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UNIQUENESS OF OPTIMAL QUADRATURE FORMULAS FOR W_1^m AND THE FUNDAMENTAL THEOREM OF ALGEBRA FOR PERIODIC MONOSPLINES

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Abstract

This paper proves that the optimal quadrature formulas of type $(r_1, \cdots r_n)$ for W_1^m is unique, and the extremal function is characterized by its oscillatory property. On the other hand, the fundamental theorem of algebra for periodic monosplines with odd multiplicities is solved

§ 1, Introduction

For pre-assigned multiplicities $(r_i)_1^n$, $1\leqslant r_i\leqslant m$, and $f\in W_q^m$, consider the quadrature formulas of type $(r_1, \cdots, r_n)$:

$$\int_0^1 f(t)dt) = \sum_{i=1}^{n}\sum_{j=0}^{r_i-1}(-1)^{j+1}a_{ij}f^{(j)}(x_i)+R(f),\ x_1=0, \tag{1.1}$$

where

$$W_q^m := \{f;\ f\in C^{m-1}[0,\ 1],\ f^{(m-1)} \text{ abs. const,. } \|f^{(m)}\|_q\leqslant 1,\ f^{(j)}(0)=f^{(j)}(1),\ j=0,\ 1,\ \cdots,\ m-1\},\ 1\leqslant q\leqslant\infty.$$

Nikolsklii[1], Schoenberg[2] and Bojanov[3, 4] studied the following nonlinear extremal problem:

(A) $$E(W_q^m;\ r_1,\ \cdots,\ r_n) := \inf_{a_{ij}, x_i}\ \sup_{f\in W_q^m} |R(f)|,$$

and they discussed the existence and uniqueness of the parameters $\{a_{ij},\ x_i\}$, which attain the infimum of (A), i. e. the existence and uniqueness of the optimal quadrature formula (OQF) of type $(r_1, \cdots, r_n)$ for W_q^m. Bojanov proved that the OQF of type $(r_1, \cdots, r_n)$ for W_q^m is existent $(1<q\leqslant\infty)$ and unique $(1<q<\infty)$. Then we have established the following

Theorem A[6]. *For given* $(r_i)_1^n$, $1<2[(r_i+1)/2]<m$, *the OQF of type* $(r_1, \cdots,$

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r_n) *for* W_1^m *is existent, and the optimal coefficients* $\{a_{ij}^*\}$ *satisfy the conditions*

$$\begin{aligned} &a_{ij}^*<0,\ j=0,\ 2,\ \cdots,\ r_i-1,\ \text{if } r_i \text{ is odd},\\ &a_{i,r_i-1}^*=0,\ a_{ij}^*<0,\ j=0,\ 2,\ \cdots,\ r_i-2,\ \text{if } r_i \text{ is even}. \end{aligned} \tag{1.2}$$

It is well-known that the extremal problem (A) is closely related to the minimal problem in $M_m(r_1, \cdots, r_n)$, which consists of all periodic monosplines $M(t)$, of the form

$$M(t)=a+\sum_{i=1}^{n}\sum_{j=0}^{r_i-1}a_{ij}D_{m-j}(t-x_i),$$

where

$$x_1<\cdots<x_n<1+x_1,\ \sum_{i=1}^{n}a_{i0}+1=0,$$

$$D_m(t)=2^{1-m}\pi^{-m}\sum_{k=1}^{\infty}k^{-m}\cos(2k\pi t-m\pi/2).$$

We solve the uniqueness of the solution of (A) in this paper. Our second goal is to discuss the fundamental theorem of algebra for $M_m(r_1, \cdots, r_n)$. Schumaker[10] Scheonberg [11], and Micchelli [12] proved the following perfect theorem for the class of non-periodic monosplines $N_m(r_1, \cdots, r_n)$,

$$N_m(r_1, \cdots, r_n)=\left\{M(t)=t^m/m!+\sum_{j=0}^{m-1}c_jt^j+\sum_{i=1}^{n}\sum_{j=0}^{r_i-1}a_{ij}(t-x_i)_+^j,\ 0<x_1<\cdots<x_n<1\right\}.$$

Theorem B. *Given* $1\leqslant r_i\leqslant m$, *let*

$$2N=\sum_{i=1}^{n}(r_i+\sigma_i), \tag{1.3}$$

where $\sigma_i=1$ (*if* r_i *is odd*) *or* 0 (*if* r_i *is even*). *Then for any* $M\in N_m(r_1, \cdots, r_n)$, *the total number of zeros of* M *is no greater than* $m+2N$. *Conversly, for given* $0\leqslant t_1<\cdots<t_{2N+m}\leqslant 1$, *there exists* $M\in N_m(r_1, \cdots, r_n)$, *such that*

$$M(t_i)=0,\ i=1,\ \cdots,\ m+2N.$$

Furthermore, $M(t)$ *is uniquely decided by* $(t_i)_1^{m+2N}$ *if and only if all* $(r_i)_1^n$ *are odd.*

But, in the periodic case, the uniqueness is not true. Zhensykbaev[9] proved

Theorem C. *Let* $M\in M_m(\underbrace{1, \cdots, 1}_{N})$. *Then* M *has at most* $2N$ *zeros in a period. Conversely, for given* $t_1<\cdots<t_{2N}<1+t_1$, *there exist exact two monosplines* $M_j(t)\in M_m(1, \cdots, 1)$, *such that*

$$M_j(t_i)=0,\ i=0,\ 1,\ \cdots,\ 2N;\ j=1,\ 2.$$

Definition 1. *We say* $(r_1, \cdots, r_n)$ *is* k*-circular if* k *is the smallest natural number that makes the equality*

$$(r_1, \cdots, r_n)=(r_{k+1}, \cdots, r_n, r_1, \cdots, r_k)$$

hold; we use the notation

$$2L:=\sum_{j=1}^{k}(r_j+\sigma_j). \tag{1.4}$$

For an n*-circular* $(r_1, \cdots, r_n)$ *we also say that that it is non-circular*

Remark 1. If $r_1=\cdots=r_n=\rho$, and ρ is odd, then $2L=\rho+1$. If $(r_1, \cdots, r_n)$ is non-circular, then $2N=2L$.

Definition 2. *Given $f\in C[a, b]$, we say f alternates r times, if there exist $a\leqslant\xi_1<\cdots<\xi_{r+1}\leqslant b$, such that*

$$f(\xi_i)=(-1)^i\sigma\|f\|_\infty,$$

where $\sigma=1$ or -1. And $(\xi_i)_1^{r+1}$ are named alternate points.

Our first main result is as follows.

Theorem 1. *Given $(r_i)_1^n$, $1<2[(r_i+1)/2]<m$, the extremal problem* (A^*)

(A^*) $\inf\{\|M\|_\infty, M\in M_m(r_1, \cdots, r_n), x_1=0<x_2<\cdots<x_n<1\}$

has a unique solution, which is characterized by (1.2) and the oscillatory property, i.e. $\phi\in M_m(r_1, \cdots, r_n)$ is the solution of (A^*) if and only if there exist $0<\xi_1<\cdots<\xi_{2N}<1$, such that

$$a_{i,r_i-1}<0, \text{ if } r_i \text{ is odd},$$

$$a_{i,r_i-1}=0, \text{ if } r_i \text{ is even}, \quad \text{odd}$$

$$\phi(\xi_i)=\sigma(-1)^i\|\phi\|_\infty,$$

where $\sigma=1$ or -1, $2N$ is defined in (1.3).

According to the duality relation, Theorem 1 is equivalent to Theorem 2.

Theorem 2. *Given $(r_i)_1^n$, $1<2[(r_i+1)/2]<m$, the OQF of type $(r_1, \cdots, r_n)$ for W_1^m is unique.*

Corollary. *Suppose that $r_1=\cdots=r_n=\rho$, $1<2[(\rho+1)/2]<m$. Then the equally spaced knots*

$$x_i^*=(i-1)/n, \quad i=1, \cdots, n$$

are uniquely optimal knots of the OQF of type $(\rho, \cdots, \rho)$ for W_1^m. Moreover,

$$E(W_1^m; \rho, \cdots, \rho)=1/n^m \min_{a_i, a}\left\| D_m(\cdot)+\sum_{j=1}^{\rho-1}a_jD_{m-j}(\cdot)+a\right\|_\infty.$$

Remark. Barrar and Loeb obtained the analogy of Theorem 1 in non-periodic case by using the fundamental thoerem of algebra for non-periodic monosplines [7] (cf. Theorem B). But the attempt to prove Theorsm 1 by establishing the fundamental theorem of algebra for periodic monosplines with odd multiplicities failed. We prove Theorem 1 by using the topological degree. Then we prove the following Theorem 3 by use of Theorem 1 and the result proved in [6].

Theorem 3. *Suppose that $(r_i)_1^n$ are all odd, $1\leqslant r_i\leqslant m$, and $(r_1, \cdots, r_n)$ is k-circular. Then for any given $0=t_1<\cdots<t_{2N}<1$, there exist exact $2L$ monosplines $M_j(t)$ in $M_m(r_1, \cdots, r_n)$, such that*

$$M_j(t_i)=0, \quad i=1, \cdots, 2N; \ j=1, \cdots, 2L.$$

Remark. In the case of $r_1=\cdots=r_n=1$, we obtain Zhensykbaev's result by use of different method.

§ 2. Preliminaries and Proof of Theorem 1

For given $x_1<\cdots<x_n<1+x_1$, $1\leqslant r_i\leqslant m$, we denote

$$y_{m_{i-1}}+1=\cdots=y_{m_i}=x_i,\ i=1,\ \cdots,\ n;\ y_{\nu k+j}=y_j+K,\ \nu\in Z, \tag{2.1}$$

where $m_0=0$, $m_i=r_1+\cdots+r_i$ ($i=1,\ \cdots,\ n$), $K=r_1+\cdots+r_n=m_n$. And we denote by $S_m\begin{pmatrix}x_1, & \cdots, & x_n\\ r_1, & \cdots, & r_n\end{pmatrix}$ the space of periodic polynomial splines with multiplicities $(r_i)_1^n$ knots $(x_i)_1^n$, respectively. Further let

$$M_m\begin{pmatrix}x_1, & \cdots, & x_n\\ r_1, & \cdots, & r_n\end{pmatrix}=-D_m(t-x_1)-s(t);\ s\in S_m\begin{pmatrix}x_1, & \cdots, & x_n\\ r_1, & \cdots, & r_n\end{pmatrix}.$$

It is easy to see that

$$M_m(r_1,\ \cdots,\ r_n)=\{M\in M_m\begin{pmatrix}x_1, & \cdots, & x_n\\ r_1, & \cdots, & r_n\end{pmatrix};\ \text{for some } x_1<\cdots<x_n<1+x_1\}.$$

Lemma 1.[3, 4] i) *If* $M\in M_m(r_1,\ \cdots,\ r_n)$, $(1\leqslant r_i\leqslant m-1,\ i=1,\ \cdots,\ n)$. *then* M *has at most* $2N$ ($2N$ *is defined in* (1.3)) *zeros in a period.*

ii) *If* $M\in M_m\begin{pmatrix}x_1, & \cdots, & x_n\\ r_1, & \cdots, & r_n\end{pmatrix}$ *has exact* $2N$ *zeros, say* $t_1<\cdots<t_{2N}<1+t_1$, *in a period, then there exists a cyclic-arrangement* $(t_i^*)_1^{2N}$ *of* $(t_i)_1^{2N}$ *such that*

$$t_{n_i}^*<x_i<t_{m+1+n_{i-1}}^*,\ i=1,\ \cdots,\ n, \tag{2.2}$$

where $n_0=0$, $n_i=\sum_{j=1}^{i}(r_j+\sigma_j)$, $i=1,\ \cdots,\ n$.

iii) *If all* $(r_i)_1^n$ *are odd,* $M\in M_m(r_1,\ \cdots,\ r_n)$ *has exact* $2N$ *zeros in a period, then the coefficients of* M *satisfy*

$$a_{i,r_i-1}<0,\ i=1,\ \cdots,\ n. \tag{2.3}$$

Lemma 2. *Suppose that* $1\leqslant r_i\leqslant m-1$, $(i=1,\ \cdots,\ n)$, $\sum_{i=1}^{n}r_i=2N-1$, f *is a continuous function of 1-periodic,* $s_f\in S_m\begin{pmatrix}x_1, & \cdots, & x_n\\ r_1, & \cdots, & r_n\end{pmatrix}$. *Then the following i) and ii) are equivalent.*

i) (*a*) $f-s_f$ *alternates at least* $2N$ *times in a period, and*

(*b*) *There exist at least* $j+1$ *alternate points of* $f-s_f$ *in each interval as* $(y_i,\ y_{i+m-1+j})$.

ii) s_f *is the strongly unique best approximation of* f *from* $S_m\begin{pmatrix}x_1, & \cdots, & x_n\\ r_1, & \cdots, & r_n\end{pmatrix}$. (*For the definition of strongly unique best approximation we refer reader to* [13]).

Proof We denote the sequence of functions

1, $\{D_m(t-x_i)-D_m(t-x_1)\}_{i=2}^n$, $\{D_{m-j}(t-x_i),\ j=1,\ \cdots,\ r_i-1;\ i=1,\ \cdots,\ n\}$ by $\{u_j(t)\}_{j=1}^{2N-1}$. Then [18] $\{u_j(t)\}$ is a base of weak Chebyshev subspace $S_m\begin{pmatrix}x_1, & \cdots, & x_n\\ r_1, & \cdots, & r_n\end{pmatrix}$, and for $t_1<\cdots<t_{2N-1}<1+t_1$, the condition

$$\det\begin{pmatrix}u_1, & \cdots, & u_{2N-1}\\ t_1, & \cdots, & t_{2N-1}\end{pmatrix}\neq 0 \tag{2.4}$$

holds if and only if there exists some cyclic-arrangement of $(t_i)_1^{2N-1}$, say $(t_i^*)_1^{2N-1}$, satisfying

$$t_i^*<y_i<t_{i+m}^*,\ i=1,\ \cdots,\ 2N-1. \tag{2.5}$$

Now the lemma may be proved similar to Corollary 1.14 of [19] by using (2.5) and Theorem 1.4 of [19]. The details are omitted.

Lemma 3. *Suppose that all* $(r_i)_1^n$ *are even,* $1\leqslant r_i\leqslant m-1$, $2N=\sum_{i=1}^n r_i$. *And let* $x=(x_1,\ \cdots,\ x_n)$ *be given,* $s(x,\ t)\in S_m\begin{pmatrix}x_1, & x_2, & \cdots, & x_n\\ r_1-1, & r_2, & \cdots, & r_n\end{pmatrix}$.

i) $M(x,\ t)=-D_m(t-x_1)-s(x,\ t)$ (2.6)

is the solution of extremal problem (B),

$$\inf\left\{\|M\|_\infty;\ M(t)=-D_m(t-x_1)-s(t),\ s(t)\in S_m\begin{pmatrix}x_1, & x_2, & \cdots, & x_n\\ r_1-1, & r_2, & \cdots, & r_n\end{pmatrix}\right\} \tag{B}$$

if and only if $M(x,\ t)$ *alternates exact* $2N$ *times in a period.*

ii) *If* (2.6) *is the minimal solution of* (B), *then* $s(x,\ t)$ *is the strongly unique best approximation of* $-D_m(t-x_1)$ *from* $S_m\begin{pmatrix}x_1, & x_2, & \cdots, & x_n\\ r_1-1, & r_2, & \cdots, & r_n\end{pmatrix}$. *Hence the solution of* (B) *is unique.*

Proof Suppose that $M(x,\ t)$ alternates $2N$ times, and the alternate points are $(\xi_i)_1^{2N}$. Then $(\xi_i)_1^{2N}$ are the zeros of $M'(x,\ t)$. According to Lemma 1, there exists some cyclic-arrangement of $(\xi_i)_1^{2N}$, say $(\xi_i^*)_1^{2N}$, such that

$$\xi_i^*<y_i<\xi_{i+m-1}^*,\ i=1,\ \cdots,\ 2N. \tag{2.7}$$

If $M(x,\ t)$ is not a solution of problem (B), then there is some

$$\overline{M}(t)\in M_m\begin{pmatrix}x_1, & x_2, & \cdots, & x_n\\ r_1-1, & r_2, & \cdots, & r_n\end{pmatrix},$$

such that

$$\|\overline{M}\|_\infty<\|M(x,\ \cdot)\|_\infty.$$

Let

$$\delta(t)=M(x,\ t)-\overline{M}(t).$$

Then $\delta(t)\in S_m\begin{pmatrix}x_1, & x_2, & \cdots, & x_n\\ r_1-1, & r_2, & \cdots, & r_n\end{pmatrix}$, and there exist $z_i\in(\xi_i,\ \xi_{i+1})$ satisfying $\delta(z_i)=0$, $(i=1,\ \cdots,\ 2N)$. We know from (2.7) that there exists some cyclic-arrangement of $(z_i)_1^{2N}$, say $(z_i^*)_1^{2N}$, such that

$$y_i<z_i^*<y_{i+m},\ i=1,\ \cdots,\ 2N. \tag{2.8}$$

Thus $(z_i)_1^{2N}$ must be isolated zeros of $\delta(t)$. This contradicts the fact that $S_m\begin{pmatrix}x_1, & x_2, & \cdots, & x_n\\ r_1-1, & r_2, & \cdots, & r_n\end{pmatrix}$ is a $2N-1$ dimensional weak-Chebyshev space.

Conversely, since $S_m\begin{pmatrix} x_1, x_2, \cdots, x_n \\ r_1-1, r_2, \cdots, r_n \end{pmatrix}$ is a weak Chebyshev space, there exists $s(x, t)\in S_m\begin{pmatrix} x_1, x_2, \cdots, x_n \\ r_1-1, r_2, \cdots, r_n \end{pmatrix}$, such that $s(t)$ is a best approxim ation of $-D_m(t-x_1)$ from $S_m\begin{pmatrix} x_1, x_2, \cdots, x_n \\ r_1-1, r_2, \cdots, r_n \end{pmatrix}$, and $\overline{\overline{M}}(t)=-D_m(t-x_1)-s(t)$ alternates $2N$ times in a period. Thus the alternate points $(\xi_i)_1^{2N}$ satisfy (2.7). Therefore there exist at least $j+1$ alternate points in each interval as $(y_i, y_{i+m-1+j})$. By use of Lemma 2, $s(t)$ is the strongly unique best approximation of $-D_m(t-x_1)$. Hence $\overline{\overline{M}}(t)=M(x, t)$. The proof is complete.

We prove the uniqueness of the solution of extremal problem (B) by use of the theory of topological degree. For the sake of completeness, we recall some properties of topological degree [17].

Let D be a non-empty open bounded set in R^n, $\overline{D}$ and ∂D be the closure and boundary of D, respectively. Let ϕ be a continuous mapping from $\overline{D}$ into R^n. Then for $c\overline{\in}\phi(\partial D)$, the degree of ϕ with respect to D and c is defined to be an integer value and denoted by $\deg(\phi, D, c)$. It has following basic properties.

i) Suppose that ϕ is differentiable at x and that $\det(\phi'(x))\neq 0$ whenever $x\in D$ and $\phi(x)=c$. Then there exist a finite number of points, say $x^i\in D$, $i\in I$, where $\phi(x^i)=c$ and $\deg(\phi, D, c)=\sum_{i\in I}\text{sign}\det(\phi'(x^i))$.

ii) If $\deg(\phi, D, c)\neq 0$ there exists at least one point $x\in D$ for which $\phi(x)=c$.

iii) Let $\phi(x, \alpha)$ be continuous on $D\times[0, 1]$; Furthermore, suppose that $\phi(x, \alpha)\neq c$ for any $x\in\partial D$, $0\leqslant\alpha\leqslant 1$. Then $\deg(\phi(\cdot, \alpha), D, c)$ is constant independent of α.

For $x=(x_1, \cdots, x_n)$, $0=x_1<\cdots<x_n<1$, denote

$$\Omega_n=\{\bar{x}\in R^{n-1}, \bar{x}=(x_2, \cdots, x_n), 0=x_1<\cdots<x_n<1\}. \tag{2.9}$$

Given even numbers $(r_i)_1^n$, $1\leqslant r_i\leqslant m-1$, define mapping

$$\phi(\bar{x})=(b_2(\bar{x}), \cdots, b_n(\bar{x})), \bar{x}\in\Omega_n, \tag{2.10}$$

where $b_i(x)$ is the coefficient a_{i,r_i-1} of $M(x, t)$, the solution of minimal problem (B) for $x=(0, \bar{x})$. In accordance with Theorem A, the uniqueness of (B) is equivalent to the uniqueness of the solution of nonlinear equation

$$\phi(\bar{x})=0. \tag{2.11}$$

We call $\bar{x}\in\Omega_n$ satisfying (2.11) the critical point.

In order to apply the topological degree, we must define an open set D, such that ϕ is continuous on $\overline{D}$ and (2.11) has no solution on ∂D. Consider the open subset $\Omega_{n\varepsilon}$ of Ω_n,

$$\Omega_{n\varepsilon}=\{\bar{x}\in\Omega_n, x_{i+1}-x_i>\varepsilon, i=1, \cdots, n, x_{n+1}=1\}.$$

The following lemma shows all critical points lie in $\Omega_{n\varepsilon}$ with some $\varepsilon>0$; therefore we may define

$$\deg(\phi, \Omega_n, 0)=\lim_{\varepsilon\to 0}\deg(\phi, \Omega_{n\varepsilon}, 0). \tag{2.13}$$

Lemma 4.[4] *Suppose that $(r_i)_n^1$ are odd multiplicities. Then $M_m^0(r_1, \cdots, r_n)$ is a compact set, where*

$$M_m^0(r_1, \cdots, r_n)=\{M\in M_m(r_1, \cdots, r_n);\ M \text{ has } 2N \text{ zeros in } [0, 1)\}.$$

Therefore there exists an $\varepsilon>0$, such that the inequalities

$$x_{i+1}-x_i>2\varepsilon,\ i=1, \cdots, n,\ x_{n+1}=1+x_1 \tag{2.14}$$

hold for all $M\in M_m^0(r_1, \cdots, r_n)$.

From Lemma 4, we know that all critical points lie in $\Omega_{n\varepsilon}$, with the ε defined in (2.14).

Lemma 5. *Suppose that even numbers $(r_i)_1^n$ satisfy $1\leqslant r_i\leqslant m-1$, $\sum_{i=1}^n r_i=2N$. For fixed $x=(x_1, \cdots, x_n)$, consider the minimal problem*

$$\inf\left\{\|M\|_{(\lambda)};\ M\in M_m\begin{pmatrix} x_1, x_2, \cdots, x_n \\ r_1-1, r_2, \cdots, r_n\end{pmatrix}\right\}, \tag{C}$$

where

$$\|\cdot\|_{(\lambda)}=\lambda\|\cdot\|_\infty+(1-\lambda)\|\cdot\|_2,\ 0\leqslant\lambda\leqslant 1.$$

Then (C) has a unique solution for each $\lambda\in[0, 1]$, the minimal function $M(x, t)$ has $2N$ zeros in a period.

Proof When $\lambda=1$, the assertion comes from Lemma 3. If $0\leqslant\lambda<1$, $\|\cdot\|_{(\lambda)}$ is a strictly convex norm; therefore (C) has unique solution $M_\lambda(x, t)$. Note that $M_\lambda(x, t)$ is a piecewise polynomial of degree m with leading term $t^m/m!$. Thus $M_\lambda(x, t)$ only has isolated zeros. Suppose that for some $\lambda\in[0, 1)$, $M_\lambda(x, t)$ only has $2q<2N$ zeros, say $(z_i)_1^{2q}$, in a period. Since $S_m\begin{pmatrix} x_1, x_2, \cdots, x_n \\ r_1-1, r_2, \cdots, r_n\end{pmatrix}$ is a $2N-1$ dimensional weak Chebyshev subspace, we can construct an $s(t)\in S_m\begin{pmatrix} x_1, x_2, \cdots, x_n \\ r_1-1, r_2, \cdots, r_n\end{pmatrix}$, $s\not\equiv 0$, such that

$$s(t)\cdot M_\lambda(x, t)\geqslant 0,\ \forall t\in[0, 1).$$

Let

$$\overline{M}_\lambda(x, t)=M_\lambda(x, t)-\delta s(t).$$

Then $\overline{M}_\lambda(x, t)\in M_m\begin{pmatrix} x_1, x_2, \cdots, x_n \\ r_1-1, r_2, \cdots, r_n\end{pmatrix}$. Evidently,

$$\|\overline{M}_\lambda(x, \cdot)\|_{(\lambda)}<\|M_\lambda(x, \cdot)\|_{(\lambda)}$$

holds for sufficiently small $\delta>0$. It contradicts the assumption of $M_\lambda(x, t)$ being a minimal function. The lemma is proved.

Lemma 6. *Let ϕ be difined in (2.10). Under the assumption of Lemma 5,*

$$\deg(\phi, \Omega_n, 0)=(-1)^{n-1}. \tag{2.15}$$

Proof Accoding to Lemma 5, for each $x=(x_1, \cdots, x_n)$ and $\lambda\in[0, 1]$, problem (C) has unique solution $M_\lambda(x, t)$. Let

$$F(A, x, \lambda)=\|M_\lambda(x, \cdot)\|_{(\lambda)}$$

where $A(x, \lambda)$ is the coefficient vector of $M_\lambda(x, t)$

$$A(x, \lambda)=(a(x, \lambda); a_{1,0}(x, \lambda), \cdots, a_{1,r_1-2}(x, \lambda); a_{2,0}(x, \lambda), \cdots, a_{2,r_2-1}(x, \lambda); \cdots;$$
$$a_{n,0}(x, \lambda), \cdots, a_{n,r_n-1}(x, \lambda)), \quad \sum_{i=1}^n a_{i,0}(x, \lambda)+1=0.$$

Then it is easy to see that the mapping

$$\Omega_n \times[0, 1]\to R^{2N}: (x, \lambda)\to A(x, \lambda)$$

is continuous (cf. Lemma 4.2 of [16]). Consider the mapping ϕ defined on $\Omega_n\times[0, 1]\to R^{n-1}$,

$$\phi(\bar{x}, \lambda)=(b_{2\lambda}(\bar{x}), \cdots, b_{n\lambda}(\bar{x})), \tag{2.16}$$

where $b_{i\lambda}(x)$ is the coefficient $a_{i,r_i-1}(x, \lambda)$ of $M_\lambda(x, t)$, $(i=2, \cdots, n)$. According to Lemma 1 and Lemma 5, all the critical points of equation (2.16) lie in $\Omega_{n\varepsilon}$ for each $\lambda\in[0, 1]$. Thus, we may prove as in [16] that $\phi(\bar{x}, \lambda)$ is a continuous function of $\Omega_{n\varepsilon}\times[0, 1]$. Moreover, by Lemma 5, $\phi(\bar{x}, \lambda)\neq 0$ for each $x\in\partial\Omega_{n\varepsilon}$, and $\lambda\in[0, 1]$. On the other hand, we know from [4]

$$\deg(\phi(\cdot, 0), \Omega_{n\varepsilon}, 0)=(-1)^{n-1},$$

since $\|\cdot\|_{(0)}=\|\cdot\|_2$. By use of the homotopy invariance of topological degree, we obtain

$$\deg(\phi(\cdot, 1), \Omega_{n\varepsilon}, 0)=(-1)^{n-1}.$$

Hence

$$\deg(\phi(\cdot, 1), \Omega\ ,\ 0)=\lim_{\varepsilon\to 0}\deg(\phi(\cdot, 1), \Omega_{n\varepsilon}, 0)=(-1)^{n-1}.$$

Lemma 7. *Suppose that all $(r_i)_1^n$ are even, $1\leqslant r_i\leqslant m-1$. Then $\phi(\bar{x})$ is differentiable at critical points, and*

$$\operatorname{sgn}\det\left(\frac{\partial\phi(\bar{x})}{\partial\bar{x}}\right)=(-1)^{n-1}.$$

Proof Assume that $x^*=(x_1^*, \cdots, x_n^*)$ is a critical point of $\phi(\bar{x})$. Then

$$M_*(t)=a^*+\sum_{i=1}^n\sum_{j=0}^{r_i-1}a_{ij}^*D_{m-j}(t-x_i^*),$$

and the minimal element of

$$\inf\left\{\|M\|_\infty;\ M\in M_m\begin{pmatrix}x_1^*, x_2^*, \cdots, x_n^*\\ r_1-1, r_2, \cdots, r_n\end{pmatrix}\right\},$$

satisfies

$$a_{i,r_i-1}^*=0,\ i=1, \cdots, n.$$

Let $h=(h_1, \cdots, h_n)$, with $h_1=0$, and $\|h\|$ be the Euclidean norm of h. By Taylor series expansion we get

$$M_*(t)=M_1(t)+o(\|h\|), \tag{2.17}$$

where

$$M_1(t)=a^*+\sum_{i=1}^n\sum_{j=0}^{r_i-2}a_{ij}^*[D_{m-j}(t-x_i^*-h_i)+h_iD_{m-j-1}(t-x_i^*-h_i)].$$

Obviously

$$M_1(t)\in M_m\begin{pmatrix} x_1^*, & x_2^*+h_2, & \cdots, & x_n^*+h_n \\ & r_1-1, & r_2, \cdots, & r_n \end{pmatrix}.$$

Suppose that $\overline{M}(t)=M(x^*+h,\ t)$ is the minimal element in

$$M_m\begin{pmatrix} x_1^*, & x_2^*+h_2, & \cdots, & x_n^*+h_n \\ & r_1-1, & r_2, \cdots, & r_n \end{pmatrix}.$$

We claim that

$$\|\overline{M}-M_1\|_\infty=o(\|h\|). \tag{2.18}$$

Assume that $(\xi_i)_1^{2N}$ are the alternate points of $M_*(t)$, i. e.

$$(-1)^i\sigma M_*(\xi_i)=\|M_*\|_\infty=:b,\ \sigma=1 \text{ or } -1. \tag{2.19}$$

From (2.17) and (2.19) we know

$$(-1)^i\sigma M_1(\xi_i)=b+o(\|h\|).$$

Assume that

$$M_*(t)=-D_m(t-x_1)-s(x^*,\ t).$$

Then $s(x^*,\ t)$ is the strongly unique best approximation of $-D_m(t-x_1)$ from $S_m\begin{pmatrix} x_1^*, x_2^*, \cdots, x_n^* \\ r_1-1, r_2, \cdots, r_n \end{pmatrix}$. From (2.4), (2.5) and (2.8) we know that for each $s\in S_m\begin{pmatrix} x_1^*, x_2^*, \cdots, x_n^* \\ r_0-1, r_2, \cdots, r_n \end{pmatrix}$, $(s\neq 0)$, there exists at laest one ξ_i, such that $(-1)^i\sigma s(\xi_i)>0$.

Therefore, if we denote

$$\rho=\min_{\|s\|=1}\max_i(-1)^i\sigma S(\xi_i),$$

then $\rho>0$. Consequently there exists a constant $C>0$, such that

$$\max_{1\le i\le 2N}(-1)^i\sigma s(\xi_i)\geq 2C\|s\|_\infty,\ \forall s\in S_m\begin{pmatrix} x_1^*, x_2^*, \cdots, x_n^* \\ r_1-1, r_2, \cdots, r_n \end{pmatrix},$$

i. e.,

$$\max_{1\le i\le 2N}(-1)^i\sigma(M_*-M)(\xi_i)\geq 2C\|M_*-M\|_\infty,\quad \forall M\in M_m\begin{pmatrix} x_1^*, x_2^*, \cdots, x_u^* \\ r_1-1, r_2, \cdots, r_n \end{pmatrix}.$$

From the continuity, we obtain

$$\max_{1\le i\le 2N}(-1)^i\sigma(M_1-M)(\xi_i)\geq C\|M_1-M\|_\infty,\quad \forall M\in M_m\begin{pmatrix} x_1^*, x_2^*, \cdots, x_n^* \\ r_1-1, r_2, \cdots, r_n \end{pmatrix}. \tag{2.20}$$

On the other hand, we know from the definition that

$$\|\overline{M}\|^\infty\leq\|M_1\|_\infty\leq b+o(\|h\|).$$

Hence

$$(-1)^i\sigma\overline{M}(\xi_i)-o(\|h\|)\leq b\leq(-1)^i\sigma M_1(\xi_i)+o(\|h\|). \tag{2.21}$$

From (2.21), we conclude that

$$(-1)^i\sigma(M_1-\overline{M})(\xi_i)\leq o(\|h\|),\ i=1,\ \cdots,\ 2N. \tag{2.22}$$

And (2.20) and (2.22) give the conclusion (2.18). Therefore $M(x^*+h,\ t)$ may be written in the form of

$$
\begin{aligned}
M(x^*+h,\ t)&=a^*+\sum_{i=1}^{n}\sum_{j=0}^{r_i-2}(a_{ij}^*+h_i a_{i,j-1}^*)D_{m-j}(t-x_i^*-h_i)\\
&\quad+\sum_{i=2}^{n}a_{i,r_i-2}^*h_iD_{m-r_i+1}(t-x_i^*-h_i)+o(\|h\|)\\
&=:g(x^*+h,\ t)+\sum_{i=2}^{n}a_{i,r_i-2}^*h_iD_{m-r_i+1}(t-x_i^*-h_i)+o(\|h\|),
\end{aligned}
$$

where $a_{i,-1}^*=0$, $(i=1, \cdots, n)$. Obviously $g(x^*+h, t)$ has no contribution to $(b_2(x^*+h), \cdots, b_n(x^*+h))$. Hence

$$
\begin{cases}
\dfrac{\partial b_i}{\partial x_i}(x^*)=a_{i,r_i-2}^*, & i=2, \cdots, n,\\[2ex]
\dfrac{\partial b_i}{\partial x_k}(x^*)=0, & i, k=2, \cdots, n;\ i\neq k,
\end{cases}
$$

i. e. $\dfrac{\partial \phi}{\partial \bar{x}}(x^*)=\left(\dfrac{\partial b_i}{\partial x_j}(x^*)\right)_{i,j=2}^{n}$ is a diagonal matrix. Since $a_{i,r_i-2}^*<0$ by (2.3), we conclude that

$$
\operatorname{sgn}\det\left(\frac{\partial \phi}{\partial x}(x^*)\right)=(-1)^{n-1}.
$$

This completes the proof.

Proof of Theorem 1 By use of Theorem A, we know that to prove the uniqueness of solution of extremal problem (A*) we only need to prove the nonlinear equation (2.11) has a unique solution in the case of $(r_i)_2^n$ being all even. According to Lemma 6,

$$
\deg(\phi(\bar{x}),\ \Omega_n,\ 0)=(-1)^{n-1}.
$$

Lemma 7 tells us that $\phi(\bar{x})$ is differentiable and $\operatorname{sgn}(\det \phi'(\bar{x}))=(-1)^{n-1}$. at each critical piont x. Hence the equation $\phi(\bar{x})=0$ has unique solution, in accordance with property i) of topological degree. Consequently, problem (A*). has unique solution.

On the other hand, suppose that $M(t)$ is a monospline with odd multiplicities $(r_i)_1^n$ knots $(x_i)_1^n$, $(x_1=0)$, and $M(t)$ has the oscillatory property. Then from Lemma 3 we know that $M(t)$ is the minimal element in $M_m\begin{pmatrix} x_1, x_2, \cdots, x_n\\ r_1, r_2+1, \cdots, r_n+1\end{pmatrix}$. Therefore the knot vector of $M(t)$ is a critical point of equation (2.11). By the uniqueness of critical point we know $M(t)$ is a solution of problem (A^*), i. e. the minimal function of extremal problem (A^*) is characterized by the oscillatory property. The theorem is proved.

§3. Proof of Theorem 3

Lemma 8.[6] *Suppose that* $1\leqslant r_i\leqslant m-2$, *and* r_i *are all odd,* $i=1, \cdots, n$. *Then for any given* $0=t_1<t_2<\cdots<t_{2N}<1$, *there exist* $K<+\infty$ *monosplines* $M_j(t)$ *in*

$M_m(r_1, \cdots, r_n)$ satisfying

$$M_j(t_i)=0,\ i=1, \cdots, 2N;\ j=1, \cdots, K. \tag{3.1}$$

Moreover, the number K is independent of the choice of points $(t_i)_1^{2N}$.

Lemma 9. [6] *Assume that r_i are odd, $1\leqslant r_i\leqslant m-2$, $(i=1, \cdots, n)$. Then there exist exact K equal-oscillating monosplines $\psi_j(t)$ in $M_m(r_1, \cdots, r_n)$, i. e., there exist $(z_i^{(j)})_{i=1}^{2N}$, $z_1^{(j)}<\cdots<z_{2N}^{(j)}<1+z_1^{(j)}$, such that*

$$\begin{cases}\psi_j(z_i^{(j)})=(-1)^i\sigma_j\|\psi_j\|_\infty,\ \sigma_j=1 \text{ or } -1,\\ \psi_j'(z_i^{(j)})=0,\\ \psi_j(0)=0.\end{cases} \tag{3.2}$$

Moreover the number K is defined in Lemma 8.

Proof of Theorem 3 When at least one of $(r_i)_1^n$ is equal to $m-1$ or m, Theorem 3 may be deduced to the non-periodic case.[4,12]. Therefore we may assume that $1\leqslant r_i\leqslant m-2$, $i=1, \cdots, n$. First we prove that $K\geqslant 2L$. By Lemma 8 we only need to prove that for the special choice of $(t_i^*)_1^{2N}$,

$$t_i^*=\frac{i-1}{2N},\ i=1, \cdots, 2N, \tag{3.3}$$

there exist at least $2L$ different monosplines $M_j(t)\in M_m(r_1, \cdots, r_n)$ satisfying

$$M_j(t_i^*)=0.\ i=1, \cdots, 2N;\ j=1, \cdots, 2L. \tag{3.4}$$

Case (i). Assume that $(r_1, \cdots, r_n)$ is non-circular, i. e. $k=n$ in Definition 1. At this time $2L=2N$. We know from Lemma 8 that there exists $M_1(t)\in M_m(r_1, \cdots, r_n)$ satisfying (3.4). Let $M_j(t)=M_1\left(t+\frac{j-1}{2N}\right)$, $j=2, \cdots, 2N$. Then $M_j(t)\in M_m(r_1, \cdots, r_n)$ and $M_j(t)$ satisfy (3.4), $j=2, \cdots, 2N$. We claim that these $2N$ monosplines $M_j(t)$ are different from each other. Otherwise, suppose that there exist $M_i(t)$ and $M_j(t)$, such that

$$M_i(t)\equiv M_j(t),\ 1\leqslant i,\ j\leqslant 2N,\ i\neq j.$$

Let the multiplicities of knots of M_i and M_j in $[0, 1)$ in the increasing order be $(r_1, \cdots, r_n)$ and $(r_{q+1}, \cdots, r_n, r_1, \cdots, r_q)$, respectively, $(0\leqslant q<n)$. Since M_i, M_j are obtained by translation of M_1, $M_i=M_j$ if and only if the knots of M_i and M_j are coincident. Therefore $(r_1, \cdots, r_n)=(r_{q+1}, \cdots, r_n, r_1, \cdots, r_q)$, and $(r_1, \cdots, r_n)$ is not non-circular, a contradiction.

Case (ii). Assume that $(r_1, \cdots, r_n)$ is k-circular, $1\leqslant k<n$. Let $s=n/k$. Then s is an integer. From Case (i) we know that there exist $2L=\sum_{i=1}^{k}(r_i+1)$ different monosplines $\overline{M}_j(t)\in M_m(r, \cdots, r_k)$ satisfying

$$\overline{M}_j\left(\frac{i-1}{2L}\right)=0,\ i,\ j=1, \cdots, 2L.$$

Let

$$M_j(t)=\overline{M}_j(st),\ j=1, \cdots, 2L.$$

Then it is obvious that $M_j\in M_m(r_1, \cdots, r_n)$, $j=1, \cdots, 2L$, and they are different.

from each other. Moreover,

$$M_j(t_i^*)=0;\ i=1,\ \cdots,\ 2N;\ j=1,\ \cdots,\ 2L.$$

In what follows we prove that there are at most $2L$ different monosplines in $M_m(r_1, \cdots, r_n)$ satisfying (3.4). Otherwise, we know from Lemma 9 that there are $K>2L$ different oscillatory monosplines in $M_m(r_1, \cdots, r_n)$ satisfying (3.2). On the other hand, Theorem 1 tells us that there exists a unique oscillatory monospilnes $\psi\in M_m(r_1, \cdots, r_n)$ with knots $0=x_1<\cdots<x_n<1$. Hence we can get $2N$ oscillatory monosplines $\psi_j(t)\in M_m(r_1, \cdots, r_n)$ satisfying (3.2). Similar to the above discussian of Case (i), it is not difficult to prove that there are at most $2L$ $\psi_j(t)$ which are different from each other. This contradiction proves our assertion.

Now the conclusion $K=2L$ is obtained and the theorem is proved.

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