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NUMBER THEORETIC METHODS IN APPLIED STATISTICS (II)

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Abstract

In this paper, the authors give some applications of F-uniformly distributed sequences, which are suggested in their previous paper under the same title, in experimental design, experiments with mixtures, geometric probability and simulation.

§1. Introduction

In our previous paper[13] we proposed a method to produce sets of points which are uniformly distributed over a domain D of R^s and gave some applications in numerical evalution of probabilties and moments of a continuous multivariate distribution and optimization. In this paper we shall give the applications of this kind of uniformly distributed sets of points in experim ental designs for both independent factors and experiments with mixtures in Section 2 and Section 3, and in geometric probability in Section 4. We also give examples to show the comparison between the number-theoretic method and some other methods.

The definitions and notations given in our paper [13] are retained hereafter.

§2. Uniform Design

If there are s factors and each factor has n levels, then the number of all possible ex-periments is n^s. The orthogonal array is to choose $O(n^2)$ experiments among them by the theory of orthogonal Latin squares and group theory. However the number of experiments in orthogonal array is still large if n is comparatively large. The number of experiments may be decreased by BIB (balanced incomplete blocks) method for the case of $s=2$ only. Hence it requires to find a method for decreasing the number of experiments.

Wang and Fang (1981) proposed a kind of experimental designs, the uniform

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designs, by the number theoretic method which has been applied satisfactorily in designs of new products in textile industry, metallurgical industry, engineering industry and agriculture in China.

We offer a set of tables $U_n(b^c)$ of uniform designs, where n denotes the number of experiments, b the number of levels and $c$the maximum number of factors. For example, if there are 3 factors A, B, C and each has 11 levels A_i, B_i, C_k in a design, then a possible choice is to use the table U_{11} (11^6) listed in Table 1.

Table 1, $U_{11}(11^6)$

numbers\columns	1	2	3	4	5	6
1	1	2	3	5	7	10
2	2	4	6	10	3	9
3	3	6	9	4	10	8
4	4	8	1	9	6	7
5	5	10	4	3	2	6
6	6	1	7	8	9	5
7	7	3	10	2	5	4
8	8	5	2	7	1	3
9	9	7	5	1	8	2
10	10	9	8	6	4	1
11	11	11	11	11	11	11

There is a table attached to each $U_n(b^c)$ which indicates how to select columns for the s factors. For $U_{11}(11^6)$, the attached table is Table 2.

Table 2. Table attached to U_{11} (11^6)

number of factors	recommended columns
2	1 5
3	1 4 5
4	1 2 4 5
5	1 2 3 4 5
6	1 2 3 4 5 6

For our problem, the columns 1, 4, 5 are recommended. Finally we list the design of exreriments in Table 3. Therefore only 11 experiments are deigned for 3 factors and each has 11 levels.

Tables of uniform design are obtained by an integer vector $(h_1, \cdots, h_s; n)$ in which $h_1 = 1$, $h_1 < h_2 < \cdots < h_s$ and g. c. d. $(h_i, n) = 1$, $i = 1, \cdots, s$. Let

$$P_n(k) = (kh_1, \cdots, kh_s) \equiv (q_{k1}, \cdots, q_{ks}) \pmod{n}, \qquad (2.1)$$

where $0 < q_{kj} \leqslant n$, $k = 1, \cdots, n$, $j = 1, \cdots, s$. Table U_n (n^s) is formed by (q_{kj}). When $n = 11$, $s = 6$, $h_1 = 1$, $h_2 = 2$, $h_3 = 3$, $h_4 = 5$, $h_5 = 7$ and $h_6 = 10$, the corresponding table of uniform design is just Table 1.

Table 3. The design of experiments

No.	A	B	C
1	A_1	B_5	C_7
2	A_2	B_{10}	C_3
3	A_3	B_4	C_{10}
4	A_4	B_9	C_6
5	A_5	B_3	C_2
6	A_6	B_8	C_9
7	A_7	B_2	C_5
8	A_8	B_7	C_1
9	A_9	B_1	C_8
10	A_{10}	B_6	C_4
11	A_{11}	B_{11}	C_{11}

Since $1 \leqslant h_i < n$ and g. c. d. $(h_i, n) = 1$, $i = 1, \cdots, s$, the number of possible h_i is given by the Euler function $\phi(n)$

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \tag{2.2}$$

where p runs over the prime factors of n. Since $-h = n - h \ (mod n)$, the rank of matrix (q_{kj}) is at most $1 + \phi(n)/2$ if $n > 2$, i. e., the number of factors must be $\leqslant \phi(u)/2 + 1$. There are at most $\binom{\phi(n)/2}{s-1}$ possible choices of $\boldsymbol{h} = (h_1, \cdots, h_s)'$ since $h_1 = 1$. We want to obtain the "best" $\boldsymbol{h}$ among $\boldsymbol{h}$'s. Wang and Fang (1981) noted that a best $\boldsymbol{h}$ is the minimum of the function

$$D(\boldsymbol{h}) = \frac{1}{n} \sum_{k=1}^{n} \sum_{v=1}^{s} \left(1 - \frac{2}{\pi} \log\left(2 \sin\left\{\frac{\pi k h_v}{n+1}\right\}\right)\right) \tag{2.3}$$

with respect to $\boldsymbol{h}$, where $\{x\}$ denote the fractional part of x, and its corresponding set of points is (2.1) which is called a uniform design.

When n is large it will cost much time for finding the best $\boldsymbol{h}$. We suggest to use a set of points of the type

$$Q_n(k) = (k, kb, \cdots, kb^{s-1}) \pmod n, \ 1 \leqslant k \leqslant n \tag{2.4}$$

instead of $P_n(k)$, where b is an integer satisfying $1 < b \leqslant n/2$ and $b^i \neq b^j \pmod n$, $1 \leqslant i < j \leqslant s-1$. Integer b is usually chosen a primitive root mod n, if n is a prime number. We call also a set of points (2.4) with minimum $D(\boldsymbol{b})$ among all those $\boldsymbol{b} = (1, b, \cdots, b_{s-1})'$ a uniform design. Most of the tables of uniform design are produced by (2.4).

A table of uniform design for the case of even n can be obtained by omitting the last row of a table for $n+1$. For instance, the table $U_{10}(10^6)$ can be obtained by omitting the last row of $U_{11}(11^6)$ (Cf. Table 1).

Data in the uniform design can not be analysed by the usual analysis of variance, because the number of experiments is too small compared with the

numbers of factors and levels. But we can treat the data by regression or stepwise regression.

Example 1. To design a Vinylon product we shall consider the following factors:

A: Temperature (C),

B: Time (m),

C: Concentration of methanol (g/l),

D: Concentration of sulphuric acid (g/l),

E: Concentration of mirabilite (g/l).

Each factor has 7 levels listed in Table 4.

Table 4. Factors and levels

factors\levels	1	2	3	4	5	6	7
A	64	66	68	70	72	74	76
B	14	16	18	20	22	24	26
C	18	20	22	24	26	28	30
D	206	212	218	224	230	236	242
E	70	70	85	85	85	100	100

If we use Latin square design, then we require 49 experiments which lead to the following linear regression equation

$$\hat{y}=-42.37+0.55x_1+0.38x_2+0.26x_3+0.10x_4-0.04x_5 \qquad (2.5)$$

with a multiple correlation coefficient $R=0.97$ and a standard deviation $\hat{\sigma}=0.83$. Here $\hat{y}$ stands for the quality of Vinylon. Now we use uniform design and choose $U_{14}(14^5)$, which designs for 14 levels. We repeat the original levels twice by the quasi-level method, and obtain the regression equation as follows

$$\hat{y}=-57.97+0.37x_1+0.46x_2+0.38x_3+0.17x_4+0.04x_5 \qquad (2.6)$$

with $R=0.96$ and $\sigma=1.13$. Equation (2.6) is close to (2.5). The result is not too bad because only 14 experiments are arranged.

§ 3. Experiments with Mixtures

If s factors $X_1, \cdots, X_s$ are non-negative and satisfy $X_1+\cdots X_s=1$, then the experiments are called the experiments with mixtures, which offen appear in designing the chemical and metallurgical products. In the last two decades, a lot of works appeared in the statistical literature have proposed some kinds of designs. Scheffe (1958) introduced the simplex-lattice designs and the polynomial models. He (Scheffe (1963)) suggested an alternative design, the simple-centroid design, to the general simplex-lattice. Draper and Lawrence (1965) proposed to use the designs

which minimized the bias in the fitted model as well as the variance through minimizing the mean square error of the estimate of response over the experimental region. Thompson and Myers (1968) considered an elliptical region inside the simplex factor space by rotatable design. Snee (1973) suggested techniques for the analysis of mixture data. Cornell (1975) gave a suggestion of axial design and he ((1973), (1981)) gave a thorough review of this subject.

In this section we give a different approach to the experiments with mixtures by a uniformly distributed set of points over the simplex. This kind of designs has the same ad-vantages as the uniform design mentioned in Section 2. We pay attention to the best formulation of ingredients, and call this design the uniform design fo r experiments with mixtures (UDEM). Let

$$T_{s-1}=\{(x_1, \cdots, x_s): x_i\geqslant 0,\ i=1, \cdots, s,\ x_1+\cdots+x_s=1\}$$

be a part of the surface of the unit simplex in R^s. The idea of uniform design for experiments with mixtures is to design n experiments which are uniformly distributed on T_{s-1}.

Let $\{q_{ki},\ i=1, \cdots, s-1,\ k=1, \cdots, n\}$ be a uniform design defined by (2.1). Then $\{b_{ki}\}$ with

$$b_{ki}=\frac{2q_{ki}-1}{2n},\ k=1, \cdots, n,\ i=1, \cdots, s-1 \tag{3.1}$$

is a set of uniformly distributed points in I^{s-1}. In Section 2 of [13] we suggest a method to produce a uniformly distributed set of points $\{\boldsymbol{P}_k,\ k=1, \cdots, n\}$ on T_{s-1} with $\boldsymbol{P}_k=(x_{k1}, \cdots, x_{ks})'$ and

$$\begin{cases} x_{kj}=\prod_{i=1}^{j-1} b_{ki}^{1/(s-i)}(1-b_{kj}^{1/(s-j)}),\ j=1, \cdots, s-1 \\ x_{ks}=\prod_{i=1}^{s-1} b_{ki}^{1/(s-i)},\ k=1, \cdots, n. \end{cases} \tag{3.2}$$

Figure 1 shows the uniformity of the set (3.2) when $s=3$ and $n=31$.

Example 2. Consider a regression model

$$Y=e+\sum_{i=1}^{s} e_iX_i+\sum_{i,j=1}^{s} e_{ij}X_i\,X_j+\epsilon,$$

where ϵ stands for a random error. Since $X_1+\cdots+X_s=1$, it can be reduced to the form

$$Y=e+\sum_{i=1}^{s-1} e_iX_i+\sum_{i,j=1}^{s-1} e_{ij}X_iX_j+\epsilon. \tag{3.3}$$

Consider the following special model

$$Y=X_1+X_2-3X_1^2-3X_2^2+X_1X_2+\epsilon, \tag{3.4}$$

where $\epsilon\sim N(0, \sigma^2)$. When σ is small (for example, $\sigma=0.005$), we get the following data for $n=17$ and $s=3$ by the use of UDEM (3.2).

The corresponding regreccion model is

Table 5. DATA

No	X_1	X_2	Y
1	.829	.076	−1.100
2	.703	.253	− .541
3	.617	.102	− .391
4	.546	.307	− .157
5	.486	.045	− .160
6	.431	.284	.038
7	.382	.564	− .230
8	.336	.215	.146
9	.293	.520	− .103
10	.252	.110	.163
11	.214	.439	.031
12	.178	.798	− .889
13	.143	.328	.134
14	.109	.708	− .644
15	.076	.190	.155
16	.045	.590	− .388
17	.015	.029	.000

$$\hat{Y} = -0.0376 + 1.1162X_1 + 1.1197X_2 - 3.0842X_1^2 - 3.0880X_2^2 + .8336X_1X_2 \tag{3.5}$$

which is close to the model (3.4). The multiple correlation coefficient of the equation (3.5) is $R=0.9999$ and the estimate of standard deviation is $\delta=0.0054$ which is close to $\sigma=0.005$ too.

When σ is getting large, we can not get so excellent results. For example, consider the model

Table 6.

No	X_1	X_2	Y
1	0.817	0.055	8.508
2	0.684	0.179	9.464
3	0.592	0.340	9.935
4	0.517	0.048	9.400
5	0.452	0.201	10.680
6	0.394	0.384	9.748
7	0.342	0.592	9.698
8	0.293	0.118	10.238
9	0.247	0.326	9.809
10	0.204	0.557	9.732
11	0.163	0.809	8.933
12	0.124	0.204	9.971
13	0.087	0.456	9.881
14	0.051	0.727	8.892
15	0.017	0.033	10.139

$$Y=10+X_1-3X_1^2-3X_2^2+X_1X_2+\epsilon \tag{3.6}$$

where $\epsilon\sim N(0,\ \sigma^2)$ with $\sigma=0.30$, and 15 experiments by (3.2). We obtain the data (Table 6) by simulation.

The correspondind regression equation now becomes

$$\hat{Y}=10.0908+0.7972X_1-3.4542X_1^2-2.6733X_2^2+0.8884X_1X_2 \tag{3.7}$$

with $R=0.9003$ and $\hat{\sigma}=0.2891$. Note that this regression equation deviates from the model (3.6), because there are high correlations between X_1 and X_1^2. In the original model the response Y reaches its maximum 10.0857 at $X_1=0.171$ and $X_2=0.0286$. From the regression equation (3.7) it is easy to show that Y reaches its maximum at $X_1=0.105$ and $X_2=0.0196$ which $\hat{Y}$ in (3.6) is 10.0728 that is close to 10.0857. Hence, from the point of view of the best formulation of ingredients, the result of this example seems nice.

§ 4. Geometric Probability and Simulation

In this section, we use two "case study" methods to illustrate the applications of uniformly ditsributed sets of points over D to the problems in geometric probability and simulation. The readers can understand the general principle from these two examples without essential difficulties. The questions come from practical problems, and have no satisfactory solution for a comparatively long time. Now we propose algorithms for finding their feasible solutions by the use of number theoretic method.

A. The area of the intersection between a fixed circle and the union of a set of random circles. Given a unit circle K with centre at the origin. There are m random circles $O_1,\cdots,\ O_m$ with centres $P_1,\ \cdots,\ P_m$ and radii $R_1,\ \cdots,\ R_m$, respectively. Assume that

$$P_i\sim N_2(\mathbf{0},\ \sigma_i^2\boldsymbol{I}_2),$$

where $\sigma_i>0$ and $\boldsymbol{I}_2$ denotes the 2×2 identity matrix. Let S be the overlap area between K and the union of all random circles, i. e.,

$$S=K\cap(O_1\cup\cdots\cup O_m).$$

It is required to know the distribution of S. It is easy to find the distribution of S for the case of $m=1$, since the overlap area of two circles can be expressed explicitly in terms of the distance between their centres. When $m>1$, it seems difficult to find a feasible method for finding the distribution of S. This is a problem of geometrical probability. A. natural way is to use simulation. The classical method is the so-called lattice points method. Let ABCD be the circumscribed square of the unit circle K as shown by Figure 3. Divide the square ABCD into n^2 equal subsquares of side $2/(n-1)$. We have n^2 lattice points

$$\left(-1+\frac{2i}{n-1},\ -1+\frac{2j}{n-1}\right), 0\leqslant i,\ j\leqslant n-1$$

in ABCD. Suppose that there are N points lying in K. We now use comupter to produce m reandom circles with centres $P_i\sim N_2(\mathbf{0},\ \sigma_i^2 I^2)$ and radii $R_i(i=1,\ \cdots,\ m)$. Suppose that M points among the N lattice points are covered by these m random circles. Then we get an obvservation $\pi M/N$ for the distribution of S. We then generate other m random circler and obtain another observation. Continuing this process, we have an empirical distribution of S. This method is called the method I. Its convergent rate is slow. The more serious thing is that its accuracy is low even if we take N large, because there are $O(\sqrt{N})$ points among the N lattice points located nearly the boundary of K.

However we may use a set of points (3.1) of $s=2$ with a linear transformation instead of the above n^2 lattic e points and do simulation as before (cf. Figure 3). We call this method the method II which gives faster convergent rate and higher accuracy than the method I. For example, we take $m=1$ and compare the results of S given by these two methods with the exact value of S. It takes more than 180 minutes by the computer IBMPC/XT and the method I to get a sample of size 1, 000 with an error 0.15, but it needs only 4 minutes by the same computer and the method II to obtain a sample of size 1, 500 with an error 0.02. In general, the method II is faster than the method I about 100$\sim$1000 times.

Note that the set of points (3.1) of $s=2$ is defined in ABCD, but not in K. Inspiring by the numerical interation over K stated in [13], it is possible to define a set of points that is uniformly distributed over K by means of the set (3.1). Let

$$\begin{cases} x=r\cos(2\pi\theta), \\ y=r\sin(2\pi\theta), \quad 0\leqslant\theta<2\pi,\ 0\leqslant r\leqslant 1, \end{cases} \tag{4.1}$$

and let $(\theta_i,\ r)$ $(1\leqslant i\leqslant N)$ be a set of points (4.1). If each point of the set

$$Q_n(i)=(r_i\cos 2\pi\theta_i,\ r_i\sin 2\pi\theta_i),\quad 1\leqslant i\leqslant N \tag{4.2}$$

has a suitable "weight" w_i, (4.2) may be considered as a uniformly distributed set in K (Cf. Figure 4). Since the transformation (4.1) has the Jacobian $2\pi r$, we define the weight for $Q_n(i)$ to be $2\pi r_i$. So if there are M points $Q_n(i_j)\,(j=1,\ \cdots,\ M)$ covered by the m random circles, we use

$$2\pi\sum_{j=1}^{M} r_{i_j}$$

to approximate the areas covered by these m random circles. We call this method the method III. In order to compare the methods II and III, we also take $m=1$ and consider the intersection area of two unit circles K and O with distance d between their centers (cf. Figure 5). We use a set of 1069 points of the type (3.1) in which

844 points are lying in K for the method II, and a set of 828 points of (3.1) is used for the method III. The result is given by Table 7.

Table 7

errors\d	0.1	0.75	0.8	1.3
method II	−0.55%	0.07%	0.19%	0.013%
method III	0.00%	0.04%	0.13%	0.08%

From Example 2 of [13] we may obtain a uniformly distributed set of points denoted by $\{\boldsymbol{x}_k,\ k=1,\ \cdots,n\}$ over K, by which we may have another method—— method IV. Now the simulation is based on $\{\boldsymbol{x}_k,\ k=1,\ \cdots,\ n\}$ with the same manner as method II. Our simulation results show that the methods III and IV almost have the same accuracy.

B. The problem of coveringthe sphere by random belts with a fixed width.

This problem comes from the steel rolling [1]. People wish to increase the life of the roller by using a random rotary ball roller instead of a fixed roller. Its mathematical model may be stated as follows: Let S be the unit sphere $x_1^2+x_2^2+x_3^2=1$ and δ a constant satisfying $0<\delta<0.3$. Let R be a great circle which is uniformly distributed on S and $G_\delta(R)$ be the belt on S with width δ and with R as the equidistant partition curve. Let G_{δ_1}, $\cdots$, G_{δ_n}, $\cdots$ be a sequential sample of the population $G_\delta(R)$. For any $\boldsymbol{x}\in S$, we denote by $D_N(\boldsymbol{x})$ the number of belts which cover $\boldsymbol{x}$ in the first N random belts. If there is a point in S that is covered by m belts where m is a given positive integer, we say that the roller is useless. For a given integer m, let T_m be the minimum of N such that $D_N(\boldsymbol{x})\geqslant m$ for some $\boldsymbol{x}\in S$, i. e.,

$$T_m=\min\{N:\ D_N(\boldsymbol{x})\geqslant m,\ \text{for some } \boldsymbol{x}\in S\}. \tag{4.3}$$

The T_m stands for the life of the roller. We wish to obtain the distribution of T_m and to find some way to increase the life of the roller.

It seems difficult to give a formula for the distribution function of T_m, which leads us to do the problem by simulation. In our simulation the following facts are used:

From Exmple 3 of [13] we can obtain a set $\{\boldsymbol{x}_k=(x_{k1},\ x_{k2},\ x_{k3})',\ k=1,\ \cdots,\ n\}$ which is uniformly distributed on S. More precisely,

$$\begin{cases} x_{k1}=\cos(\pi c_{k1}), \\ x_{k2}=\sin(\pi c_{k1})\cos(2\pi c_{k2}), \quad k=1,\ \cdots,\ n, \\ x_{k3}=\sin(\pi c_{k1})\sin(2\pi c_{k2}), \end{cases} \tag{4.4}$$

where $F_i(c_{ki})=b_{ki}$, $i=1,\ 2$, $k=1,\ \cdots,\ n$, $\{\boldsymbol{b}_k=(b_{k1},\ b_{k2})',\ k=1,\ \cdots,\ n\}$ is a uniformly distributed set on I^2, and

$$F_i(x)=\frac{\pi}{B\left(\frac{1}{2},\ \frac{3-i}{2}\right)}\int_0^x(\sin\pi t)^{(3-i-1)}dt,\ i=1,\ 2,$$

i. e.,
$$F_1(x)=\frac{1}{2}(1-\cos(\pi x))$$

and
$$F_2(x)=x.$$

Therefore
$$b_{k1}=\frac{1}{2}(1-\cos(\pi c_{k1}))$$

$$b_{k2}=c_{k2}$$

and
$$\begin{cases} x_{k1}=1-2b_{k1}, \\ x_{k2}=2\sqrt{b_{k1}-b_{k1}^2}\cos(2\pi b_{k2}), \\ x_{k3}=2\sqrt{b_{k1}-b_{k1}^2}\sin(2\pi b_{k2}),\ \ k=1,\ 2,\ \cdots,\ n. \end{cases} \tag{4.5}$$

Let
$$T_m^*=\min\{N:\ D_N(\boldsymbol{x}_k)\geqslant m,\ \text{for some } k,\ 1\leqslant k\leqslant n\}.$$

When n is sufficiently large, T_m^* is close to T_m and the distribution of T_m^* is close to that of T_m. Our simulation is based on T_m^*.

Given a point $\boldsymbol{v}$ on S, it corresponds a great circle R such that the normal direction of the plane including R is the direction of $\overrightarrow{o\boldsymbol{v}}$. If we identify the points $\boldsymbol{v}$ and $-\boldsymbol{v}$, then it has a one-one correspondence between the points on S and the great circles and consequently the belts with width δ on S. Thus, to generate a random belt $G_\delta(R)$ which is uniformly distributed on S is equival ent to generate a point $\boldsymbol{v}\in S$ which is uniformly distributed on S. We also use $G_\delta(\boldsymbol{v})$ to denote the belt corresponding to $\boldsymbol{v}$. Our problem of simulation includes the following steps:

Step 1. Give m and δ, for instance, $m=20$ and $\delta=0.2$.

Step 2. Choose a suitable n (for example, $n=1069$) and produce a set of n points $\{\boldsymbol{x}_k,\ k=1,\ \cdots,\ n\}$ which are uniformly distributed on S.

Step 3. By a standard technique of the simulation, generate sequentially the points $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \cdots$, which are independent and uniformly distributed on S and consequently we have the corresponding belts $G_\delta(\boldsymbol{v}_1),\ G_\delta(\boldsymbol{v}_2),\ \cdots$.

Step 4. If there are $N(N=1,\ 2,\ \cdots)$ random belts to be generated in the current step, account the number of belts covered $\boldsymbol{x}_k$ and denote by $D_N(\boldsymbol{x}_k)$. If $D_N(\boldsymbol{x}_k)=m$ for some k, go to Step 5, otherwise go back to Step 3 and generate the $(N+1)th$ random belt.

Step 5. Account the number of random belts generated already. This number is an obcervation of T_m^*.

Repeat the above process n_0 times and obtain a sample of size n_0 of T_m^*. We take $n_0=5000$, the respective sample mean and the sample standard deviation are

$$\overline{T}_m^*=99.7 \text{ and } \sigma(T_m^*)=9.8.$$

Furthermore, the corresponding empirical distribution is close to the normal distribution.

By the same way, we obtain 20 samples of size $n_0=5000$ (total 100,000 observations) and find that the results are very close to each other.

Since the T_m (*or* T_m^*) stands for the life of roller, we note that sometimes T_m^* can be reached at 125 in 100.000 obcervations by the above simulation. We denote the corresponding normal directions by $\boldsymbol{v}_1^*$, $\boldsymbol{v}_2^*$, $\cdots$, $\boldsymbol{v}_{125}^*$. This means that if $\boldsymbol{v}_i=\boldsymbol{v}_i^*$, $i=1$, 2, $\cdots$, are fixed, we always have $T_m^*=125$ in the case of $\delta=0.2$ and $m=20$, which is better than the above random choices of $\{\boldsymbol{v}_i\}$. Is it possible to find another set of $\boldsymbol{x}_1^{**}$, $\boldsymbol{v}_2^{**}$, $\cdots$ to beat the above $\{\boldsymbol{v}_i^*, i=1, \cdots, 125\}$? We may use the uniformly distributed sets of points on I^2 to produce the sets of $\{\boldsymbol{v}_k^{**}, k=1, \cdots, n\}$ on S. First we choose $n=126$ and find the corresponding $T_m^*=126$. Then we increase n one by one untill the T_m^* can not be increased any more. Finally we find a set $\{\boldsymbol{v}_k^{**}=(v_{k1}, v_{k2}, v_{k3})', k=1, 2, \cdots, 155\}$ by which $T_m^*=155$! The $\boldsymbol{v}_k^{**}$ is given by

$$\begin{cases} v_{k1}=1-2b_{k1}, \\ v_{k2}=2\sqrt{b_{k1}-b_{k1}^2}\cos(2\pi b_{k2}), \\ v_{k3}=2\sqrt{b_{k1}-b_{k2}^2}\sin(2\pi b_{k2}), \end{cases}$$

where the set $\{\boldsymbol{b}_k=(b_{k1}, b_{k2})', k=1, \cdots, 155\}$ is produced by $(h_1, h_2; n)=(1, 20; 155)$ (See [13], Section 3 for details).

This example indicates that the number theoretic method wins the champion of 100, 000 experiments by Monte carlo method in our simulation.

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