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A RIGIDITY THEOREM FOR NON-NEGATIVELY IMMERSED SUBMANIFOLDS

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Abstract

The paper is to generalize the rigidity theorem that the special Weingarten surface is the sphere to the case of submanifolds. It is proved that a non-negatively immersed compact submaifnold in space form of constant curvature is a Riemannian product of several totally umbilical submanifolds if the mean curvature and the scalar curvature of the submanifold satisfy a certain function relation.

§ 1. Introduction

After the original work of J. Simons[7] who first studied the minimal submanifolds by calculating the Laplacian of the second fundamental form, the submanifolds with parallel mean curvature vector immersed in a Riemannian manifold with constant sectional curvature have been extensively considered. K. Nomizu and B. Smyth[1] classified the non-negatively immersed compact hypersurfaces with constant mean curvature in space form of constant curvature. A well consequence is to classify the non-negatively immersed compact minimal hypersurfaces in sphere. J. Erbacher[6], K. Yano and Ishihara[5] generalized it to high codimension case simultaneously. S. T. Yau[3] showed that the compact hypersurfaces with constant scalar curvature and positive sectional curvature immersed in a space form of constant curvature are totally umbilical. Later, S. Y. Cheng and S. T. Yau[4] classified the non-negatively immersed compact hypersurfaces with constant scalar curvature in space of constant curvature. In this paper, we not only generalize Cheng and Yau's resultto high codimension case but also unify the works of Erbacher, Yano-Ishihara and Cheng-Yau. We have the following.

Main Theorem. *Let M be an n-dimensional compact submanifold with non-negative sectional curvature immersed in an $(n+p)$-dimensional Riemannian manifold with constant sectional curvature c. Suppose the normal bundle is flat and the*

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unit mean curvatvre vector is locally parallel. If the mean curvature H and the scalar curvature R of M satisfy the following function relatlon

$$f(H, R)=0, \tag{1.1}$$

$$\left(\frac{\partial f}{\partial H}\right)^2+\frac{4nH}{n-1}\frac{\partial f}{\partial H}\frac{\partial f}{\partial R}+\frac{4n}{n-1}(R-c)\left(\frac{\partial f}{\partial R}\right)^2>0, \tag{1.2}$$

then M is a Riemannian product of totally umbilical submanifold, i.e.

$$M=M_1\times M_2\times\cdots\times M_m,\ m\leqslant\min(n, p+1)$$

where M_i $(1\leqslant i\leqslant m)$ are totally umbilical.

§2. Local Formulas

We shall state the structure equations of Riemannian manifold as Chern has done[2]. Let N be an $(n+p)$-dimensional Riemannian manifold with constant sectional curavture c and M an immersed n-dimensional submanifold. We shall make use of the convention of the range of indices:

$$1\leqslant A, B, C, \cdots\leqslant n+p;\ 1\leqslant i, j, k, \cdots\leqslant n;$$
$$n+1\leqslant\alpha, \beta, \gamma, \cdots\leqslant n+p.$$

On N, we choose orthonormal local frames $\{e_1, \cdots, e_{n+p}\}$ such that, restrited to M, $e_1, \cdots, e_n\in T(M)$. Let $\{\omega_1, \cdots, \omega_{n+p}\}$ be the coframes of $\{e_1, \cdots, e_{n+p}\}$. The structure equations of N are written as

$$\begin{aligned}
&d\omega_A=\sum_B\omega_{AB}\wedge\omega_B,\ \omega_{AB}+\omega_{BA}=0,\\
&d\omega_{AB}=\sum_C\omega_{AC}\wedge\omega_{CB}+\Phi_{AB},\\
&\Phi_{AB}=-\frac{1}{2}\sum_{CD}K_{ABCD}\omega_C\wedge\omega_D,\\
&K_{ABCD}=c(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}).
\end{aligned}\tag{2.1}$$

Restricted to M, we have $\omega_\alpha=0$ and $0=d\omega_\alpha=\sum_i\omega_{\alpha i}\wedge\omega_i$. By Cartan's lemma, we get

$$\omega_{i\alpha}=\sum_j h^\alpha_{ij}\omega_j,\ h^\alpha_{ij}=h^\alpha_{ji}. \tag{2.2}$$

Of counrse, we also have

$$\begin{aligned}
&d\omega_i=\sum_j\omega_{ij}\wedge\omega_j,\ \omega_{ij}+\omega_{ji}=0,\\
&d\omega_{ij}=\sum_k\omega_{ik}\wedge\omega_{kj}+\Omega_{ij},\\
&\Omega_{ij}=-\frac{1}{2}\sum_{k,l}R_{ijkl}\omega_k\wedge\omega_l,\\
&R_{ijkl}=K_{ijkl}+\sum_\alpha(h^\alpha_{ik}h^\alpha_{jl}-h^\alpha_{il}h^\alpha_{jk}),\\
&dh^\alpha_{ij}-\sum_k h^\alpha_{kj}\omega_{ik}-\sum_k h^\alpha_{ik}\omega_{jk}+\sum_\beta h^\beta_{ij}\omega_{\beta\alpha}=\sum_k h^\alpha_{ijk}\omega_k,\\
&h^\alpha_{ijk}=h^\alpha_{ikj},\\
&dh^\alpha_{ijk}-\sum_l h^\alpha_{ljk}\omega_{il}-\sum_l h^\alpha_{ilk}\omega_{jl}-\sum_l h^\alpha_{ijl}\omega_{kl}+\sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}=\sum_l h^\alpha_{ijkl}\omega_l,
\end{aligned}\tag{2.3}$$

$$h^\alpha_{ijkl}-h^\alpha_{ijlk}=-h^\alpha_{im}R_{jmkl}-h^\alpha_{mj}R_{imkl}+\sum h^\beta_{ij}R_{\beta\alpha kl}.$$

The second fundamental form of M is

$$II=\sum_{i,j,\alpha} h^\alpha_{ij}\omega_i\otimes\omega_j\otimes e_\alpha. \tag{2.4}$$

Letting h be the mean curvature vector, H the mean curvature, R the scalar curvature of M and S the square of the length of the second fundamental form, we have

$$\begin{aligned} h&=\frac{1}{n}\sum_\alpha\sum_i h^\alpha_{ii}e_\alpha,\\ H^2&=\frac{1}{n^2}\sum_\alpha\left(\sum_i h^\alpha_{ii}\right)^2,\\ R&=\frac{1}{n(n-1)}\sum_{i\neq j}R_{ijij},\\ S&=\sum_{i,j,\alpha}(h^\alpha_{ij})^2. \end{aligned} \tag{2.5}$$

If the normal bundle of M is flat, then the second fundamental form can be diagolized simutaneously. We write $h^\alpha_{ij}=\lambda^\alpha_i\delta_{ij}$ $\lambda_i=(\lambda^{n+1}_i, \cdots, \lambda^{n+p}_i)$ and $\langle\lambda_i, \lambda_j\rangle=\sum_\alpha\lambda^\alpha_i\lambda^\alpha_j$. At the moment, the Gauss equation is

$$R_{ijij}=\sum_\alpha\lambda^\alpha_i\lambda^\alpha_j+c=\langle\lambda_i, \lambda_j\rangle+c, \ i\neq j. \tag{2.6}$$

Now, let us establish a fundamental identity about the Laplacian ΔS. From (2.3), we have

$$\begin{aligned} \frac{1}{2}\Delta S&=\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2+\sum_{i,j,k,\alpha}h^\alpha_{ij}h^\alpha_{ijkk}\\ &=\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2+\sum_{i,j,k,\alpha}h^\alpha_{ij}h^\alpha_{ijkk}-\sum_{i,j,k,\alpha}h^\alpha_{ij}h^\alpha_{kkij}-\sum_{i,j,k,\alpha}h^\alpha_{ijj}h^\alpha_{kki}+\sum_{i,j,k,\alpha}(h^\alpha_{ij}h^\alpha_{kki})_j\\ &=\sum_{i,k,\alpha}((h^\alpha_{ijk})^2-h^\alpha_{ijj}h^\alpha_{kki})+\sum_{i,j,k,\alpha}h^\alpha_{ij}(h^\alpha_{ijkk}-h^\alpha_{kkij})+\sum_{i,j,k,\alpha}(h^\alpha_{ij}h^\alpha_{kki})_j \end{aligned} \tag{2.7}$$

and

$$h^\alpha_{ijkk}-h^\alpha_{kkij}=\sum_m h^\alpha_{im}R_{mkjk}+\sum_m h^\alpha_{mk}R_{mijk}-\sum_\beta h^\beta_{ik}R_{\alpha\beta jk}. \tag{2.8}$$

If the normal bundle of M is flat, from (2.6) and (2.8), we obtain

$$\sum_{i,j,k,\alpha}h^\alpha_{ij}(h^\alpha_{ijkk}-h^\alpha_{kkij})=\frac{1}{2}\sum_{i,j}\|\lambda_i-\lambda_j\|^2(\langle\lambda_i, \lambda_j\rangle+c). \tag{2.9}$$

§ 3. The Proof of Main Theorem

In order to prove the main theorem, we need the following

Lemma. *Let M be a compact n-dimensional submanifold with flat normal bundle and non-negative sectional curvature immersed in an $(n+p)$-dimensional Riemannian manifold N with constant sectional curvature c. Suppose*

$$\sum_{i,j,k,\alpha}(h^\alpha_{i,j,k})^2=0, \ \sum_{i,j}\|\lambda_i-\lambda_j\|^2(\langle\lambda_i, \lambda_j\rangle+c)=0. \tag{3.1}$$

There

$$M=M_1\times\cdots\times M_m,\ m\leqslant\min(n,\ p+1),$$

where M_i $(1\leqslant i\leqslant m)$ *are totally umbilical.*

Proof Because the normal bundle of M is flat, we can diagonalize the second fundamental form simultaneously, i. e.

$$h_{ij}^\alpha=\lambda_i^\alpha\delta_{ij}.$$

We may write

$$\lambda_1=\cdots=\lambda_{k_1},\ \lambda_{k_1+1}=\cdots=\lambda_{k_1+k_2},\ \cdots,\ \lambda_{k_1+\cdots k_{m-1}+1}=\cdots=\lambda_n$$

and we then, without loss of generality, assume λ_i $(1\leqslant i\leqslant m)$ are the distinct vectors of eigenvalues with the multiplicity k_i.

According to (3.1) and (2.3), we have $(\lambda_i^\alpha-\lambda_j^\alpha)\omega_{ij}=0$ and $\omega_{ij}=0$ if $\lambda_i\neq\lambda_j$. Thus the distributions D_i^* defined by $\omega_{k_1+\cdots+k_{i-1}+1}=\cdots=\omega_{k_1+\cdots+k_i}=0$ and the orthogonal complement D_i are integrable. Denote the integral manifolds by M_i^* and M_i respectively. Since all the $\lambda_i's$ are constant vectors, the leaves of the foliation are all closed and hence compact. It is easy to see that M_i are totally umbilical and the dimension of M_i is k_i. Then M is a Riemannian product $M_1\times\cdots\times M_m$.

From (3.1) and $\langle\lambda_i,\ \lambda_j\rangle+c\geqslant0$, we have $\langle\lambda_i,\ \lambda_j\rangle+c=0$ if $\lambda_i\neq\lambda_j$. Suppose $\lambda_1,\ \cdots,\ \lambda_{p+2}$ are distinct vectors of eigenvalues and let $\lambda_1,\ \cdots,\ \lambda_q(q\leqslant p)$ be linear independent vectors. Then $\lambda_{p+1}=h_1\lambda_1+\cdots+a_q\lambda_q$. By inner product,

$$\langle\lambda_{p+1},\ \lambda_{p+2}\rangle=\sum_\alpha a_\alpha\langle\lambda_\alpha,\ \lambda_{p+2}\rangle,$$

$$\langle\lambda_{p+1},\ \lambda_\alpha\rangle=a_1\langle\lambda_1,\ \lambda_\alpha\rangle+\cdots+a_q\langle\lambda_q,\ \lambda_\alpha\rangle,\ 1\leqslant\alpha\leqslant q,$$

and then

$$-c=-c\sum_\alpha a_\alpha.$$

If $c=0$, by $\langle\lambda_i,\ \lambda_j\rangle=0$, $i\neq j$, we get $\langle\lambda_{p+1},\ \lambda_{p+1}\rangle=0$, $\langle\lambda_{p+2},\ \lambda_{p+2}\rangle=0$; these contradict the independence. So

$$1=\sum_\alpha a_\alpha,\ 1\leqslant\alpha\leqslant q,$$

and hence

$$-c=a_\alpha\langle\lambda_\alpha,\ \lambda_\alpha\rangle-c(1-a_\alpha),$$

i.e.

$$0=a_\alpha(\langle\lambda_\alpha,\ \lambda_\alpha\rangle+c),\ 1\leqslant\alpha\leqslant q.$$

Clearly, there are at least two a_α being not zero. Therefore, there exist $\lambda_1,\ \cdots,\ \lambda_t$ $(1<t\leqslant q)$ such that $\langle\lambda_i,\ \lambda_j\rangle=-c$ $(1\leqslant i,\ j\leqslant t)$. This contradicts the independence. Hence $m\leqslant p+1$.

Proof of Main Theorem Let us consider 1-form

$$\omega=\sum_j\left(\frac{1}{2}S_j-\sum_{i,k,\alpha}h_{ij}^\alpha h_{kki}^\alpha\right)\omega_j.$$

Because M is compact, by Stokes' formula, we have

$$0=\int_M d*\omega=\int_M\left(\frac{1}{2}\Delta S-\sum_{i,j,k,\alpha}(h_{ij}^\alpha h_{kki}^\alpha)_j\right)*1$$

and, from (2.7) and (2.9), we get

$$0=\int_M\{\sum_{i,j,k,\alpha}((h^\alpha_{ijk})^2-h^\alpha_{ijj}h^\alpha_{kki})+\sum_{i<j}\|\lambda_i-\lambda_j\|^2(\langle\lambda_i,\ \lambda_j\rangle+c)\}*1. \tag{3.2}$$

Let $h=He_{n+1}$. Hence

$$\sum_i h^\alpha_{ii}=0 \text{ if } \alpha\neq n+1. \tag{3.3}$$

From (2.3) we have

$$\sum_{i,k} h^{n+1}_{iik}\omega_k=ndH,$$
$$\sum_{i\ k} h^\alpha_{iik}\omega_k=nH\omega_{\alpha n+1} \text{ if } \alpha\neq n+1. \tag{3.4}$$

Because e_{n+1} is parallel in the normal bundle, i.e. $\omega_{n+1\alpha}=0$, from (3.4), we then get

$$\sum_i h^\alpha_{iik}=0 \text{ if } \alpha\neq n+1. \tag{3.5}$$

Since the normal bundle of M is flat, we may write $h^\alpha_{ij}=\lambda^\alpha_i\delta_{ij}$. From (2.5), we get

$$R-c=\frac{1}{n(n-1)}\sum_{i\neq j}\sum_\alpha\lambda^\alpha_i\lambda^\alpha_j,\ H^2=\frac{1}{n^2}\sum_\alpha(\sum_i\lambda^\alpha_i)^2,\ S=\sum_{i,\alpha}(\lambda^\alpha_i)^2,$$

and then

$$n^2H^2-S=n(n-1)(R-c). \tag{3.6}$$

By taking differentiation of (1.1), we get

$$\frac{\partial f}{\partial H}H_k+\frac{\partial f}{\partial R}R_k=0$$

and, from (3.6),

$$\frac{\partial f}{\partial H}H_k+\frac{\partial f}{\partial R}\left(\frac{2n^2HH_k-S_k}{n(n-1)}\right)=0,$$

i. e.

$$\left(\frac{\partial f}{\partial H}+\frac{2nH}{n-1}\frac{\partial f}{\partial R}\right)nH_k=\frac{1}{n-1}\frac{\partial f}{\partial R}S_k.$$

From (3.4) and Cauchy inequality,

$$\left(\frac{\partial f}{\partial H}+\frac{2nH}{n-1}\frac{\partial f}{\partial R}\right)^2(\sum_i h^{n+1}_{iik})^2=\left(\frac{2}{n-1}\frac{\partial f}{\partial R}\right)^2(\sum_{i,\alpha}h^\alpha_{ii}h^\alpha_{iik})^2$$
$$\leqslant\left(\frac{2}{n-1}\frac{\partial f}{\partial R}\right)^2 S\sum_{i,\alpha}(h^\alpha_{iik})^2.$$

According to (1.2), we have

$$\left(\frac{\partial f}{\partial H}+\frac{2nH}{n-1}\frac{\partial f}{\partial R}\right)^2>\left(\frac{2}{n-1}\frac{\partial f}{\partial R}\right)^2S.$$

Then we get

$$(\sum_i h^{n+1}_{iik})^2\leqslant\sum_{i,\alpha}(h^\alpha_{iik})^2, \tag{3.7}$$

and the equality holds if and only if $\sum_{i,\alpha}(h^\alpha_{iik})^2=0$. Otherwise, we get

$$\left(\frac{\partial f}{\partial H}+\frac{2nH}{n-1}\frac{\partial f}{\partial R}\right)^2\leqslant\left(\frac{2}{n-1}\frac{\partial f}{\partial R}\right)^2S, \text{ a contradiction.}$$

From (3.5) and (3.7), we also have

$$\sum_{\alpha}(\sum_{i} h_{iik}^{\alpha})^2 \leqslant \sum_{i,\alpha}(h_{iik}^{\alpha})^2$$

and hence

$$\sum_{i,k,\alpha}(h_{iik}^{\alpha})^2 - \sum_{k,\alpha}(\sum_{i} h_{iik}^{\alpha})^2 \geqslant 0, \tag{3.8}$$

and the equality holds if and only if $h_{iik}^{\alpha}=0$.

From (3.2) and (3.8), we conclude

$$\sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2=0, \quad \sum_{i<j}\|\lambda_i-\lambda_j\|^2(\langle\lambda_i, \lambda_j\rangle+c)=0.$$

We have completed the proof by the lemma.

§ 4. Other Results

We shall prove some further results about hypersurface. Let the codimension $p=1$ and $\varkappa_1, \varkappa_2, \cdots, \varkappa_n$ be the principal curvatures of M. Define

$$H_r = \sum_{1\leqslant i_1\leqslant \cdots \leqslant i_r\leqslant n} \varkappa_{i_1}\varkappa_{i_2}\cdots\varkappa_{i_r}, \quad H_0=1. \tag{4.1}$$

Theorem 1. *Let M be a compact hypersurface with non-negative sectional curvature immersed in a manifold with constant sectional curvature c. If the fundamental symmetric functions of the principal curvatures of M satisfy the following functional relation*

$$f(H_1, H_2, \cdots, H_n)=0, \tag{4.2}$$

$$\sum_{r}\frac{\partial f}{\partial H_r}H_{r-1}\sum_{r}(n-2r+3)\frac{\partial f}{\partial H_r}H_{r-1} > \sum_{i}\left(\frac{\partial f}{\partial \varkappa_i}\right)^2, \tag{4.3}$$

then M is either a totally umbilical hypersurface or a Riemannian product of two totally umbilical submanifolds.

Proof From (3.2) we have

$$0=\int_M\{\sum_{i,j,k}(h_{ijk}^{n+1})^2-\sum_{j}(\sum_{i} h_{iikj}^{n+1})^2+\sum_{i<j}(\varkappa_i-\varkappa_j)^2(\varkappa_i\varkappa_j+c)\}*1. \tag{4.4}$$

Let $h_{ij}=h_{ij}^{n+1}=\varkappa_i\delta_{ij}$. From (2.3) we get

$$H_{rk}=\sum_{i}\frac{\partial H_r}{\partial \varkappa_i}h_{iik}. \tag{4.5}$$

By (4.2), we obtain

$$\sum_{r}\frac{\partial f}{\partial H_r}H_{rk}=0$$

and substituting (4.5) into it, we obtain

$$\sum_{r,i}\frac{\partial f}{\partial H_r}\frac{\partial H_r}{\partial \varkappa_i}h_{iik}=0. \tag{4.6}$$

We shall derive a recursion relations of $\frac{\partial H_r}{\partial \varkappa_i}$. By identity

$$(\lambda-\varkappa_1)\cdots(\lambda-\varkappa_n)=\lambda^n-H_1\lambda^{n-1}+\cdots+(-1)^nH_n \tag{4.7}$$

we get

$$(\lambda_1-\varkappa_1)\cdots(\lambda-\varkappa_{i-1})(\lambda-\varkappa_{i+1})\cdots(\lambda-\varkappa_n)=\lambda^{n-1}-\frac{\partial H_2}{\partial \varkappa_i}\lambda^{n-2}+\cdots+(-1)^n\frac{\partial H_n}{\partial \varkappa_i}. \tag{4.8}$$

Comparing (4.7) with (4.8), we obtain the following recursion relations

$$H_r = \frac{\partial H_{r+1}}{\partial \varkappa_i} + \varkappa_i \frac{\partial H_r}{\partial \varkappa_i},\ 1 \leqslant r \leqslant n-1,\ H_n = \varkappa_i \frac{\partial H_n}{\partial \varkappa_i}. \tag{4.9}$$

Inserting (4.9) into (4.6) yields

$$\sum_{i,r} \frac{\partial f}{\partial H_r}\left(H_{r-1} - \varkappa_i \frac{\partial H_{r-1}}{\partial \varkappa_i}\right) h_{iik} = 0,$$

i.e.

$$\sum_r \frac{\partial f}{\partial H_r} H_{r-1} \sum_i h_{iik} = \sum_{i,r} \frac{\partial f}{\partial H_r} \frac{\partial H_{r-1}}{\partial \varkappa_i} \varkappa_i h_{iik}.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}\left(\sum_r \frac{\partial f}{\partial H_r} H_{r-1}\right)^2 \left(\sum_i h_{iik}\right)^2 &= \left(\sum_{i,r} \frac{\partial f}{\partial H_r} \frac{\partial H_{r-1}}{\partial \varkappa_i} \varkappa_i h_{iik}\right)^2 \\ &\leqslant \sum_i \left(\sum_r \frac{\partial f}{\partial H_r} \frac{\partial H_{r-1}}{\partial \varkappa_i} \varkappa_i\right)^2 \sum_i h_{iik}^2 \\ &= \sum_i \left(\sum_r \frac{\partial f}{\partial H_r}\left(H_{r-1} - \frac{\partial H_r}{\partial \varkappa_i}\right)\right)^2 \sum_i h_{iik}^2 \\ &= \sum_i \left(\sum_r \frac{\partial f}{\partial H_r} H_{r-1} - \frac{\partial f}{\partial \varkappa_i}\right)^2 \sum_i h_{iik}^2.\end{aligned} \tag{4.10}$$

From (4.3) and (4.10), we get

$$\left(\sum_r \frac{\partial f}{\partial H_r} H_{r-1}\right)^2 > \sum_i \left(\sum_r \frac{\partial f}{\partial H_r} H_{r-1} - \frac{\partial f}{\partial \varkappa_i}\right)^2 \tag{4.11}$$

and

$$\left(\sum_i h_{iik}\right)^2 \leqslant \sum_i h_{iik}^2,$$

and the equality holds if and only if $h_{iik} = 0$.

Then, from (4.4) and (4.11), we get

$$\sum_{i,j,k} h_{ijk}^2 = 0,\ \sum_{i<j} (\varkappa_i - \varkappa_j)^2 (\varkappa_i \varkappa_j + c) = 0.$$

By lemma, we have done.

Theorem 2. *Let M be a compact surface with non-negative sectional curvature immersed in a 3-dimensional manifold with constant sectional curvature c. If the mean curvature H and the Gauss curvature R satisfy equation*

$$f(H, R) = 0 \tag{4.12}$$

and the inequality

$$\left(\frac{\partial f}{\partial H}\right)^2 + 4H \frac{\partial f}{\partial H} \frac{\partial f}{\partial R} + 4(R-c)\left(\frac{\partial f}{\partial R}\right)^2 > 0 \tag{4.13}$$

holds, then M is either a totally umbilical surface or a Riemannian product of two totally umbilical submanifolds.

Proof We first note that

$$\frac{\partial f}{\partial \varkappa_1} = \frac{1}{2} \frac{\partial f}{\partial H} + \varkappa_2 \frac{\partial f}{\partial R},$$

$$\frac{\partial f}{\partial \varkappa_2} = \frac{1}{2} \frac{\partial f}{\partial H} + \varkappa_1 \frac{\partial f}{\partial R}.$$

Hence

$$\frac{\partial f}{\partial \varkappa_1}\frac{\partial f}{\partial \varkappa_2}=\frac{1}{4}\left(\frac{\partial f}{\partial H}\right)^2+H\frac{\partial f}{\partial H}\frac{\partial f}{\partial R}+(R-c)\left(\frac{\partial f}{\partial R}\right)^2>0. \tag{4.14}$$

From (4.6),

$$\sum_{i=1}^{2}\frac{\partial f}{\partial \varkappa_i}h_{iik}=0,\ k=1,\ 2.$$

Then by (4.14) there exist ρ, σ such that

$$h_{111}=\rho\frac{\partial f}{\partial \varkappa_2},\quad h_{221}=-\rho\frac{\partial f}{\partial \varkappa_1},$$

$$h_{112}=\sigma\frac{\partial f}{\partial \varkappa_2},\ h_{222}=-\sigma\frac{\partial f}{\partial \varkappa_1}.$$

Therefore,

$$\sum_{i,j,k}h_{ijk}^2-\sum_j\Big(\sum_i h_{iij}\Big)^2=2\sum_j(h_{12j}^2-h_{11j}h_{22j})$$

$$=2\sum_j h_{12j}^2+2(\rho^2+\sigma^2)\frac{\partial f}{\partial \varkappa_1}\frac{\partial f}{\partial \varkappa_2}\geqslant 0. \tag{4.15}$$

From (4.4) and (4.15) we obtain

$$\sum_{i,j,k}h_{ijk}^2=0,\ \sum_{i<j}(\varkappa_i-\varkappa_j)^2(\varkappa_i\varkappa_j+c)=0.$$

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