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SOME TWO-INDEPENDENT-VARIABLE INEQUALITIES INVOLVING IMPROPER INTEGRALS

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Abstract

In the paper the authors establish four theorems on two-independent-variable Gronwall-type inequalities involving improper integrals with infinite integration limits. The results obtained improve and generalize the main results proved in the recent paper [5] by A. Corduneanu. Two classes of nonlinear continuous functions defined by F. M. Dannan [6] are applied in this article. And the results can be used as handy tools in the study of many Volterra integral and integro-differential equations with improper integral functionals.

§1. Introduction

As well-known, various kinds of inequalities have played a vital role in the development of mathematics. Particularly, the theory of Gronwall type integral inequalities, both in one and more than one variable, is crucial and indispensable in the study of almost all kinds of analytic equations such as the differential and integral equations, integro-differential equations, and functional-differential equations. In the recent years, in addition to a large number of research papers many interesting monographs devoted to such inequalities have also been published. See the books by Beesack[2], Bellman and Cooke[4], Lakshmikantham and Leela[7] and Walter[11]. However, in spite of the frequent appearance in literature of many equations involving improper integrals with infinite integration limit, only a few investigation work have been done so far for Gronwall type inequalities with improper integrals. To the authors' knowledge, the articles by Antonishin[1], Pachpatte[9], Corduneanu[5], and Singare[10] are the only papers devoted to the establishment of such inequalities. Among them the first two works are devoted to the one-variable case and the others are concerned with two-variable inequalities. As for application, we note that the inequality obtained in Pachpatte[9] has been successfully applied by Máté and Nevai[8] in the study of n-th order linear differential equations. It is also indicated in Singare[10] that two-variable

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inequalities with improper integrals are very useful in the investigation of many Volterra type integral equations.

In the recent paper [5], Corduneanu discussed the following two-independent-variable inequality and some of its special cases

$$u(x, y) \leqslant f(x, y) + h(x, y) F\left[\int_x^\infty \int_y^\infty a(s, t) w(u(s, t))\, ds\, dt\right] < \infty, \quad x, y \geqslant 0, \tag{A}$$

where the functions $a, f, h, u \in C(R_+ \times R_+, R_+)$, $F, w \in C(R_+, R_+)$, and $R_+ = [0, \infty)$. Here we denote by $C(N, M)$ the class of all continuous functions defined on set N with range in set M. He gave some upper bounds on the function $u(x, y)$ satisfying (A) under various hypotheses.

The aim of the present paper is not only to obtain some new bounds on function $u(x, y)$ from (A) but also to discuss the following generalization of inequality (A)

$$\begin{aligned} u(x, y) \leqslant f(x, y) &+ g(x, y) \int_x^\infty \int_y^\infty a(s, t) u(s, t)\, ds\, dt \\ &+ h(x, y) F\left[\int_x^\infty \int_y^\infty b(s, t) w(u(s, t))\, ds\, dt\right] < \infty, \quad x, y \geqslant 0, \end{aligned} \tag{B}$$

where the functions $a, b, u, f, g, h \in C(R_+ \times R_+, R_+)$, and $F, w \in C(R_+, R_+)$.

§ 2. Linear Case

In this section we discuss the linear case of inequality (A). Two theorems obtained here are different from the Corollary 4.1 of Corduneanu[5]. Consider the linear inequality

$$u(x, y) \leqslant f(x, y) + h(x, y) \int_x^\infty \int_y^\infty b(s, t) u(s, t)\, ds\, dt < \infty, \quad x, y \geqslant 0. \tag{1}$$

Theorem 1. *Suppose the functions b, u, f, h belong to $C(R_+ \times R_+, R_+)$, with $f(x, y)$ non-increasing and $f(x, y) \geqslant A > 0$ for $x, y \geqslant 0$, where A is a constant. Suppose that*

$$\int_0^\infty \int_0^\infty b(s, t)(1 + h(s, t))\, ds\, dt < \infty \tag{C_1}$$

holds and inequality (1) is satisfied. Then we have

$$u(x, y) \leqslant f(x, y)(1 + h(x, y)\theta(x, y)) \quad \text{for } x, y \geqslant 0, \tag{2}$$

where

$$\theta(x, y) := -1 + \exp \int_x^\infty \int_y^\infty b(s, t)(1 + h(s, t))\, ds\, dt. \tag{3}$$

In addition, if $h(x, y) \geqslant 1$ for $x, y \geqslant 0$, then the factor $1 + h(s, t)$ contained both in (C_1) and (3) can be replaced by $h(s, t)$.

Proof Define $v(x, y) := u(x, y)/f(x, y)$. Then we have

$$\int_0^\infty \int_0^\infty b(s, t) v(s, t)\, ds\, dt < \infty,$$

since $f(x, y) \geqslant A > 0$ for $x, y \geqslant 0$. We observe from inequality (1)

$$v(x, y)\leqslant 1+h(x, y)\int_x^\infty\int_y^\infty b(s, t)\,\frac{u(s, t)}{f(x, y)}\,ds\,dt\leqslant 1+h(x, y)r(x,y) \text{ for } x, y\geqslant 0, \tag{4}$$

where

$$r(x, y):=\int_x^\infty\int_y^\infty b(s, t)v(s, t)ds\,dt,\ x, y\geqslant 0, \tag{5}$$

since $f(x, y)$ is non-increasing for $x, y\geqslant 0$. We obtain from (5), by differentiation,

$$r'_x(x, y)=-\int_y^\infty b(x, t)v(x, t)dt\leqslant 0,$$

$$r'_y(x, y)=-\int_x^\infty b(s, y)v(s, y)ds\leqslant 0,$$

and

$$r''_{xy}(x, y)=b(x, y)v(x, y)\leqslant b(x, y)\,[1+h(x, y)r(x, y)]$$
$$\leqslant b(x, y)(1+h(x,y))(1+r(x, y)),\ x, y\geqslant 0;$$

herein we have used inequality (4). We notice here that the factor $(1+h(x, y))$ contained in the last inequality can be replaced by $h(x, y)$ if the condition $h(x, y)\geqslant 1$ holds for $x, y\geqslant 0$. Noting r'_y, r'_x are non-positive, we derive from the above inequality

$$\frac{\partial}{\partial y}\left(\frac{r'_x(x, y)}{1+r(x, y)}\right)\leqslant b(x, y)(1+h(x, y)),\ x, y\geqslant 0. \tag{6}$$

Setting $y=t$ in (6) and integrating with respect to t from y to ∞, we get

$$\frac{-r'_x(x, y)}{1+r(x, y)}\leqslant\int_y^\infty b(x, t)(1+h(x, t))dt,\ x, y\geqslant 0,$$

since $r'_x(x, \infty)\equiv 0$ holds. Now set $x=s$ in the last inequality and integrate with respect to s over $[x, \infty)$. Using $r(\infty, y)\equiv 0$, we then obtain

$$\ln(1+r(u, y))\leqslant\int_x^\infty\int_y^\infty b(s, t)(1+h(s, t))ds\,dt,\ x, y\geqslant 0,$$

or

$$r(x, y)\leqslant\theta(x, y) \text{ for } x, y\geqslant 0, \tag{7}$$

where function $\theta(x, y)$ is given by (3). Finally, substituting the last bound on $r(x, y)$ in (4) and using $v(x, y)=u(x, y)/f(x, y)$, we have the desired estimate in (2).

We remark here that the upper bound in (2) is better than that given in the Corollary 4.1 of [5] by inequality (3.4). The next result gives a very useful new bound for the function $u(x, y)$ satisfying inequality (1).

Theorem 2. *Suppose the function b, u, f, $h\in C(R_+\times R_+, R_+)$ and the condition*

$$\int_0^\infty\int_0^\infty b(s, t)(f(s, t)+h(s, t))ds\,dt<\infty \tag{C_2}$$

holds. Suppose further that inequality (1) is satisfied. Then

$$u(x, y)\leqslant f(x, y)+h(x, y)\psi(x, y) \text{ for } x, y\geqslant 0, \tag{8}$$

where

$$\psi(x, y):=-1+\exp\int_x^\infty\int_y^\infty b(s, t)(f(s, t)+h(s, t))ds\,dt.$$

This theorem can be derived from Theorem 3 below by letting $F(q)\equiv q$ and $w(q)\equiv q$. We leave out details to the reader.

Remark 1. The monotomicity of $f(x, y)$ and $f(x, y)\geqslant A>0$ and $h(x, y)\geqslant 1$ arenotrequired in Theorem 2; thus the last result is more general than the Corall ary 4.1 in [5].

§ 3. Nonlinear Cases

We now prove some new nonlinear extensions of inequality (1), which are very useful in many situations of applications. In the sequel, the following function class $H(\varphi)$ defined by F. M. Dannan[6] will be used.

Definition[6]. *A function $w\in C(R_+, R_+)$ is said to belong to the class $H(\varphi)$ if*

(i) *$w(u)$ is nondecreasing and positive for $u>0$,*

(ii) *there exists a function $\varphi\in C(R_+, R_+)$ such that*

$$w(vu)\leqslant\varphi(v)w(u) \text{ for } v>0,\ u\geqslant 0.$$

We are awared that any submultiplicative function w in the class $C(R_+, R_+)$ satisfying above condition (i) must belong to the class $H(w)$.

Let us first consider the nonlinear inequality

$$u(x, y)\leqslant f(x, y)+h(x, y)F\left[\int_x^\infty\int_y^\infty b(s, t)w(u(s, t))\,ds\,dt\right]<\infty,\ x, y\geqslant 0, \tag{A}$$

where $b, u, f, h\in C(R_+\times R_+, R_+)$ and $F, w\in C(R_+, R_+)$. Throughout this section we define a strictly increasing, nonnegative and continuous function G on R_+ by

$$G(r):=\int_0^r\frac{dv}{w(1+F(v))},\ r\geqslant 0, \tag{D}$$

and denote by G^{-1} the inverse function of G. Clearly, G^{-1} is also strictly increasing, nonnegative, continuous on R_+.

Theorem 3. *Suppose $b, u, f, h\in C(R_+\times R_+, R_+)$, and $F, w\in C^1(R_+, R_+)$ are nondecreasing, with $w\in H(\varphi)$. Suppose that the condition*

$$\int_0^\infty\int_0^\infty b(s, t)\varphi(f(s, t)+h(s, t))\,ds\,dt<\infty \tag{C_3}$$

and the inequality (A) are satisfied. Then we have

$$u(x, y)\leqslant f(x, y)+h(x, y)F\left\{G^{-1}\left[\int_x^\infty\int_y^\infty b(s, t)\varphi(f(s, t)+h(s, t))\,ds\,dt\right]\right\}, \tag{9}$$

provided that

$$\int_x^\infty\int_y^\infty b(s, t)\varphi(f(s, t)+h(s, t))\,ds\,dt<G(\infty). \tag{10}$$

Proof We define a function $R(x, y)\in C(R_+\times R_+, R_+)$ by

$$R(x, y):=\int_x^\infty\int_y^\infty b(s, t)w(u(s, t))\,ds\,dt,\ x, y\geqslant 0. \tag{11}$$

Then inequality (A) yields

$$u(x, y) \leqslant f(x, y) + h(x, y) F[R(x, y)], \quad x, y \geqslant 0. \tag{12}$$

We derive from (11), by differentiation,

$$R'_x(x, y) = -\int_y^\infty b(x, t) w(u(x, t))\, dt \leqslant 0,$$

$$R'_y(x, y) = -\int_x^\infty b(s, y) w(u(s, y))\, ds \leqslant 0,$$

and

$$R''_{xy}(x, y) = b(x, y) w(u(x, y)) \text{ for } x, y \geqslant 0.$$

Since $w \in H(\varphi)$, and b, u, F, $R(x, y)$ are nonnegative, we obtain from the last inequality

$$\begin{aligned} R''_{xy}(x, y) &\leqslant b(x, y) w\{f(x, y) + h(x, y) F[R(x, y)]\} \\ &\leqslant b(x, y) w\{(f(x, y) + h(x, y))(1 + F[R(x, y)])\} \\ &\leqslant b(x, y) \varphi(f(x, y) + h(x, y)) w\{1 + F[R(x, y)]\}, \quad x, y \geqslant 0, \end{aligned}$$

since (12) holds. This inequality implies that

$$\frac{\partial}{\partial y}\left(\frac{R'_x(x, y)}{w(1 + F[R(x, y)])}\right) \leqslant b(x, y) \varphi(f(x, y) + h(x, y)), \quad x, y \geqslant 0, \tag{13}$$

since R'_x, $R'_y \leqslant 0$, F', $w' \geqslant 0$, and $w(u) > 0$ for $u > 0$. Setting $y = t$ in (13) and integrating with respect to t over $[y, \infty)$, we then get

$$-\frac{R'_x(x, y)}{w(1 + F[R(x, y)])} \leqslant \int_y^\infty b(x, t) \varphi(f(x, t) + h(x, t))\, dt, \quad x, y \geqslant 0$$

since $R(x, \infty) \equiv R(x, \infty) \equiv 0$ and $F(0) \geqslant 0$. Hold $y \geqslant 0$ fixed in the last inequality and then rewrite it as

$$-\frac{d}{dx} G[R(x, y)] \leqslant \int_y^\infty b(x, t) \varphi(f(x, t) + h(x, t))\, dt, \quad x, y \geqslant 0,$$

where function G is defined by (D). Setting $x = s$ in the above inequality and integrating with respect to s over $[x, \infty)$, we then obtain

$$G[R(x, y)] \leqslant \int_x^\infty \int_y^\infty b(s, t) \varphi(f(s, t) + h(s, t))\, ds\, dt, \quad x, y \geqslant 0,$$

since $G(0) = 0$ and $R(\infty, y) \equiv 0$. Now, we may derive from the last inequality

$$R(x, y) \leqslant G^{-1}\left[\int_x^\infty \int_y^\infty b(s, t) \varphi(f(s, t) + h(s, t))\, ds\, dt\right] \tag{14}$$

provided that condition (10) is satisfied. Thus the desired bound in (9) follows from (12) and (14) immediately.

Remark 2. The following conditions are required in the Proposition 4 of Corduneanu[5]:

(1) $f(x, y) \geqslant 1$, $h(x, y) \geqslant 1$ for $x, y \geqslant 0$ and $f(x, y)$ is nonincreasing,

(2) the function w is submultiplicative and $v^{-1} w(u) \leqslant w(u/v)$ holds for all $v \geqslant 1$, $u \geqslant 0$.

Because all these conditions are either eliminated or relaxed in above Theorem 3, Theorem 3 should have a much wider range of application than the Proposition 4 of

[5].

We now turn to the following generalization of inequality (A):

$$u(x, y)\leqslant f(x, y)+g(x, y)\int_x^{\infty}\int_y^{\infty}a(s, t)u(s, t)ds\,dt$$
$$+h(x, y)F\left[\int_x^{\infty}\int_y^{\infty}b(s, t)w(u(s, y))ds\,dt\right]$$
$$<\infty, \; x, y\geqslant 0. \tag{B}$$

Theorem 4. *Suppose the functions a, b, u, f, g, and h belong to the class $C(R_+\times R_+, R_+)$, with f, h non-increasing for x, $y\geqslant 0$ and $f(x, y)\geqslant A>0$, where A is a constant. Suppose F, $w\in C^1(R_+, R_+)$ are nondecreasing and $w\in H(\varphi)$. Suppose that the conditions*

$$\int_0^{\infty}\int_0^{\infty}a(s, t)(1+g(s, t))ds\,dt<\infty, \tag{C_4}$$

$$\int_0^{\infty}\int_0^{\infty}b(s, t)\varphi(f^*(s, t)+h^*(s, t))ds\,dt<\infty, \tag{C_5}$$

and the inequality (B) are satisfied. Then we have

$$u(x, y)\leqslant f^*(x, y)+h^*(x, y)F\left\{G^{-1}\left[\int_x^{\infty}\int_y^{\infty}b(s, t)\varphi(f^*(s, t)+h^*(s, t))ds\,dt\right]\right\}, \tag{15}$$

provided that

$$\int_x^{\infty}\int_y^{\infty}b(s, t)\varphi(f^*(s, t)+h^*(s, t))ds\,dt<G(\infty), \tag{16}$$

where G^{-1} is the same as defined in Theorem 3

$$\begin{aligned} f^*(x, y)&:=f(x, y)[1+g(x, y)\xi(x, y)],\\ h^*(x, y)&:=h(x, y)[1+g(x, y)\xi(x, y)], \end{aligned} \tag{17}$$

and

$$\xi(x, y):=-1+\exp\int_x^{\infty}\int_y^{\infty}a(s, t)(1+g(s, t))ds\,dt. \tag{18}$$

In addition, if $g(x, y)\geqslant 1$ then the factor $[1+g(s, t)]$ contained in (C_4) and (18) can be replaced by $g(s, t)$.

Proof Rewrite inequality (B) as

$$u(x, y)\leqslant J(x, y)+g(x, y)\int_x^{\infty}\int_y^{\infty}a(s, t)u(s, t)ds\,dt, \; x, y\geqslant 0, \tag{19}$$

where function $J(x, y)$ is defined by

$$J(x, y):=f(x, y)+h(x, y)F\left[\int_x^{\infty}\int_y^{\infty}b(s, t)w(u(s, t))ds\,dt\right].$$

It is a simple matter to verify that $J(x, y)\in C(R_+\times R_+, R_+)$ is non-increasing for x, $y\geqslant 0$ and $J(x, y)\geqslant A>0$ holds. Note condition (C_4), and then a suitable application of Theorem 1 to (19) yields

$$u(x, y)\leqslant J(x, y)[1+g(x, y)\xi(x, y)], \; x, y\geqslant 0,$$

i. e.,

$$u(x, y)\leqslant f^*(x, y)+h^*(x, y)F\left[\int_x^{\infty}\int_y^{\infty}b(s, t)w(u(s, t))ds\,dt\right], \; x, y\geqslant 0, \tag{20}$$

where $\xi(x, y)$ and $f^*(x, y)$, $h^*(x, y)$ are given by (17) and (18), respectively. Finally, in view of condition (C_5), an application of Theorem 3 to (20) completes the proof of Theorem 4.

The next corollary is a consequence of Theorem 4 which establishes a very useful new inequality.

Corollary 5. *Let all functions a, b, u, f, g, and h be the same as defined in Theorem 4. Suppose condition* (C_4) *and*

$$\int_0^\infty\int_0^\infty b(s, t)[f(s, t)+h(s, t)]p(s, t)ds\,dt<\infty \tag{C_6}$$

hold, where

$$p(x, y):=-1+\exp\int_x^\infty\int_y^\infty (1+g(s, t))ds\,dt.$$

Suppose further that the linear inequality

$$u(x, y)\leqslant f(x, y)+g(x, y)\int_x^\infty\int_y^\infty a(s, t)u(s, t)ds\,dt$$
$$+h(x, y)\int_x^\infty\int_y^\infty b(s, t)(u(s, t))^q ds\,dt<\infty,\ x, y\geqslant 0 \tag{21}$$

is satisfied, where $q\in(0, 1]$ *is a constant. Then we have*

$$u(x, y)\leqslant f(x, y)p(x, y)+h(x, y)p(x, y)[\sigma(x, y)-1],\ x, y\geqslant 0, \tag{22}$$

where

$$\sigma(x, y):=\left\{1+(1-q)\int_x^\infty\int_y^\infty b(s, t)[f(s, t)+h(s, t)]p(s, t)ds\,dt\right\}^{1/(1-q)}.$$

Proof Taking $F(u)\equiv u$ and $w(u)\equiv u^q$ in inequality (B), we then get inequality (21). Noting $w\in H(w)$, we obtain here, by simple computation,

$$G(r)=\int_0^r\frac{du}{(1+v)^q}=\frac{1}{1-q}[(1+r)^{1-q}-1],\ r\geqslant 0.$$

Hence we have $G(\infty)=\infty$ and

$$G^{-1}(p)=-1+[1+(1-q)p]^{1/(1-q)} \text{ for all } p\geqslant 0.$$

Thus, an application of Theorem 4 to inequality (21) completes the proof of this corollary.

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