{(1,k)} - \bar{X}_{(2,k)})^2 / (\hat{\lambda}^{(1)}(k) + \hat{\lambda}^{(2)}(k))^2 \\ &< (\bar{X}_{(1,k)} - \bar{X}_{(2,k)})^2 / (\hat{\lambda}^{(1)}(k))^2, \end{aligned}$$

which implies that

$$|\hat{\lambda}^{(1)}(k) - A_{1,k,N}| < (\bar{X}_{(1,k)} - \bar{X}_{(2,k)})^2. \tag{6.19}$$

In the same way

$$|\hat{\lambda}^{(2)}(k) - A_{2,k,N}| < (\bar{X}_{(1,k)} - \bar{X}_{(2,k)})^2. \tag{6.20}$$

Write

$$G_N^*(k) = \frac{m}{2} \log A_{1,k,N} + \frac{m}{2} \log A_{2,k,N} - m \log B_{k,N}. \tag{6.21}$$

Then

$$|G_N(k) - G_N^*(k)| \leqslant \frac{m}{2} \left| \log \frac{\hat{\lambda}^{(1)}(k)}{A_{1,k,N}} \right| + \frac{m}{2} \left| \log \frac{\hat{\lambda}^{(2)}(k)}{A_{2,k,N}} \right|. \tag{6.22}$$

By (4.12), (4.14) with $p=1$, and (6.19)—(6.22), it is easy to verify that for any $\varepsilon > 0$,

$$P\{|G_N(k) - G_N^*(k)| \geqslant \varepsilon \log^2 N\} < C_1 N^{-3}, \tag{6.23}$$

where $C_1 > 0$ is a constant independent of $k \in K_N$.

Since $G_N^*(k)$ is just $G_N(k)$ of (4.2) with $p=1$, the rest of the proof has already been given in the second part of the proof of Theorem 4.1. Thus we have with probability one for N large, $\hat{q} \leqslant q$, $\hat{q} = q$, and

$$\lim_{N\to\infty} \hat{t}_j = t_j, \text{ a.s., } j=1, 2, \cdots, q.$$

Theorem 6.2 is proved.

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