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THE ERDÖS-RÉNYI LAWS OF LARGE NUMBERS FOR NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES**

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Abstract

The Erdös-Rényi law of large numbers (1970) is the first important result for asymptotic behaviours of increments of partial sums of a sequence of random variables with span $[C\log N]$. Some generalizations have been done since then, such as convergence rate of the limit, some results when order of span being either higher or lower than $\log N$. But all these results are only obtained in the case of i. i. d. random variables. This paper aims at the generalization of these results to the case when random variables are independent, but not necessarily identically distributed. To this end Chernoff Theorem is generalized to the corresponding case at first.

§ 1. Introduction

For the almost sure asymptotic behaviours of increments of partial sums of a sequence of random variables, the first important result is well-known Erdös-Rényi law of large numbers (1970):

Theorem A. *Let $\{X_n\}$ be a sequence of i. i. d. random variables satisfying the conditions*

(1) $EX_1=0,\ EX_1^2=1,$

(2) *there exists $t_0>0$ such that $R(t)=E\exp(tX_1)<\infty$ for $|t|<t_0$. Put $S_n=\sum_{i=1}^{n}X_i$. $\rho(x)=\inf_t \exp(-tx)\cdot R(t)$. Then for $\alpha\in\{R'(t)/R(t),\ t\in(0,\ t_0)\}$ and $C=C(\alpha)$ such that $\rho(\alpha)=\exp(-C^{-1})$,*

$$\lim_{N\to\infty}\max_{0\le n\le N-[C\log N]}\frac{S_{n+[C\log N]}-S_n}{[C\log N]}=\alpha \qquad \text{a. s.}$$

M. Csörgö and Steinebach (1981) generalized this result. First, they gave convergence rate of the limit in Theorem A.

Theorem B. *Under the conditions of Theorem A,*

$$\lim_{N\to\infty}\Big(\max_{0\le n\le N-[C\log N]}\frac{S_{n+[C\log N]}-S_n}{[C\log N]^{1/2}}-[C\log N]^{1/2}\alpha\Big)=0 \qquad a.\ s.$$

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$$\lim_{N\to\infty}\left(\max_{0\le n\le N-[C\log N]}\ \max_{0\le k\le[C\log N]}\frac{S_{n+k}-S_n}{[C\log N]^{1/2}}-[C\log N]^{1/2}\alpha\right)=0 \qquad a.\ s.$$

And further, they investigated the case when the order of span a_N is higher than $C\log N$:

(i) $0<a_N\le N$,

(ii) a_N/N is nonincreasing,

(iii) $a_N/(\log N)^p\to 0$ for some $p>2$,

(iv) $a_N/\log N\to\infty$.

Theorem C. *Suppose $\{X_n\}$ satisfies the conditions in Theorem B. Then for $\{a_N\}$ satisfying* (i) —(iv), *we have*

$$\lim_{N\to\infty}\left(\max_{0\le n\le N-a_N}\frac{S_{n+a_N}-S_n}{a_N^{1/2}}-\alpha_N\right)=0 \qquad a.\ s.,$$

$$\lim_{N\to\infty}\left(\max_{0\le n\le N-a_N}\ \max_{0\le k\le a_N}\frac{S_{n+k}-S_n}{a_N^{1/2}}-\alpha_N\right)=0 \qquad a.\ s.,$$

where $\alpha_N>0$ is the solution of the equation $\rho^{a_N}(\alpha_N a_N^{-1/2})=a_N/N$.

Recently, Huse and Steinebach investigated the case of $a_N/\log N\to 0$: (v) $a_N/\log N$ is nonincreasing to 0, (vi) $a_N/(\log N)^{1/2}$ is nondecreasing to ∞.

Theorem D. *Suppose $\{X_n\}$ satisfies condition* (1) *and*

(2)′ $R(t)<\infty$ *for all* $t\ge 0$,

(3) $\lim\limits_{t\to\infty}\psi''(t)=\sigma_0^2,\ 0<\sigma_0^2<\infty$,

where $\psi(t)=\log R(t)$. Then for $\{a_N\}$ satisfying (i), (v) *and* (iv) *we have the conclusions in Theorem C.*

The purpose of this paper is to generalize these results to the case of independent, but not necessarily identically distributed, random variables.

Let $\{X_n\}$ be a sequence of independent random variables. Without loss of generality, we assume $EX_n=0$ $(n\ge 1)$. Put $\sigma_n^2=EX_n^2$, $r_n(t)=E\exp(tX_n)$, $\psi_n(t)=\log r_n(t)$, $\sigma_{nN}^2=\sigma_{n+1}^2+\cdots+\sigma_{n+N}^2$, $R_{nN}(t)=E\exp(t(S_{n+N}-S_n))$. If there exists $t_0>0$ such that $R_{nN}(t)<\infty$. for $t\in[0,t_0)$, put $\rho_{nN}(x)=\inf\limits_t \exp(-t\sigma_{nN}^2x)R_{nN}\cdot(t)$. $\alpha_{nN}=\alpha_{na_NN}$ is the solution of the equation $\rho_{na_N}(x)=N^{-1}$. $t_{nN}=t_{na_NN}$ satisfies the equation

$$R'_{na_N}(t_{nN})/R_{na_N}(t_{nN})=\sigma_{na_N}^2\alpha_{nN}. \qquad (1)$$

§ 2. $a_N=[C\log N]$

Theorem 1. *Suppose that $\{X_n\}$ satisfies*

(a) there exists such an increasing function $M(t)$ that for some $T\in(0,\infty]$, $M(t)\uparrow\infty$ as $t\uparrow T$ and $r_n(t)\le M(t)$ for every n and any $t\in[0,T)$. And further if $T<\infty$, then there exists an increasing function $m(t)\uparrow\infty$ as $t\uparrow T$ such that $\varliminf\limits_{N\to\infty}\dfrac{N_n}{N}\ge\alpha>0$

for every n, where $N_n=\#\{j,\ m(t)\leqslant r_j(t),\ t\in[0,\ T),\ n<j\leqslant n+N\}$.

(b) *there exists a continuous function $v(t)>0$, $t\in[0,\ T)$, which is not integrable on $[0,\ T)$ if $T=\infty$, such that $\varliminf_{N\to\infty}\frac{N^{(n)}}{N}\geqslant\beta>0$ for every n, where $N^{(n)}=\#\{j,\ v(t)\leqslant \psi_j''(t),\ t\in[0,\ T),\ n<j\leqslant n+N\}$. Then*

$$\lim_{N\to\infty}\left\{\max_{0<n<N-a_N}\left(\frac{S_{n+a_N}-S_n}{\sigma_{na_N}}-\sigma_{na_N}\alpha_{nN}\right)\right\}=0 \qquad a.\ s.,$$

$$\lim_{N\to\infty}\left\{\max_{0<n<N-a_N}\max_{0<k<a_N}\left(\frac{S_{n+k}-S_n}{\sigma_{na_N}}-\sigma_{na_N}\alpha_{nN}\right)\right\}=0 \qquad a.\ s..$$

Proof First of all, we prove that there exist $0<t_1<\infty$, $0<t_2<\infty$ (or $0<\delta<T$) such that for every n and all large N,

$$t_{nN}\geqslant t_1, \tag{2}$$

and

$$t_{nN}\leqslant\begin{cases}t_2 & \text{as } T=\infty,\\ T-\delta & \text{as } T<\infty.\end{cases} \tag{3}$$

Conditions (a) and (b) $(\psi_n''(0)=\sigma_n^2)$ imply that there exist $0<\sigma_0^2\leqslant\sigma_0'^2<\infty$ such that for all large N

$$a_N\,\sigma_0^2\leqslant\sigma_{na_N}^2\leqslant a_N\sigma_0'^2. \tag{4}$$

Furthermore, from the definitions of α_{nN} and t_{nN}, we can write

$$\exp(-\log N)=\rho_{na_N}(\alpha_{nN})=\exp(-t_{nN}\sigma_{na_N}^2\alpha_{nN})\,E\exp(t_{nN}(S_{n+a_N}-S_n)). \tag{5}$$

Since $EX_n=0\,(n\geqslant1)$, $E\exp(t_{nN}(S_{n+a_N}-S_n))\geqslant1$. Hence (5) implies $C\sigma_0'^2t_{nN}\alpha_{nN}\geqslant1$. If there exists subsequence N_k such that $\lim_{k\to\infty}t_{nN_k}=0$ for some n, $\lim_{k\to\infty}\alpha_{nN_k}=\infty$. Moreover, definition (1) can be rewritten as

$$\sigma_{na_N}^2\alpha_{nN}=\sum_{j=n+1}^{n+a_N}\frac{r_j'(t_{nN})}{r_j(t_{nN})}=\sum_{j=n+1}^{n+a_N}\psi_j'(t_{nN}). \tag{6}$$

From these and (4), we have

$$a_{N_k}=O(\sigma_{na_{N_k}}^2\alpha_{nN_k})=O\left(\sum_{j=n+1}^{n+a_{N_k}}\frac{r_j'(t_{nN_k})}{r_j(t_{nN_k})}\right). \tag{7}$$

But $r_j'(t_{nN_k})\leqslant c$ (in this paper, c denotes a positive constant which can take different values at different places) for all large k and every j since $t_{nN_k}\to0$ and condition (a). On the other hand, $r_j(t_{nN_k})\geqslant1$ always. Therefore

$$\sum_{j=n+1}^{n+a_{N_k}}\frac{r_j'(t_{nN_k})}{r_j(t_{nN_k})}=O(a_{N_k}), \tag{8}$$

which contradicts (7). This proves (2).

Turn to (3). By condition (a) and $EX_n=0$ $(n\geqslant1)$, there exists $T_1\in(0,\ T)$ for given $g>1$, such that

$$E\exp(tX_n)\leqslant\exp(g\sigma_n^2t^2/2),\quad 0\leqslant t\leqslant T_1 \tag{9}$$

for every n (e. g., cf. Lemma 5 of Chapter 3 in [4]). Combining (5) and noting the definition of t_{nN}, we obtain

$$\exp(-\log N)\leqslant\exp(-t\sigma_{na}^2\alpha_{nN}+gt^2\sigma_{na_N}/2),\quad t\in[0,\ T_1].$$

Then $\sigma_0^2 Ct(\alpha_{nN}-gt/2)\leqslant 1$, i. e.

$$\alpha_{nN}\leqslant\frac{1}{\sigma_0^2 Ct}+\frac{1}{2}gt,\quad t\in[0,\ T_1]. \tag{10}$$

If $T<\infty$, from (5) and condition (a),

$$(m(t_{nN}))^{\alpha a_N/2}\leqslant E\exp(t_{nN}(S_{n+a_N}-S_n))=\frac{1}{N}\exp(t_{nN}\sigma^2_{na_N}\alpha_{nN})\leqslant N^{C}$$

for all large N. Thus

$$(m(t_{nN}))\leqslant N^{C/(\alpha\log N)}=e^{C/\alpha}.$$

Then, noting $m(t)\uparrow\infty(t\uparrow T)$, we see that there exists $\delta>0$ such that $t_{nN}\leqslant T-\delta$. Assuming $T=\infty$ by condition (b) we have $\psi_j'(t)\uparrow\infty$ $(t\to\infty)$ for a great many j, which together with (6) and (10) implies that there exists $t_2\in(0,\ \infty)$ such that $t_{nN}\leqslant t_2$.

In order to prove the Theorem, we verify, as the first step, for arbitrarily given $\varepsilon>0$

$$\limsup_{N\to\infty}\left\{\max_{0\leqslant n\leqslant N-a_N}\ \max_{0\leqslant k\leqslant a_N}\left(\frac{S_{n+k}-S_n}{\sigma_{na_N}}-\sigma_{na_N}\alpha_{nN}\right)\right\}\leqslant\varepsilon.\qquad a.\ s. \tag{11}$$

Estimate the probability, using the submartingale inequality,

$$\begin{aligned}
&P\left\{\max_{0\leqslant n\leqslant N-a_N}\ \max_{0<k\leqslant a_N}\left(\frac{S_{n+k}-S_n}{\sigma_{na_N}}-\sigma_{na_N}\alpha_{nN}\right)\geqslant\varepsilon\right\}\\
&\leqslant\sum_{n=0}^{N-a_N}P\{\max_{0\leqslant k\leqslant a_N}(S_{n+k}-S_n)\geqslant\sigma^2_{na_N}\alpha_{nN}+\sigma_{na_N}\varepsilon\}\\
&\leqslant\sum_{n=0}^{N-a_N}\exp(-t_{nN}(\sigma^2_{na_N}\alpha_{nN}+\sigma_{na_N}\varepsilon))E\exp(t_{nN}(S_{n+a_N}-S_n))\\
&\leqslant\sum_{n=0}^{N-a_N}\rho_{na_N}(\alpha_{nN})\exp(-t_{nN}\sigma_{na_N}\varepsilon)\leqslant\max_{0\leqslant n\leqslant N-a_N}\exp(-t_{nN}\sigma_{na_N}\varepsilon)\\
&\leqslant\exp(-t_1\sigma_0\varepsilon a_N^{1/2})\leqslant\exp(-c\sqrt{\log N}).
\end{aligned} \tag{12}$$

For $i\geqslant 3$, define $N_i=\sup\{N,\ C\log N\leqslant\log^3 i\}$ as $[\log^3 i]=[\log^3(i+1)]$ and $\sup\{N,$ $C\log N\leqslant[\log^3(i+1)]\}$ as $[\log^3 i]<[\log^3(i+1)]$. By (12) and Borel-Cantelli lemma, we obtain

$$\limsup_{i\to\infty}\left\{\max_{0\leqslant n\leqslant N_i-a_{N_i}}\ \max_{0\leqslant k\leqslant a_{N_i}}\left(\frac{S_{n+k}-S_n}{\sigma_{na_{N_i}}}-\sigma_{na_{N_i}}\alpha_{nN_i}\right)\right\}\leqslant\varepsilon.\qquad a.s. \tag{13}$$

We have $a_N=a_{N_i}$ for $N_{i-1}<N\leqslant N_i$. Noting convex propertyof $-\log\rho_{nN}(x)$ and $(-\log\rho_{na_N}(x))'_{x=\alpha_{nN}}=t_{nN}\sigma^2_{na_N}\geqslant t_1\sigma^2_{na_N}$ as N large enough, by the definition of α_{nN} (assume $\alpha_{nN_i}\geqslant\alpha_{nN}$, the contrary can be dealt similarly with), we have

$$\begin{aligned}
\log N_i-\log N&=-\log\rho_{na_{N_i}}(\alpha_{nN_i})+\log\rho_{na_N}(\alpha_{nN})\\
&=-\log\rho_{na_N}(\alpha_{nN_i})+\log\rho_{na_N}(\alpha_{nN})\geqslant t_1\sigma^2_{na_N}(\alpha_{nN_i}-\alpha_{nN}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sigma_{an_{N_i}}\alpha_{nN_i}-\sigma_{na_N}\alpha_{nN}=\sigma_{na_N}(\alpha_{nN_i}-\alpha_{nN})\\
&\leqslant\frac{1}{\sigma_0 t_1 a_{N_i}^{1/2}}(\log N_i-\log N)\leqslant c\log^{-3/2}(i-1)\{\log^3(i+1)-\log^3(i-1)\}\\
&\leqslant c\log^{1/2}(i+1)\log\left(1+\frac{2}{i-1}\right)\leqslant\frac{c\log^{1/2}(i+1)}{i-1}\to 0\quad(i\to\infty).
\end{aligned} \tag{14}$$

Combine it with (13), (11) is proved.

The second step is to show

$$\liminf_{N\to\infty}\left\{\max_{0\leqslant n\leqslant N-a_N}\left(\frac{S_{n+a_N}-S_n}{\sigma_{na_N}}-\sigma_{na_N}\alpha_{nN}\right)\right\}\geqslant-\varepsilon. \tag{15}$$

Write

$$P\left\{\max_{0<n<N-a_N}\left(\frac{S_{n+a_N}-S_n}{\sigma_{na_N}}-\sigma_{na_N}\alpha_{nN}\right)\leqslant-\varepsilon\right\}$$
$$\leqslant\prod_{i=0}^{[N/a_N]-1}\left\{1-P\left\{\frac{S_{(i+1)a_N}-S_{ia_N}}{\sigma_{ia_N,a_N}}-\sigma_{ia_N,a_N}\alpha_{ia_N,N}\geqslant-\varepsilon\right\}\right\}. \tag{16}$$

Put $\xi_{iN}=\frac{S_{(i+1)a_N}-S_{ia_N}}{\sigma_{ia_N,a_N}}-\sigma_{ia_N,a_N}\alpha_{ia_N,N}$. Applying associated probability measure

$$P_{iN}(E)=\int_E\exp\{t_{ia_N,N}(S_{(i+1)a_N}-S_{ia_N})/R_{ia_N,a_N}(t_{ia_N,N})\}dP,$$

we have

$$P(\xi_{iN}\geqslant-\varepsilon)=R_{ia_N,a_N}(t_{ia_N,N})\int_{(\xi_{iN}\geqslant-\varepsilon)}\exp\{-t_{ia_N,N}(S_{(i+1)a_N}-S_{ia_N})\}dP_{iN}$$
$$\geqslant R_{ia_N,a_N}(t_{ia_N,N})\exp\left\{-t_{ia_N,N}\left(\sigma^2_{ia_N,a_N}\alpha_{ia_N,N}-\frac{\varepsilon}{2}\sigma_{ia_N,a_N}\right)\right\}P_{iN}\left(-\varepsilon<\xi_{iN}\leqslant-\frac{\varepsilon}{2}\right). \tag{17}$$

By (6) and noting $E_{iN}\left(X_j-\frac{r_j'(t_{ia_N,N})}{r_j(t_{ia_N,N})}\right)=0\,(j=ia_N+1,\cdots,(i+1)a_N)$, we have

$$E_{iN}\xi_{iN}=\frac{1}{\sigma_{ia_N,a_N}}E_{iN}\left\{\sum_{j=ia_N+1}^{(i+1)a_N}\left(X_j-\frac{r_j'(t_{ia_N,N})}{r_j(t_{ia_N,N})}\right)\right\}=0.$$

And further, it is easy to verify that $X_j-\frac{r_j'(t_{ia_N,N})}{r_j(t_{ia_N,N})}$, $j=ia_N+1,\cdots,(i+1)a_N$, are mutually independent under P_{iN}, so

$$E_{iN}\xi^2_{iN}=\frac{1}{\sigma^2_{ia_N,a_N}}\sum_{j=ia_N+1}^{(i+1)a_N}E_{iN}\left(X_j-\frac{r_j'(t_{ia_N,N})}{r_j(t_{ia_N,N})}\right)^2$$
$$=\frac{1}{\sigma^2_{ia_N,a_N}}\sum_{j=ia_N+1}^{(i+1)a_N}\left\{\frac{r_j''(t_{ia_N,N})}{r_j(t_{ia_N,N})}-\left(\frac{r_j'(t_{ia_N,N})}{r_j(t_{ia_N,N})}\right)^2\right\}$$
$$=\frac{1}{\sigma^2_{ia_N,a_N}}\sum_{j=ia_N+1}^{(i+1)a_N}\psi_j''(t_{ia_N,N}). \tag{18}$$

By (a), there exists $M>0$ such that $r_j''(t)\leqslant M$ as well as $\psi_j''(t)\leqslant M$ for every j and any $t\in[t_1,t_2]$ (or $[t_1,T-\delta]$). And by condition (b), there exists $v>0$ such that $\psi_j''(t)\geqslant v$ for the same t and a lot of j. Then

$$0<c_1\leqslant E_{iN}\xi^2_{iN}\leqslant c_2<\infty. \tag{19}$$

Similarly to calculation in (18), we have $E_{iN}\left(X_j-\frac{r_j'(t_{ia_N,N})}{r_j(t_{ia_N,N})}\right)^4=\psi_j^{(IV)}(t_{ia_N,N})$. And $\psi_j^{(IV)}(t_{ia_N,N})\leqslant M'<\infty$. Thus $\xi_{iN}/(E_{iN}\xi^2_{iN})^{1/2}$ converges to standard normal variable in distribution by Lyapou-nov's center limit theorem. Combining it with (19) implies that for every i and all large N, the probability in the right hand side of (17) is larger than a positive number p_ε. Thus the left hand side of (17)

$$P(\xi_{iN}\geqslant -\varepsilon)\geqslant p_\varepsilon \rho_{i a_N, a_N}(\alpha_{i a_N, N})\exp\left(\frac{\varepsilon}{2}t_1\sigma_0\sqrt{a_N}\right)=\frac{p_\varepsilon}{N}\exp\left(\frac{\varepsilon}{2}t_1\sigma_0\sqrt{a_N}\right)$$

Inserting it into (16), we obtain

$$P\left\{\max_{0\leqslant n\leqslant N-a_N}\left(\frac{S_{n+a_N}-S_n}{\sigma_{na_N}}-\sigma_{na_N}\alpha_{nN}\right)\leqslant -\varepsilon\right\}$$

$$\leqslant \exp\left\{-p_\varepsilon\frac{1}{2a_N}\exp\left(\frac{\varepsilon}{2}t_1\sigma_0\sqrt{a_N}\right)\right\}\triangleq J_N. \tag{20}$$

It is easy to see that $\sum_{N=1}^{\infty}J_N<\infty$. This proves (15). Combining (15). with (11) implies the conclusions of the theorem.

§ 3. The Order of a_N is Larger than $\log N$.

Suppose that $\{a_N\}$ satisfies conditions (i)—(iv) in the introduction. At this time, the restrictions on $\{X_n\}$ can be relaxed.

Theorem 2. *Suppose that $\{X_n\}$ satisfies*

(a)′ *there exists $t_0>0$, such that $r_n(t)\leqslant M<\infty$ for every n and $t\in[0, t_0)$,*

(b)′ *there exists $\sigma_0^2>0$ such that $\varliminf_{N\to\infty}\frac{N_n}{N}\geqslant\alpha>0$ for every n where $N_n=\#\{j: \sigma_0^2\leqslant\sigma_j^2,\ n<j\leqslant n+N\}$.*

And suppose that $\{a_N\}$ is a sequence of positive integers satisfying conditions (i)—(iv). *Then the conclusions in Theorem* 1 *remain true.*

Proof First, we prove

$$t_{nN}\to 0 \quad (N\to\infty) \tag{21}$$

uniformly in n. In fact, (10) can be written as $\varlimsup_{N\to\infty}\alpha_{nN}\leqslant gt/2$ ($t\in[0, T_1]$) under condition (iv). Thus $\lim_{N\to\infty}\alpha_{nN}=0$ uniformly in n. Furthermore, $\psi_n'(t)$ is nondecreasing because $\psi_n''(t)\geqslant 0$. Hence, noting condition (b)′, we see that for $t_1\in(0, t_0)$ there exists $h>0$ such that $\psi_j'(t)\geqslant h$ for a lot of j and $t\in[t_1, t_0)$. Therefore, by using (6) and $\alpha_{nN}\to 0$, (21) is true.

Since $\psi_n(t)=\frac{t^2}{2}\sigma_n^2+O(t^3)$ $(t\to 0)$, noting condition (a)′ (implies that for every k, k-th moments are bounded uniformly in n), we can write

$$\psi_{na_N}(t)\triangleq\log R_{na_N}(t)=\sum_{j=n+1}^{n+a_N}\psi_j(t)=\frac{t^2}{2}\sigma_{na_N}^2(1+O(t)) \quad (t\to 0).$$

And the order $O(t)$ in the right hand side tends to zero uniformly in n and N as $t\to 0$. Comparing the above expression with (6) implies

$$t_{nN}/\alpha_{nN}\to 1 \quad (N\to\infty) \tag{22}$$

uniformly in n. Furthermore, from (5) we have

$$t_{nN}\sigma_{na_N}^2\alpha_{nN}\geqslant\log N.$$

Hence there exists $d>0$ such that for every n and all large N, $\alpha_{nN}^2\sigma_{na_N}^2\geqslant d\log N$, i. e.

$$\alpha_{nN}\geqslant\frac{(d\log N)^{1/2}}{\sigma_{na_N}}.$$

Put $N_i=\sup\{N,\ d\log N\leqslant\log^3 i\}$. If N_i is such that $a_{N_i}-a_{N_{i-1}}=1$ (by condition (iii), $a_{N_i}-a_{N_{i-1}}\leqslant 1$ for all large i), we modify the definition of N_{i-1} by setting $N_{i-1}=\sup\{N,\ a_N=a_{N_i}-1\}$ if necessary. Then we have also (13) and $a_N=a_{N_i}$ for $N_{i-1}<N\leqslant N_i$. In imitation of (14),

$$\begin{aligned}&\sigma_{na_{N_i}}\alpha_{nN_i}-\sigma_{na_N}\alpha_{nN}=\sigma_{na_{N_i}}(\alpha_{nN_i}-\alpha_{nN})\\&\leqslant\frac{1}{t_{nN_i}\sigma_{na_{N_i}}}(\log N_i-\log N)\leqslant\frac{c}{(\log N_i)^{1/2}}(\log N_i-\log N)\\&\leqslant c\log^{-1/2}N_i(\log N_i-\log N)\leqslant c\log^{-3/2}(i-1)(\log^3(i+1)-\log^3(i-1))\to 0\\&\qquad(i\to\infty).\end{aligned}$$

Hence (11) in Theorem 1 is also true.

In order to prove (15), (16)—(18) are still employed. Now we have (21) and (a)′, (b)′, so (19) remains true. The rest is the same as the corresponding part in Theorem 1. The proof of Theorem 2 is completed.

§ 4. The Order of a_N is Lower than $\log N$

Theorem 3. *Suppose that $\{X_n\}$ satisfies*

(a)″ *there exist $T\in(0,\infty]$ and an increasing function $M(t)\uparrow\infty$ $(t\uparrow T)$ such that $r_n(t)\leqslant M(t)$ for every n and $t\in[0,T)$,*

(b)″ *there exist $0<b_1\leqslant b_2<\infty$ such that for every n $\overline{\lim\limits_{t\to T}}\,\psi_n''(t)\leqslant b_2$ and $\underline{\lim\limits_{N\to\infty}}\,N_n/N\geqslant\alpha>0$ where $N_n=\#\{j,\ b_1\leqslant\underline{\lim\limits_{t\to T}}\psi_j''(t),\ n<j\leqslant n+N\}$.*

And suppose that $\{a_N\}$ satisfies conditions (i), (ii) *and*

(iii)′ $a_N/\log N\downarrow 0(N\to\infty)$,

(iv)′ *there exists $\tau\in(0,1)$ such that $a_N/\log^\tau N\uparrow\infty$ as $N\to\infty$.*

Then the conclusions of Theorem 1 *remain true.*

Proof First, we prove

$$t_{nN}\to T\quad(N\to\infty)\tag{23}$$

uniformly in n. In fact, from (5), $t_{nN}\alpha_{nN}\to\infty$ $(N\to\infty)$ uniformly in n by condition (iii)′. It is enough to consider the case of $\alpha_{nN}\to\infty$. At this time, from (6), for any $M>0$ there exist some j such that $\psi_j'(t_{nN})>M$ for all large N. Noting condition (a)″ (implies that there exists a function $A(t)\uparrow\infty$, $t\uparrow T$, such that $r_n'(t)\leqslant A(t)$), we have proved (23).

Put $t_N=\min\limits_{0\leqslant n\leqslant N-a_N}t_{nN}$. Certainly $t_N\to T$ as well. Let $N_i=\sup\{N,\ \log^\tau N\leqslant\log^3 i\}$. If N_i is such that $a_{N_i}-a_{N_{i-1}}=1$, we modify the definition of N_{i-1} into $N_{i-1}=\sup\{N,\ a_N=a_{N_i}-1\}$ if necessary. Then we have also (13) and $a_N=a_{N_i}$ for $N_{i-1}<N\leqslant N_i$. In imitation of (14)

$$\begin{aligned}\sigma_{na_{N_i}}\alpha_{rN_i}-\sigma_{na_N}\alpha_{rN} &\leqslant \frac{1}{t_{rN_{i-1}}\sigma_{na_{N_i}}}(\log N_i-\log N)\\ &\leqslant c\log^{-\tau/2}N_i(\log N_i-\log N)\\ &\leqslant c\log^{-3/2}(i-1)(\log^{3/\tau}(i+1)-\log^{3/\tau}(i-1))\to 0 \quad (i\to\infty).\end{aligned}$$

Hence (11) remains true.

Recalling (23) and applying condition (b)′, we have (19) for large N by (18). Then (15) still holds. The theorem is proved.

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