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ON CONDITIONAL EXPECTATION OPERATORS ON $L(\mu, X)$ $(1\leqslant p\leqslant +\infty, p\neq 2)$

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Abstract

Some characterizations of the conditional expectation operators on Lebesgue–Bochner spaces $L_p(\mu, X)$ are given, where $1\leqslant p<\infty$, $p\neq 2$. Also an example is given to show that the characterizations of the conditional expectation operators on $L_p(\mu, X)$ are different from that on $L_p(\mu)$. Finally, a representation of the constant-preserving contractive projection on spaces $L_p(\mu, X)$ is got when $0<p<1$.

§1. Introduction and Preliminaries

It is well known that a lot of immanent relations between the convergence of that martingales in Lebesgue–Bochner spaces $L_p(\mu, X)$ $(1<p<\infty)$ and the structure properties (e.g. Radon–Nikodym property) of Banach spaces X have been discovered (cf. [1]). But every convergent martingale in $(L_p(\mu, X)$ is generated by the conditional expectations of an element in $L_p(\mu, X)$ relative to a monotone increasing net of sub-σ-fields (cf. [1]). Therefore charaterizing the conditional expectation operators on $L_p(\mu, X)$ is an important problem. Following this direction, a lot of results have been obtained for case $X=\mathbf{R}$ (cf. [2—7]). Recently, P. Landers and L. Rogger in [8] showed that every constant-preserving contractive linear projection on $L_1(\mu, X)$ is a conditional expectation operator, where μ is a probability measure and X is a strictly convex Banach space. They also gave an example to demonstrate that, even if X is a uniformly rotund Banach space, the above result does not hold for $L_p(\mu, X)$ when $1<p<\infty$. In [9] the author gave some characterizations of conditional expectation operators on $L_1(\mu, L_1(\lambda))$. In this paper, we study the characterizations of the conditional expectation operators on Lebesgue–Bochner spaces $L_p(\mu, X)$ for $1\leqslant p<\infty$.

Throughout the rest of this paper, we always assume that (Ω, Σ, μ) is a probability space, all operations of sets work in the modulo μ-null sense; and X is a Banach space. $L_p(\mu, X)$ denotes the Lebesgue–Bochner spaces of p-th integrable

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X-valued functions (cf. [1]), and $L_p(\mu, \mathbf{R})=L_p(\mu)$, $0\leqslant p<\infty$. The definition and elementary properties can be found in [1].

Theorem 1.1. [6,8] *A linear operator T on $L_p(\mu)$ $(1\leqslant p<\infty,\ p\neq 2)$ is a conditional expectation operator if and only if it satisfies the following conditions.*
i) $T^2=T$, ii) $T\chi_\Omega=\chi_\Omega$, iii) $\|T\|=1$.

§2 Some Characterizations of the Conditional Expectation Operator on Spaces $L(\mu, X)$ $(1\leqslant p<\infty,\ p\neq 2)$

Theorem 2.1. *A linear operator T on $L_p(\mu, X)$ $(1\leqslant p<\infty,\ p\neq 2)$ is a conditional expectation operator if and only if it satisfies the following conditions:*

i) $T^2=T$;

ii) $\|T\|=1$;

iii) $Ta\chi_\Omega=a\chi_\Omega$ *for all* $a\in X$;

iv) *For arbitrary $x^*, y^*\in S(X^*)$ (the unit sphere of X^*) and $f, g\in L_p(\mu, X)$, if $x^*(f)=y^*(g)$ a. e., then $x^*(Tf)=y^*(Tg)$ a. e., where $(x^*(f))(t)=x^*(f(t))$.*

Proof The necessity can be deduced from the elementary properties of the conditional expectation operators easily. Now we prove the sufficency.

First, suppose that T is a linear operator on $L_p(\mu, X)$ satisfying i) to iv). For each $x^*\in S(X^*)$, we define a linear operator T_{x^*}: $L_p(\mu)\to L_p(\mu)$ by

$$T_{x*}(X^*(f))=X^*(Tf) \text{ for all } f\in L_p(\mu, X).$$

It is easy to see that for each $\tilde{f}\in L_p(\mu)$ there exists an $f\in L_p(\mu, X)$ such that $\tilde{f}=x^*(f)$. If there also exists a $g\in L_p(\mu, X)$ such that $x^*(f)=x^*(g)=\tilde{f}$, then by iv), we have $T(x^*(f))=T(x^*(g))$. It follows that T is well defined. Clearly, T is a linear operator.

Second, we claim $T_{x*}=E(\,|\mathscr{B}_{x*})$ for some sub-σ-field $\mathscr{B}_{x*}$ of Σ.

In fact, for each $f\in L_p(\mu, X)$

$$T^2_{x*}(X^*(f))=X^*(T^2f)=X^*(Tf)=T_{x*}(X^*(f)).$$

Hence $T^2_{x*}=T$.

For each $\tilde{g}\in L_p(\mu)$ such that $\|\tilde{g}\|=1$, there exists, for each $\varepsilon>0$, a $g\in L_p(\mu, X)$ such that

$$\|g\|<1+\varepsilon \text{ and } x^*(g)=\tilde{g}.$$

Then

$$\|T_{x*}(\tilde{g})\|=\|x^*(TG)\leqslant\|Tg\|\leqslant\|g\|<1+\varepsilon.$$

Since ε is arbitrary, we have $\|T_{x*}\|<1$.

For each $a\in X$ such that $x^*(a)=1$, we have $x^*(a)\chi_\Omega=\chi_\Omega$, and so

$$T_{x*}(\chi_\Omega)=x^*(Ta\chi_\Omega)=x^*(a\chi_\Omega)=\chi_\Omega.$$

It follows that $T_{x*}(\chi_\Omega)=\chi_\Omega$ and $\|T_{x*}\|=1$.

By Theorem 1.1, there exists a sub-σ-field $\mathscr{B}_{x*}$ of Σ such that $T_{x*}=E(\,|\mathscr{B}_{x*})$.

Thirdly, we claim $T_{x^*}=E(\,|\mathscr{B})$ where $\mathscr{B}$ is a sub-σ-field of Σ which does not depend on the choice of x^*.

In fact, suppose $y^*\in S(X^*)$, then there exists a sub-σ-field $\mathscr{B}_{y^*}$ of Σ such that $T_{y^*}=E(\,|\mathscr{B}_{y^*})$. For each $\tilde{f}\in L_p(\mu)$, there exists f_1, $f_2\in L_p(\mu, X)$ such that $\tilde{f}=x^*(f)=y^*(f)$. Then by iv) we have

$$T_{x^*}(f)=x^*(Tf)=y^*(Tf)=T_{y^*}(f).$$

It follows that $E(\,|\mathscr{B}_{x^*})=E(\,|\mathscr{B}_*)$ in $L_p(\mu)$. By the definition of the conditional expectation operator we have $\mathscr{B}_{x^*}=\mathscr{B}_{y^*}=\mathscr{B}$. Hence $\mathscr{B}$ does not depend on the choice of x^*.

Finally, we prove that $T=E(\,|\mathscr{B})$ on $L_p(\mu, X)$.

Indeed, by a standard argument of an approximation sequence of simple functions, for each $f\in L_p(\mu, X)$ and $x^*\in S(X^*)$, we have $x^*(Tf)=x^*(E(f|\mathscr{B}))$. Hence $T=E(\,|\mathscr{B})$.

Proposition 2.2. *A linear operator T on $L_p(\mu, X)$ $(1\leqslant p<\infty,\ p\neq 2)$ is a conditional expectation operator if and only if T satisfies the following conditions:*

i) $T^2=T$;

ii) $Ta\chi_\Omega=a\chi_\Omega$ *for all* $a\in X$;

iii) $\|T\|=1$;

iv) $T(ga)=g'a$ *for all* $a\in X$ *and* $g\in L_p(\mu)$, *where* $g'\in L_p(\mu)$.

Proof The necessity is obvious. Now we prove the sufficiency.

First, we claim that g' does not depend on the choice of a.

In fact, if $b=ka$ for some $k\in \mathbf{R}$, then $Tgb=g'b$ follows from the linearity of T. Let a, b be two linear independent elements in X. Then we have

$$Tga=g'_a a,\ Tgb=g'_b b \text{ and } Tg(a+b)=g'_{(a+b)}(a+b).$$

It follows that

$$g'_a a+g'_b b=g'_{(a+b)}(a+b).$$

Since a, b are linear independent, we have

$$g'_a=g'_b=g'_{(a+b)}.$$

Second, let $\hat{T}$: $L_p(\mu)\to L_p(\mu)$ be defined by

$$Tg=g' \text{ for all } g\in L_p(\mu),$$

where $Tga=g'a$ for some $a\neq 0$ in X. It is easy to prove that $\hat{T}=\hat{T}^2$, $\hat{T}\chi_\Omega=\chi_\Omega$, and $\|\hat{T}\|=1$. By appealing to the Theorem 1.1 we see that there exists a sub-σ-field $\mathscr{B}$ of Σ such that $\hat{T}=E(\,|\mathscr{B})$ on $L_p(\mu)$.

Since $Tga=E(g|\mathscr{B})a=E(ga|\mathscr{B})$ for all $g\in L_p(\mu)$ and $a\in X$, the last equality can be found in [1] on page 123. Notice that T and $E(\,|\mathscr{B})$ are bounded operators. By passing to a standard argument of an approximation sequence of simple functions, we have

$$Tg=E(g|\mathscr{B}) \text{ for all } g\in L_p(\mu, X).$$

Therefore $T=E(\cdot|\mathscr{B})$ is a conditional expectation operator in $L_p(\mu, X)$.

§ 3. A Counterexample

Theorem 3.1. *Let X be a Banach space such that $X=(X_1\oplus X_2)_{l_p}$ where X_1 and X_2 are nonzero closed subspace of X. Let $P:X\to X_1$ be the projection from X onto X_1. Then for arbitray sub-σ-fields $\mathscr{B}_1$, $\mathscr{B}_2$ of Σ, the operator T on $L_p(\mu, X)$ defined by*

$$Tf=E(Pf|\mathscr{B}_1)+E((I-P)f|\mathscr{B}_2) \text{ for all } f\in L_p(\mu, X),$$

where $p\neq 2$, $1\leq p<\infty$, and $(Pf)(t)=P(g(t))$, is a linear operator satisfying the following conditions:

i) $T^2=T$;

ii) $Ta\chi_\Omega=a\chi_\Omega$ *for all* $a\in X$, *and* $\|T\|=1$.

Moreover, if $\mathscr{B}_1\neq\mathscr{B}_2$, then T is not a conditional expectation operator.

Proof First, we will show $T^2=T$. By the definition of T, for each $f\in L_p(\mu, X)$, we have

$$Tf=E(Pf|\mathscr{B}_1)+E((I-P)f|\mathscr{B}_2).$$

By passing to a standard argument of an approximation sequence of simple functions, we get for almost all t in Ω $E(Pf|\mathscr{B}_1)(t)$ and $E((I-P)f|\mathscr{B}_2)(t)$ are in X_1 and X_2 respectively. Therefore by the definition of T and P, we have

$$\begin{aligned}T^2f&=E(P(Tf)|\mathscr{B}_1)+E((I-P)(Tf)|\mathscr{B}_2)\\&=E(E(Pf|\mathscr{B}_1)|\mathscr{B}_1)+E(E((I-P)f|\mathscr{B}_2)|\mathscr{B}_2)\\&=E(Pf|\mathscr{B}_1)+E((I-P)f|\mathscr{B}_2)=Tf.\end{aligned}$$

Hence $T^2=T$.

Second, for each $a\in X$, we have

$$T(a\chi_\Omega)=E(Pa\chi_\Omega|\mathscr{B}_1)+E((I-P)a\chi_\Omega|\mathscr{B}_2)=Pa\chi_\Omega+(I-P)a\chi_\Omega=a\chi_\Omega.$$

Thirdly, since $X=(X_1\oplus X_2)_{l_p}$, we have

$$\|f\|_X=(\|Pf\|_X^p+\|(I-P)f\|_X^p)^{1/p} \text{ a.e. for all } f\in L_p(\mu, X).$$

It follows that

$$\|f\|_p=(\|Pf\|_p^p+\|(I-P)f\|_p^p)^{1/p}.$$

Furthermore

$$\|Tf\|_p=(\|E(Pf|\mathscr{B}_1)\|_p^p+\|E((I-P)f|\mathscr{B}_2)\|_p^p)^{1/p}\leq(\|Pf\|_p^p+\|(I-P)f\|_p^p)^{1/p}=\|f\|_p.$$

Therefor $\|T\|=1$.

Finally, if $\mathscr{B}_1\neq\mathscr{B}_2$, we claim that T is not a conditional expectation operator.

Otherwise, there exists a sub-σ-field $\mathscr{B}$ of Σ such that $T=E(\cdot|\mathscr{B})$. Then taking $0\neq a\in X_1$, for each $f\in L_p(\mu)$ by the definition of T, we have

$$E(f|\mathscr{B})a=E(fa|\mathscr{B})=Tfa=E(fa|\mathscr{B}_1)=E(f|\mathscr{B}_1)a.$$

Hence

$$E(f|\mathscr{B})=E(f|\mathscr{B}_1) \text{ for all } f\in L_p(\mu).$$

It follows that $\mathscr{B}_1=\mathscr{B}$. Similarly, we have $\mathscr{B}_2=\mathscr{B}$. Therefore $\mathscr{B}_1=\mathscr{B}_2$. This contradicts the hypothesis $\mathscr{B}_1\neq\mathscr{B}_2$, and the proof is completed.

Remark 3.1. In contrast with Theorems 1.1, 2.1 and 3.1, we see the difference between the characterizations of the conditional expectation operators on $L_p(\mu, X)$ and that on $L_p(\mu)$.

Remark 3.2. Let (Ω, Σ, μ) and $(W, \mathscr{B}, \lambda)$ be probability spaces, where W is not an atom of λ, $1\leqslant p<\infty$, $p\neq 2$. Then there exists a constant-preserving contractive projection T on $L_p(\mu, L_p(\lambda))$, but T is not a conditional expectation operator on $L_p(\mu, L_p(\lambda))$. However, T is a conditional expectation operator on $L_p(\mu\times\lambda)$.

In fact, let $\mathscr{B}_1\neq\mathscr{B}_2$ be two sub-σ-fields of Σ. Since W is not an atom of λ, there exist W_1, $W_2\in\mathscr{B}$ such that $\lambda(W_i)>0$, $i=1, 2$, $W_1\cap W_2=\phi$, $W_1\cup W_2=W$. Then $L_p(\lambda)=(L_p(\lambda|_{W_1})+L_p(\lambda|_{W_2}))_{l_p}$. Let P be the projection from $L_p(\lambda)$ onto $L_p(\lambda|_{W_1})$. Then T defined by

$$Tf=E(Pf|\mathscr{B}_1)+E((I-P)f|\mathscr{B}_2) \text{ for all } f\in L_p(\mu, L_p(\lambda))$$

is a constant-preserving contractive projection on $L_p(\mu, L_p(\lambda))$, but T is not a conditional expectation operator on $L_p(\mu, L_p(\lambda))$ (by Theorem 3.1). Howerver, by Fubini theorem $L_p(\mu, L_p(\lambda))$ is isometric isomorphic to $L_p(\mu\times\lambda)$. It is easy to check that T on $L_p(\mu\times\lambda)$ has the following properties: i)T is linear, ii)$T^2=T$, iii) $T\chi_{\Omega\times W}=\chi_{\Omega\times W}$, iv) $\|T\|=1$. By Theorem 1.1, T is a conditional expectation operator on $L_p(\mu\times\lambda)$. This fact illustrate the difference between the product of σ-field and its factor σ-field.

§ 4. Case $0<p<1$

In general, the conditional expectation operator on $L_p(\mu)$ for $0<p<1$ need not exist. However we have the following results.

Definition 4.1. *$\Phi:\Sigma\to\Sigma$ is said to be a regular set isomorphism, if it satisfies:*

1) $\Phi(\Omega\backslash A)=\Phi(\Omega)\backslash\Phi(A)$, $\forall A\in\Sigma$,

2) $\Phi\left(\bigcup_{n=1}^{\infty} A_n\right)=\bigcup_{n=1}^{\infty}\Phi(A_n)$, $A_n\in\Sigma$, $A_n\cap A_m=\phi$ *if* $n\neq m$, *and*

3) $\mu(\Phi(A))=0$ *if and only if* $\mu(A)=0$.

Moreoxer, if in addition, $\mu(\Phi(A))=\mu(A)$ *for all* $A\in\Sigma$, *then* Φ *is said to be a measure-preserving regular set isomorphism.*

It is easy to see that for each measure-preserving regular set isomorphism of the measure space (Ω, Σ, μ), there exists a (unique in the sense a. e.) operator on $L_p(\mu, X)$ satisfying the following conditions (we also denote the operator by Φ):

i) $\Phi(a\chi_E)=a\chi_E$, $\forall a\in X$, $E\in\Sigma$,

2) $\Phi(f+g)=\Phi(f)+\Phi(g)$, $f, g\in L_p(\mu, X)$,

3) $f_n \xrightarrow{\|\|_p} f$ implies $\Phi(f_n)\xrightarrow{\|\|_p}\Phi(f)$ $f_n, f\in L_p(\mu, X)$.

It is easy to deduce the following result.

Lemma 4.1.[10] *Let* $0<p<1$. *Then* $f, g\in L_p(\mu, X)$ *are disjoint (in the sense* $(\mathrm{supp} f)\cap(\mathrm{supp} g)=\phi$) *if and only if* $\|f+g\|_p=\|f\|_p+\|g\|_p$.

Theorem 4.2. *Let* $0<p<1$. *If* T *is a linear operator on* $L_p(\mu, X)$ *satisfying the following conditions:* i) $T^2=T$, ii) $Ta\chi_\Omega=a\chi_\Omega$ *for all* $a\in X$, iii) $\|T\|=1$, *then* $T=\Phi$, *where* Φ *is a linear operator determined by a measure-preserving regular set isomorphism satisfying* $\Phi^2=\Phi$, *i. e.* $\Phi(\Phi(A))=\Phi(A)$ *for all* $A\in\Sigma$.

Proof For each $0\neq a\in X$ and $E\in\Sigma$, we have

$$a\chi_\Omega=Ta\chi_\Omega=Ta\chi_E+Ta\chi_{E^c}\quad (E^c=\Omega\setminus E).$$

Then

$$\|a\chi_\Omega\chi_p=\int_\Omega\|Ta\chi_E+Ta\chi_{E^c}\|^p d\mu\leqslant\int_\Omega(\|Ta\chi_E\|+\|Ta\chi_{E^c}\|^p)d\mu=\|Ta\chi_E\|_p+\|Ta\chi_{E^c}\|_p$$
$$\leqslant\|a\chi_E\|_p+\|a\chi_{E^c}\|_p=\|a\chi_\Omega\|.$$

It follows that

$$\|Ta\chi_E+Ta\chi_{E^c}\|_p=\|Ta\chi_E\|_p+\|Ta\chi_{E^c}\|_p \text{ and } \|Ta\chi_E\|_p=\|a\chi_E\|_p. \tag{2}$$

By Lemma 4.1, we have

$$\mathrm{sup}\, p\|Ta\chi_E\|\cap\mathrm{sup}\, p\|Ta\chi_{E^c}\|=\varnothing. \tag{3}$$

Let $\Phi_a(E)=\mathrm{sup}\,p\|Ta\chi_E\|$ for all $E\in\Sigma$. By (1) and (3) we have

$$Ta\chi_E=a\chi_{\Phi_a}(E). \tag{4}$$

Let $a, b\in X$. If $b=ka$ for some $k\in\mathbf{R}$, then it is easy to see that $\Phi_a=\Sigma_b$; if a, b are linear independent, then we have

$$T(a+b)\chi_E=(a+b)\chi_{\Phi_{(a+b)}}(E),\ Ta\chi_E+Tb\chi_E=a\chi_{\Phi_a(E)}+b_{\Phi_b(E)}.$$

Since T is linear and a, b are linear independent, we have

$$\Phi_a(E)=\Phi_b(E) \text{ for all } E\in\Sigma.$$

Hence if we fix a $0=b\in X$, then

$$\Phi_a=\Phi_b=\Phi \text{ for all } a\in X. \tag{5}$$

By (2), (4) and (5), we have

$$Ta\chi_E=a\chi_{\Phi(E)}, \text{ for all } a\in X, \text{ and } \mu(E)=\mu(\Phi(E)), \text{ for all } E\in\Sigma. \tag{6}$$

For $E, F\in\Sigma$ with $E\cap F=\varnothing$, let $0\neq a_1\in X$. Then we have

$$\|a_1\chi_{\Phi(E\cap F)}\|_p=\|Ta_1\chi_{E\cap F}\|=\|Ta_1\chi_E+Sa_1\chi_F\|\leqslant\|Ta_1\chi_E\|_p+\|Ta_1\chi_F\|_p\leqslant\|a_1\chi_E\|_p+\|a_1\chi_F\|_p$$
$$=\|a_1\chi_{E\cup F}\|=\|a_1\|^p\mu(E\cup F)\|a_1\|^p\mu(E\cup F)=\|a_1\|^p\mu(\Phi(E\cup F)$$
$$=\|a_1\chi_{\Phi(E\cap F)}\|_p.$$

It follows that

$$\|Ta_1\chi_E+Ta_1\chi_F\|_p=\|Ta_1\chi_E\|_p+\|Ta_1\chi_F\|_p.$$

By Lemma 4.1 and (6), we have

$$\Phi(E)\cap\Phi(F)=\varnothing. \tag{7}$$

By (1), (6) and (7), we have

$$\Phi(E^c)=\Omega\backslash\Phi(E). \tag{8}$$

Let $A_i\cap A_j=\varnothing$. If $i\neq j$, $A_i\in\Sigma$, take $x=\sum_{i=1}^{\infty}a_1\chi_{A_i}\in L_p(\mu, X)$, where $0\neq a_1\in X$. Then

$$Tx=\sum_{i=1}^{\infty}Ta_1\chi_{A_i}=\sum_{i=1}^{\infty}a_1\chi_{\Phi(A_i)}$$

follows from the fact that T is contractive and (6). By (6) and (7), we have

$$\Sigma\left(\bigcup_{i=1}^{\infty}A_i\right)=\sum_{i=1}^{\infty}\Phi(A_i). \tag{9}$$

Hence Φ is a measure-preserving regular set isomorphism.

By (6), if $f=\sum_{i=1}^{n}a_i\chi_{A_i}$, $A_i\cap A_j=\varnothing, i\neq j$, $A_i\in\Sigma$, $a_i\in X$, then we have

$$Tf=\Phi(f).$$

For each $f\in L_p(\mu, X)$ there exists a sequence (f_n) of simple functions such that $f_n \xrightarrow{\|\|_p} f$, $Tf_n\xrightarrow{\|\|_p} Tf$, so we have

$$Tf=\lim_{n\to\infty}Tf_n=\lim_{n\to\infty}\Phi(f_n)=\Phi(f).$$

Hence $T=\Phi$.

By 1) for each $E\in\Sigma$ and $0\neq a_1\in X$, we have

$$\chi_{\Phi(\Phi(E))}a_1=\Phi^2(a_1\chi_E)=T^2(a_1\chi_E)=\Phi(a_1\chi_E)=\chi_{\Phi(E)}a_1.$$

It follows that $\Phi^2=\Phi$. This completes the proof.

Remark 4.1. By the representation, we know that the operator in Theorem 4.1 is an isometry.

Remark 4.2. If (Ω, Σ, μ) is a finite measure space, all results in this paper hold as well.

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