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DIRICHLET FORMS AND SYMMETRIC DIFFUSIONS ON A BOUNDED DOMAIN IN R^d**

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Abstract

Let D be a bounded C^3-domain in R^d and (a_{ij}) be a bounded symmetric matrix defined on D. Consider the symmetric form

$$\mathscr{E}(u, v)=\frac{1}{2}\sum_{i,j=1}^{d}\int_D a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\,dx\,,\ u, v\in H^1(D).$$

Under some assumptions it is shown that the diffusion process associated with the regular Dirichlet space $(\mathscr{E}, (H^1(D))$ on $L^2(\overline{D})$ can be characterized as a unique solution of a certain stochastic differential equation.

§ 0. Introduction

Let D be a bounded domain in R^d and $a_{ij}(x)$ $1\leqslant i,j\leqslant d$, be bounded Borel functions on D such that the matrix (a_{ij}) is symmetric and uniformly positive definite on D. Consider the following form

$$\mathscr{E}(u, v)=\frac{1}{2}\sum_{i,j=1}^{d}\int_D a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\,dx,\ u,\ v\in H^1(D), \tag{0.1}$$

where $H^1(D)$ is the Soblev space of order 1. It is easy to see that $(\mathscr{E}, H^1(D))$ is a Dirichlet space on $L^2(D)$ which is local but not regular. However, if $C^\infty(\overline{D})$ is dense in $H^1(D)$ and if we regard $(\mathscr{E}, H^1(D))$ as a Dirichlet spaceon $L^2(\overline{D})$ rather than on $L^2(D)$, then it is regular because the $\mathscr{E}_1$-norm $(\mathscr{E}_1(u, v)=\mathscr{E}(u, v)+(u,v))$ is equivalent to the Sobolev norm on $H^1(D)$. Thus according to Fukushima's Theorems ([2], Theorems 6.2.1 and 6.2.2) there exists uniquely (up to the equivalence) a dx-symmetric diffusion process $\mathscr{M}=\{\Omega, \mathscr{F}, \mathscr{F}_t, \overline{X}_t, P_x, x\in\overline{D}\}$ on $(\overline{D}, \mathscr{B}(\overline{D}))$ whose Dirichlet space is $(\mathscr{E}, H^1(D))$.

The purpose of this paper is to characterize $\overline{X}_t$, under some assumptions on D and on (a_{ij}), as a unique solution of an SDE (stochastic differential equation) on $\overline{D}$. Our proof of this result is essentially based on a generalized Stokes formula for functions in $H^1(D)$ which isestablished in § 1. In § 2 we show that for each $v\in H^1(D)$, the restriction on ∂D of its quasi-continuous modification $\tilde{v}$ on $\overline{D}$ is a

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version of its trace $\nu(v)$ on ∂D. The main result is proved in § 3.

§ 1. A Generalized Stokes Formula

Let D be a bounded C^3-domain in R^d. It is well known that $C^\infty(\overline{D})$ is dense in $H^1(D)$ and D is situated on one side of ∂D, where $H^1(D)$ denotes the Sobolev space of order 1, i. e.

$$H^1(D)=\left\{u\in L^2(D):\ \frac{\partial u}{\partial x_i}\in L^2(D),\ 1\leqslant i\leqslant d\right\}. \tag{1.1}$$

Let $n(x)=(n_1(x), \cdots, n_d(x))$ denote the unit outward normal vector of D at $x\in\partial D$. It is well known that for $v\in C^1(D)$ we have so-called "Stokes formula"

$$\int_D \frac{\partial v(x)}{\partial x_i}\,dx=\int_{\partial D} v(x)n_i(x)\sigma(dx), \tag{1.2}$$

where $\sigma(dx)$ stands for the "area measure" on the boundary ∂D. The following lemma extends this formula to the case where $v\in H^1(D)$:

Lemma 1.1. *Let $v\in H^1(D)$. We denote by $\nu(v)$ the trace of v on the boundary ∂D. Then we have*

$$\int_D \frac{\partial v(x)}{\partial x_i}\,dx=\int_{\partial D} \nu(v)(x)n_i(x)\sigma(dx). \tag{1.3}$$

Proof Consider the following symmetric form on $H^1(D)$

$$a(u, v)=\sum_{i=1}^d\int_D \frac{\partial u(x)}{\partial x_i}\,\frac{\partial v(x)}{\partial x_i}\,dx+\int_D u(x)v(x)dx.$$

It is well known that there exists a unique $u\in H^1(D)$ such that

$$a(u, v)=\int_{\partial D}\nu(v)d\sigma,\ \forall v\in H^1(D), \tag{1.4}$$

where $\nu(v)$ is the trace of v on ∂D which is in $H^{1/2}(\partial D)$ (see [1], § 37). Since $a(|v|, |v|)\leqslant a(v, v)$, we have

$$\int_{\partial D}|\nu(v)|d\sigma\leqslant\int_{\partial D}\nu(|v|)d\sigma=a(u, |v|)\leqslant\sqrt{a(u, u)}\,\sqrt{a(|v|, |v|)}\leqslant\|u\|_{H^1}\|v\|_{H^1}. \tag{1.5}$$

This means the mapping $v\to\nu(v)$ is continuous from $H^1(D)$ into $L^1(\partial D, d\sigma)$.

Now suppose $v\in H^1(D)$. Let $v_n\in C^2(\overline{D})$, $n\geqslant1$, such that $\|v_n-v\|_{H^1}\to0$ $(n\to\infty)$. As $\nu(v_n)=v_n|_{\partial D}$, we have by (1.2)

$$\int_D \frac{\partial v_n(x)}{\partial x_i}\,dx=\int_{\partial D}\nu(v_n)(x)n_i(x)\sigma(dx),$$

from which and by letting $n\to\infty$ we get (1.3).

The following result is essential for proving our main result in § 3.

Lemma 1.2. *Let $a_{ij}\in C^1(\overline{D})$, $i, j=1, \cdots, d$. Then for any $u\in C^1(\overline{D})$ and $v\in H^1(D)$ we have*

$$\sum_{i,j=1}^{d}\int_D a_{ij}(x)\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx$$
$$=-\int_D v(x)Au(x)dx+\int_{\partial D}\nu(v)(x)\sum_{i,j=1}^{d}a_{ij}(x)\frac{\partial u}{\partial x_j}n_i(x)\sigma(dx) \tag{1.6}$$

where

$$Au(x)=\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right].$$

Proof Let $h_i(x)=v(x)\sum_{j=1}^{d}a_{ij}(x)\frac{\partial u(x)}{\partial x_j}$, $i=1,\cdots,d$. It isobvious that $h_i\in H^1(D)$ and we have

$$\nu(h_i)(x)=\nu(v)(x)\sum_{j=1}^{d}a_{ij}(x)\frac{\partial u(x)}{\partial x_j},\ x\in\partial D.$$

Thus by (1.3), we get

$$\sum_i\int_D\frac{\partial h_i(x)}{\partial x_i}dx=\sum_i\int_{\partial D}\nu(v)(x)\sum_{j=1}^{d}a_{ij}(x)\frac{\partial u(x)}{\partial x_j}n_i(x)\sigma(dx),$$

which is just (1.6).

§ 2. A Remark on Quasi C^3-Continuous Modifications

Let D be a bounded C^3-domain in R^d. Consider the following form

$$\mathscr{E}(u,v)=\sum_{i=1}^{d}\int_D\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}dx.$$

It is known that $(\mathscr{E}, H^1(D))$ is a regular local Dirichlet space on $L^2(\overline{D})$. Thus each $v\in H^1(D)$ admits a modification $\tilde{v}$ on $\overline{D}$ which is quasi-continuous on $\overline{D}$. On the otherhand, if D is a C^1-domain such that D is situated on one side of ∂D, then each $v\in H^1(D)$ has its trace $\nu(v)$ on ∂D. The following lemma tells us that we have actually $\tilde{v}|_{\partial D}=\nu(v)$, $\sigma(dx)$-a. e.

Lemma 2.1. *Let D be a bounded C^3-domain in R^d.*

(1) *The "area measure" σ on ∂D is of finite energy integral in the sense of* [2].

(2) *For each $v\in H^1(D)$, the restriction on ∂D of its quasi-continuous modification $\tilde{v}$ on $\overline{D}$ is a version of its trace $\nu(v)$ on ∂D with respect to $\sigma(dx)$.*

Proof Put

$$\mathscr{E}_1(u,v)=\mathscr{E}(u,v)+\int_D u(x)v(x)dx,\ u,\ v\in H^1(D).$$

By (1.4) there exists a unique $u\in H^1(D)$ such that

$$\mathscr{E}_1(u,v)=\int_{\partial D}v\,d\sigma,\ \forall v\in C(\overline{D})\cap H^1(D).$$

Thus by definition σ is of finite energy integral and we have

$$\mathscr{E}_1(u,v)=\int_{\partial D}\tilde{v}(x)\sigma(dx),\ v\in H^1(D)$$

(see [2], Theorem 3.2.2). This means in particular that the mapping $v\to\tilde{v}|_{\partial D}$ is continuous from $H^1(D)$ into $L^1(\partial D, d\sigma)$ (see the proof of Lemma 1.1). But this mapping coincides with the trace mapping ν on $C^2(\bar{D})$, so they are the same on $H^1(D)$. Thus we have $\tilde{v}|_{\partial D}=\nu(v)$, $\sigma(dx)$-a. e., for each $v\in H^1(D)$.

§ 3. The Main Result

Let D be a bounded C^3-domain in R^d. Let $a_{ij}(x)$, $1\leqslant i, j\leqslant d$, be functions in $C^1(\bar{D})$ such that the matrix (a_{ij}) is symmetric and uniformly positive definite on D. Consider the regular Dirichlet space $(\mathscr{E}, H^1(D))$ on $L^2(\bar{D})$ which is defined by (0.1). Let $\mu=\{\Omega, \mathscr{F}, \mathscr{F}_t, \bar{X}_t, P_x, x\in D\}$ be a dx-symmetric diffusion process associated with $(\mathscr{E}, H^1(D))$.

Let $\sigma=(\sigma_{ij})$ denote the square root of (a_{ij}) and put

$$b(x)=\frac{1}{2}\sum_{j=1}^{d}\frac{\partial a_{ij}(x)}{\partial x_j},\quad c_i(x)=\sum_{j=1}^{d}a_{ij}(x)n_j(x), \tag{3.1}$$

where $n(x)=(n_1(x), \cdots, n_d(x))$ denotes the unit outward normal vector of D at $x\in \partial D$. Consider the following SDE on $\bar{D}$:

$$X_t=X_0+\int_0^t\sigma(X_s)\,dB_s+\int_0^t b(X_s)\,dx-\int_0^t c(X_s)dL_s, \tag{3.2}$$

where $X_t=(X_t^1, \cdots, X_t^d)$, $B_t=(B_t^1, \cdots, B_t^d)$ and L_t are stochastic processes satisfying the following conditions:

(i) (B_t) is a d-dimensional Brownian motion,

(ii) (L_t) is a continuous non-negative increasing process satisfying

$$\int_0^t I_{\partial D}(X_s)dL_s=L_t, \tag{3.3}$$

(iii) (X_t) is a continuous process taking values in $\bar{D}$.

A solution of (3.2) is understood as a system $\{X_t, B_t, L_t\}$ defined on a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ and satisfying the conditions (i)—(iii) and (3.2). The following theorem is the main result of this paper.

Theorem 3.1. *Let* $\mu=\{\Omega, \mathscr{F}_t, \bar{X}_t, P_x, x\in\bar{D}\}$ *be a dx-symmetric diffusion process associated with* $(\mathscr{E}, H^1(D))$.

(1) *The "area measure"* $\sigma(dx)$ *on* ∂D *is of finite energy integral with respect to* $(\mathscr{E}, H^1(D))$.

(2) *Let* (L_t) *be the PCAF (positive continuous additive functional) associated with the measure* σ. *Then there is a d-dimensional process* (B_t) *and a Borel set* $N\subset\bar{D}$ *of zero* $\mathscr{E}_1$*-capacityssuch that, for each* $x\in\bar{D}\backslash N$, *the system* $(\bar{X}_t, B_t, L_t, P_x)$ *is a solution of SED* (3.2).

(3) *Let* $\mathscr{L}$ *be the* L^2*-generator of the Dirichlet form* $\mathscr{E}$ *on* $L^2(\bar{D})$. *Put*

$$\mathscr{D}_0=\left\{u\in C^2(\bar{D}):\ \frac{\partial u}{\partial n}\Big|_{\partial D}=0\right\},$$

where

$$\frac{\partial u}{\partial n}(x)=\sum_{i=1}^{d}\frac{\partial u}{\partial x_i}n_i(x),\ x\in\partial D.$$

Then $\mathscr{D}_0\subset\mathscr{D}(\mathscr{L})$ *and, for* $u\in\mathscr{D}_0$, *we have*

$$\mathscr{L}u=\frac{1}{2}Au,\ \textit{where}\ Au=\sum_{i,j}\frac{\partial}{\partial x_i}\left[a_{ij}\frac{\partial u}{\partial x_j}\right].$$

Proof (1) By Lemma 2.1 (i) or (1.5) we have

$$\int_{\partial D}|v(x)|\sigma(dx)\leqslant C\sqrt{\mathscr{E}_1(v,v)},\ v\in C(\overline{D})\cap H^1(D), \tag{3.4}$$

which means $\sigma(dx)$ is a measure of finite energy integral with respect to $(\mathscr{E}, H^1(D))$

(2) Let (L_t) be the PCAF associated with the measure $\sigma(dx)$. Then by [2, lemma 5.1.4], one has

$$P_x\left[L_t=\int_0^t I_{\partial D}(\overline{X}_t)dL_s,\ \forall t\geqslant 0\right]=1 \tag{3.5}$$

for each $x\in\overline{D}\setminus N_1$, where N_1 is a set of zero capacity.

For any $f, g\in\mathscr{B}_b(\overline{D})$, ($\mathscr{B}_b(\overline{D})$ stands for the set of all bounded Borel functions on $\overline{D}$) put

$$\nu(A)=\int_{\partial D\cap A}f(x)\,\sigma(dx)+\int_{D\cap A}g(x)dx,\ A\in\mathscr{B}(\overline{D}). \tag{3.6}$$

Then ν is a signed measure of finite energy integral and the CAF associated with ν is

$$A_t=\int_0^t f(\overline{X}_s)dL_s+\int_0^t g(\overline{X}_s)ds. \tag{3.7}$$

Now let $u\in C^2(\overline{D})$, put

$$Au=\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left[a_{ij}\frac{\partial u}{\partial x_j}\right]. \tag{3.8}$$

Then $Au\in C(\overline{D})$. Put

$$\nu(dx)=-\sum_{i=1}^{d}c_i(x)\frac{\partial u}{\partial x_i}\sigma(dx)+\frac{1}{2}Au(x)dx. \tag{3.9}$$

Then the CAF associated with ν is

$$A_t=-\int_0^t\sum_{i=1}^{d}c_i(\overline{X}_s)\frac{\partial u}{\partial x_i}(\overline{X}_s)dL_s+\frac{1}{2}\int_0^t Au(\overline{X}_s)ds. \tag{3.10}$$

According to [2, Lemma 5.1.4], one has for any $v\in H^1(D)$

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D E_x[A_t]v(x)dx=\int_D\tilde{v}(x)\nu(dx). \tag{3.11}$$

On the other hand, by Lemma 1.2 and Lemma 2.1 (2) we have for any $v\in H^1(D)$

$$\mathscr{E}(u,v)=-\frac{1}{2}\int_D\tilde{v}(x)Au(x)dx+\int_{\partial D}\tilde{v}(x)\sum_i c_i(x)\frac{\partial u}{\partial x_i}(x)\sigma(dx)=-\int_D\tilde{v}(x)\nu(dx). \tag{3.12}$$

Thus (3.11) and (3.12) give us

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D E_x[A_t]v(x)dx=-\mathscr{E}(u,v),\ v\in H^1(D). \tag{3.13}$$

According to [2, Theorem 5.3.1] there is a set $N_2\in\overline{D}$ of zero capacity such that, for each $x\in\overline{D}\backslash N_2$, the process

$$M_t^{[u]}=u(X_t)-u(X_0)-A_t \tag{3.14}$$

is a P_x-martingale. In addition, by [2, Theorem 5.2.3], we have for any $u,\ v,\ f\in C^2(\overline{D})$,

$$\int_{\overline{D}} f(x)\mu\langle M^{[u]},\ M^{[v]}\rangle(dx)=\mathscr{E}(uf,\ v)+\mathscr{E}(vf,\ u)-\mathscr{E}(uv,\ f)$$
$$=\int_{\overline{D}} f(x)\sum_{i,j=1}^d a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}dx,$$

where $\mu\langle M^{[u]},\ M^{[v]}\rangle$ denotes the signed smooth measure associated with the CAF $\langle M^{[u]},\ M^{[v]}\rangle$. Therefore we have

$$\langle M^{[u]},\ M^{[v]}\rangle_t=\int_0^t\Big(\sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\Big)(\overline{X}_s)ds. \tag{3.15}$$

Now let $u_i(x)=x_i$, $M^i=M^{[u_i]}$. Then we have by (3.10), (3.14) and (3.15)

$$M_t^i=\overline{X}_t^i-\overline{X}_0^i-\int_0^t b_i(\overline{X}_s)ds+\int_0^t c_i(\overline{X}_s)dL_s, \tag{3.16}$$

$$\langle M^i,\ M^j\rangle_t=\int_0^t a_{ij}(\overline{X}_s)ds. \tag{3.17}$$

Let (α_{ij}) be the inverse of (σ_{ij}) and put

$$B=\begin{cases}\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{2^n}\alpha_{ij}(\overline{X}_{(k-1)t/2^n}(M^i_{kt/2^n}-M^i_{(k-1)t/2^n}\}, & \text{if the limt exsits,}\\ 0, & \text{otherwise.}\end{cases}$$

Then for $x\in\overline{D}\backslash N_2$ the process $B_t=(B_t^1,\ \cdots,\ B_t^d)$ is a d-dimensional Brownian motion under P_x, and one has

$$M_t^i=\sum_{j=1}^d\int_0^t\sigma_{ij}(\overline{X}_s)dB_s^j,\ P_x\text{-a. e.} \tag{3.18}$$

From (3.16) and (3.18) we see that, for $x\in\overline{D}\backslash(N_1\cup N_2)$, the system $(\overline{X}_t,\ B_t,\ L_t,\ P_x)$ is a solution of SDE (3.2).

(3) Let $u\in\mathscr{D}_0$. We have $Au\in L^2(\overline{D})$ and by (1.6)

$$\mathscr{E}(u,\ v)=\frac{1}{2}(-Au,\ v),\ v\in H^1(D).$$

This means $u\in\mathscr{D}(\mathscr{L})$ and $\mathscr{L}u=\frac{1}{2}Au$ (see [2], p. 19).

Remark. If the SDE (3.2) admits a unique weak solution for any initial distribution, then the corresponding family of probability measure $\{Q_x,\ x\in D\}$ on the sample space is called the reflecting diffusion on $\overline{D}$ with the directions $c(x)$ of oblique reflection at the boundary. In general, such a reflecting diffusion does not exist. But the above theorem tells us that the family of probability $\{P_x, x\in\overline{D}\}$ can be regarded as ananalogue of such a reflecting diffusion.

The next theorem gives us a sufficient condition under which the above mentioned reflecting diffusion on $\bar{D}$ exsits and canbe regarded as the dx-symetric diffusion associated with $(\mathscr{E}, H^1(D))$.

Theorem 3.2. *If furthermore each a_{ij} is in $C^2(\bar{D})$, then the SDE (3.2) admits a unique strongsolution. We denote by Ω the spacce of the continuous functions on R_+ with values in $\bar{D}$, and let $\mathscr{F}_t=\sigma\{X_s, s\leqslant t\}$, $\mathscr{F}=\bigvee_{t\in R_+}\mathscr{F}_t$, where $X_s(w)=w(s)$ is the coordinate process. Then $\{\Omega, \mathscr{F}, \mathscr{F}_t, X_t, Q_x, x\in\bar{D}\}$ is a diffusion process associated with $(\mathscr{E}, H^1(D))$, where Q_x is the law of the solution of (3.2) with $X_0=x$.*

Proof Since $\langle c(x), n(x)\rangle=\sum_{i,j=1}^{d} a_{ij}(x)n_i(x)n_j(x)\geqslant\delta>0$, the first assertion of the theorem is a fact proved in [3]. Without lossof the generality, the diffusion process associated with $(\mathscr{E}, H^1(D))$ can be constructed on $(\Omega, \mathscr{F}, \mathscr{F}_t)$ with the coordinate process (X_t) and a fimily of probability measures $\{P_x, x\in\bar{D}\}$. Thus by Theorem 3.1, for each $x\in\bar{D}\backslash N$ where N is a Borel set of zero capacity, one has $P_x=Q_x$. Therefore $\{\Omega, \mathscr{F}, \mathscr{F}_t, X_t, Q_x, x\in\bar{D}\}$ is a diffusion process associated with $(\mathscr{E}, H^1(D))$.

Remark. Under each Q_x, (X_t) is a semimartingale, because it satisfies the SDE (3.2), where (L_t) and (B_t) can be contructed as follows. Since we see easily

$$\int_0^t I_{\partial D}(X_s)ds=0,$$

by (3.3) we have

$$\int_0^t c(X_s)dL_s=-\int_0^t I_{\partial D}(X_s)dX_s.$$

Let $d_i(x)=\sum_{j=1}^{d}\beta_{ij}(x)n_j(x)$, where (β_{ij}) is the inverse of (a_{ij}). Then $\langle c(x), d(x)\rangle=1$, and

$$L_t=-\sum_i\int_0^t I_{\partial D}(X_s)d_i(X_s)dX_s^i. \tag{3.19}$$

After that, we put

$$M_t=X_t-X_0-\int_0^t b(X_s)ds+\int_0^t c(X_s)dL_s.$$

Then we can construct the process (B_t) just as in the proof of Theorem 3.1. In this way we have constructed a family of solutions $\{X_t, B_t, L_t, P_x, x\in D\}$ of SDE (3.2) on the same space $(\Omega, \mathscr{F}, \mathscr{F}_t)$ with the same processes (X_t), (B_t) and (L_t). According to Theorems 3.1 and 3.2 the process (L_t) is necessarily the PCAF associated with the measure σ. This fact can also be proved by the following argument. Let $u\in C^2(\bar{D})$, $v\in H^1(D)$. From (3.2) and by Ito's formula we have ((T_t) denotes the L^2-semigroup of the diffusion)

$$\mathscr{E}(u, v)=\lim_{t\to0}\frac{1}{t}(u-T_tu, v)=\lim_{t\to0}\frac{1}{t}\int_D v(x)E_x[u(x)-u(X_t)]dx$$
$$=-\lim_{t\to0}\frac{1}{2t}\int_D v(x)E_x\Big[\int_0^t Au(X_s)ds\Big]dx$$

$$+\lim_{t\to 0}\frac{1}{t}\int_D v(x)E_x\Big[\int_0^t\sum_{i=1}^d\frac{\partial u}{\partial x_i}(X_s)c_i(X_s)dL_s\Big]dx$$
$$=\frac{1}{2}(-Au,\ v)_{L^2}+\lim_{t\to 0}\frac{1}{t}\int_D v(x)E_x\Big[\int_0^t\sum_{i=1}^d\frac{\partial u}{\partial x_i}(X_s)c_i(X_s)dL_s\Big]dx,$$

from which and (3.12) we get

$$\lim_{t\to 0}\frac{1}{t}\int_D v(x)E_x\Big[\int_0^t\sum_{i=1}^d\frac{\partial u}{\partial x_i}(X_s)c_i(X_s)dL_s\Big]dx$$
$$=\int_{\partial D}\tilde{v}(x)\sum_{i=1}^d\frac{\partial u(x)}{\partial x_i}c_i(x)\sigma(dx).$$

In particular, if we take $u\in C^2(\bar{D})$ such that $\sum_{i=1}^d\frac{\partial u}{\partial x_i}c_i(x)=1$ for $x\in\partial D$, then we have

$$\lim_{t\to 0}\frac{1}{t}\int_D v(x)E_x(L_t)dx=\int_{\partial D}\tilde{v}(x)\sigma(dx).$$

This means (L_t) is the PCAF associated with the measure σ of finite energy integral (see [Lemma 5.1.4).

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