Chin. Ann. of Math.
11B: 4 (1990), 438—444.

ON THE EXISTENCE OF NONTRIVIAL PERIODIC SOLUTIONS OF DIFFERENTIAL DIFFERENCE EQUATIONS**

WANG KE(王 克)*

Abstract

This paper cosiders the existence of nontrivial periodic solutions of the differential diffeıence equations

$$x'(t)=-f(x(t-1)),$$
$$x'(t)=-(f(x(t-1)+f(x(t-2))),$$

and

$$\begin{cases}x'(t)=f(x(t),\ y(t),\ x(t-1),\ y(t-1)),\\ y'(t)=g(x(t),\ y(t),\ x(t-1),\ y(t-1)).\end{cases}$$

Some new existence criteria are obtained.

In the recent years, the existence of non-trivial peroibio solutions of differential difference equations has attracted much attention of mathematicians[1-6]. Fixed point theorems are the principal tool to conclude the existence of such solutions. Using the technique of [4, 6], in this paper we obtain some new results withoutany fixed point theorems.

We consider the differential difference equation

$$x'(t)=-f(x(t),\ x(t-1)). \tag{1}$$

We suppose

1°. $f:R^2\to R$ is continuous, $xf(y,\ x)>0$, for $x\neq 0$, $y\in R$,

2°. $f(-y,\ x)=f(y,\ x)$, $f(y,\ -x)=-f(y,\ x)$,

3°. $|f(y,\ x)|\leqslant r(|x|)$, where $r(s)\geqslant 0$ is continuous in s with $r(0)=0$ and $r(s)>0$, for $s>0$,

4°. $\int_0^\infty f(y,\ x)dx=+\infty$, for any fixed $y\in R$,

5°. there is a constant $M>0$ such that

$$|Mf(y_1,\ x)|\geqslant|f(y_2,\ x)|,\ \text{for } y_1\geqslant y_2\geqslant 0 \text{ or } 0\geqslant y_2\geqslant y_1.$$

We construct the coupled system of ordinary differential equation

$$\begin{cases}x'=-f(x,\ y),\\ y'=f(y,\ x).\end{cases} \tag{2}$$

Manuscript received November 3, 1987.

* Department of Mathematics, Northeast Normal University, Changchun, Jilin, China.

**Supported by the National Science Foundation of China.

Lemma 1. *All solutions of* (2) *are periodic.*

Proof If $(x_0, y_0)\neq(0, 0)$, say, $x_0\neq 0$, and $f(x_0, y_0)=f(y_0, x_0)=0$, then $x_0 f(y_0, x_0)=0$; this is a contradiction, so the point (0, 0) is the unique singular point of (2). From the symmetry of the field of directions defined by equation (2), it suffices to prove that there is no unbounded trajectory of (2).

If the positive half-trajectory $(x(t), y(t))$ of (2) passing through the point $(a, 0)$, $a>0$, is unbounded on the first quadrant, it must have a vertical asymptote $x=b$, $0\leqslant b<a$. That is, $x(t)\to b$, as $t\to+\infty$. Since $x'(t)=-f(x(t), y(t))$,

$$\begin{aligned}|x(t)-x(t_0)| &= \left|-\int_{t_0}^{t} f(x(t), y(t))dt\right| = \int_{t_0}^{t} f(x(t), y(t))dt \\ &\geqslant \frac{1}{M}\int_{t_0}^{t} f(b, y(t))dt = \frac{1}{M}\int_{y_0}^{y} \frac{f(b, y)}{f(y, x(y))}dy \\ &\geqslant \frac{1}{M}\int_{y_0}^{y} \frac{f(b, y)}{r(x(y))}dy\end{aligned}$$

where $y_0=y(t_0)$, $y=y(t)$, $t\geqslant t_0$.

There is a $\delta>0$ such that $|r(s)-r(b)|<r(b)/2$, for $|s-b|<\delta$. Taking $t_0>0$ so large that $x(t)-b<\delta$ for $t\geqslant t_0$, we have

$$|x(t)-x(t_0)|\geqslant\frac{1}{M}\int_{y_0}^{y}\frac{f(b, y)}{r(x(y))}dy\geqslant\frac{2}{3M}\int_{y_0}^{y} f(b, y)dy\to+\infty, \text{ as } t\to+\infty,$$

and this is a contradiction.

Ifthe negative half-trajectory $(x(t), y(t))$ of (2) passing through the point $(0, a)$, $a>0$, is unbounded on the first quadrant, it must have a horizontal asymptote $y=b$, $0\leqslant b<a$. That is, $y(t)\to b$, as $t\to-\infty$. We have

$$\begin{aligned}|y(t)-y(t_0)| &= \int_{t}^{t_0} f(y(t), x(t))dt \geqslant \frac{1}{M}\int_{t}^{t_0} f(b, x(t))dt \\ &= \frac{1}{M}\int_{t}^{t_0}\frac{f(b, x(t))}{-f(x(t), y(t))}(-f(x(t), y(t))dt \\ &= -\frac{1}{M}\int_{x}^{x_0}\frac{f(b, x)}{f(x, y(x))}dx \\ &\geqslant \frac{1}{M}\int_{x_0}^{x}\frac{f(b, x)}{r(y(x))}dx, \text{ where } x_0=x(t_0),\ x=x(t).\end{aligned}$$

The remainder of the argument process as above.

The other case can be proved by the same argument.

Lemma 2. *Suppose L is a simple closed curve on R^2, and the origin is in the interior of the region G enclosed by L. Let T: $R^2\to R^2$ be a rotation transformation, and the center of rotation be the origin. Then $TL\cap L\neq\varnothing$.*

Proof The region enclosed by TL is TG. If $TL\cap L=\emptyset$, since $(0, 0)\in G\cap TG$, we have $L\subset TG$ provided any point of L belongs to TG. Thus, the area of TG is more than the one of G, and this is a contradiction for the areas of them are equal. If $TL\subset G$, the proof is the same as above.

Lemma 3. *If $(x(t), y(t))$ is a $4w$-periodic solution of (2), $w>0$, then $y(t)=x(t-w)$.*

Proof Since $(-x(t))'=-x'(t)=f(x(t), y(t))=-f(-x(t), -y(t))$ and $(-y(t))'=-y'(t)=-f(y(t),x(t))=f(-y(t), -x(t))$, we see that $(-x(t), -y(t))$ is also a solution of (2). From Lemma 2, we assert that the two solutions $(x(t), y(t))$ and $(-x(t), -y(t))$ have the same trajectory, so there exists $\lambda>0$, $\lambda\in(0, 4w)$ such that $x(t-\lambda)=-x(t)$, $x(t-2\lambda)=-x(t-\lambda)=x(t)$. Hence, $\lambda=2w$, that is, $x(t-2w)=-x(t)$. Since $y'(t)=f(y(t), x(t))=-f(-y(t), -x(t))$ and $(-x(t))'=-x'(t)=f(x(t), y(t))=f(-x(t), y(t))$, we see that $(y(t), -x(t))$ is a solution of (2). By the same argument, the solution $(y(t), -x(t))$ has the same trajectory as $(x(t), y(t))$ and $(-x(t), -y(t))$ have, so there is a $\sigma>0$, $\sigma\in(0, 4w)$ such that $x(t-\sigma)=y(t)$, $x(t-2\sigma)=y(t-\sigma)=-x(t)$. Therefore, $2\sigma=4nw+2w$, and $\sigma=w$ or $\sigma=3w$. By examining the order of maxima and minime as guaranteed by (2), we have $\sigma=w$. The Lemma is proved.

Let $X(t, \lambda)=(x(t, \lambda), y(t, \lambda))$, $\lambda\geqslant 0$, denote the trajectory of (2) passing through the point (λ, λ). Suppose the period of $X(t, \lambda)$ is T_λ.

Lemma 4. *If*

$$\alpha=\lim_{x\to 0}\frac{f(y, x)}{x},\quad \beta=\lim_{x\to\infty}\frac{f(y, x)}{x},$$

then we have

$$\begin{aligned}
&T_\lambda\to 2\pi/\alpha,\ \lambda\to 0,\ \text{if } \alpha\neq 0,\\
&T_\lambda\to\infty,\ \lambda\to 0,\ \text{if } \alpha=0,\\
&T_\lambda\to 2\pi/\beta,\ \lambda\to\infty,\ \text{if } \beta\neq 0,\\
&T_\lambda\to\infty,\ \lambda\to\infty,\ \text{if } \beta=0.
\end{aligned}$$

Proof Define

$$\theta_\lambda(t)=\operatorname{arc\,tg}\left(-\frac{x(t, \lambda)}{y(t, \lambda)}\right).$$

We have

$$\begin{aligned}
\frac{d}{dt}\theta_\lambda(t)&=\frac{x(t, \lambda)f(y(t, \lambda), x(t, \lambda))+y(t, \lambda)f(x(t, \lambda), y(t, \lambda))}{x^2(t, \lambda)+y^2(t, \lambda)}\\
&\overset{\text{def.}}{=\!=} R(x(t, \lambda), y(t, \lambda)).
\end{aligned}$$

It follows that

$$\begin{aligned}
2\pi&=\int_0^{T_\lambda}\theta'(t)dt=\int_0^{T_\lambda}R(x(t, \lambda), y(t, \lambda))dt\\
&=\int_0^{T_\lambda}\frac{x^2(t, \lambda)\dfrac{f(y(t, \lambda), x(t, \lambda))}{x(t, \lambda)}+y^2(t, \lambda)\dfrac{f(x(t, \lambda), y(t,\lambda))}{y(t, \lambda)}}{x^2(t, \lambda)+y^2(t, \lambda)}dt\\
&\overset{\text{def.}}{=\!=}\int_0^{T_\lambda}J_\lambda(t)dt. \qquad (3)
\end{aligned}$$

If $0<\alpha<+\infty$, we have

$$\int_0^{T_\lambda} J_\lambda(t)\,dt=\left(\int_0^{2\pi/\alpha}+\int_{2\pi/\alpha}^{T_\lambda}\right)J_\lambda(t)\,dt.$$

It is easy to see that

$$\lim_{\lambda\to 0} J_\lambda(t)=\alpha, \tag{4}$$

and

$$\lim_{\lambda\to 0}\int_{2\pi/\alpha}^{T_\lambda} J_\lambda(t)\,dt=0.$$

Therefore, for any given $\varepsilon>0$, there is a $\delta>0$ such that

$$\left|\int_{2\pi/\alpha}^{T_\lambda} J_\lambda(t)\,dt\right|<\varepsilon,\quad |J_\lambda(t)-\alpha|<\varepsilon,\ \text{for } \lambda<\delta,$$

and for $\varepsilon<\alpha/2$, there is a $\delta>0$ such that

$$\alpha|T_\lambda-2\pi/\alpha|=\left|\int_{2\pi/\alpha}^{T_\lambda}\alpha\,dt\right|$$
$$=\left|\int_{2\pi/\alpha}^{T_\lambda}(\alpha-J_\lambda(t))\,dt+\int_{2\pi/\alpha}^{T_\lambda}J_\lambda(t)\,dt\right|\leqslant|T_\lambda-2\pi/\alpha|+\varepsilon,$$

and

$$(\alpha-\varepsilon)\left|T_\lambda-\frac{2\pi}{\alpha}\right|<\varepsilon,\quad \left|T_\lambda-\frac{2\pi}{\alpha}\right|<\frac{\varepsilon}{\alpha-\varepsilon}<\frac{2\varepsilon}{\alpha},$$

that is, $\lim\limits_{\lambda\to 0}T_\lambda=\dfrac{2\pi}{\alpha}$.

If $\alpha=0$, from (3) for any $\varepsilon>0$, there is a $\delta>0$ such that

$$\left|\int_0^{T_\lambda} J_\lambda(t)\,dt-2\pi\right|<\varepsilon,\ \text{for } \lambda<\delta.$$

Noticing (4), we see that for $\varepsilon<\pi$, there is a $\delta>0$ such that

$$2\pi-\varepsilon<\int_0^{T_\lambda} J_\lambda(t)\,dt<\varepsilon T_\lambda,\quad T_\lambda>\frac{2\pi-\varepsilon}{\varepsilon}>\frac{\pi}{\varepsilon},$$

that is, $\lim\limits_{\lambda\to 0}T_\lambda=+\infty$.

If $\alpha=+\infty$, from (4) for any given $\varepsilon>0$, there is a $\delta>0$ such that $J_\lambda(t)>\varepsilon^{-1}$ for $\lambda<\delta$. From (3), for $\varepsilon<1$ there is a $\delta>0$ such that

$$\frac{T_\lambda}{\varepsilon}\leqslant\int_0^{T_\lambda} J_\lambda(t)\,dt<2\pi+\varepsilon,\ \text{for } \lambda<\delta,$$
$$T_\lambda<(2\pi+\varepsilon)\varepsilon<(2\pi+1)\varepsilon,$$

that is, $\lim\limits_{\lambda\to 0}T_\lambda=0$.

By the same argument, we can prove the limits when λ tends to $+\infty$.

Theorem 1. *Let k be a non-negative integer, if*

$$\alpha<(4k+1)\pi/2<\beta,\ \text{or}\ \beta<(4k+1)\pi/2<\alpha. \tag{5}$$

Then (1) *has a non-trivial periodic solution with period* $4/(4k+1)$.

Proof Acooding to the given conditions, we can suppose

$$\alpha<(4k+1)\pi/2<\beta$$

and hence

$$\frac{2\pi}{\beta}<\frac{4}{4k+1}<\frac{2\pi}{\alpha}.$$

From Lemma 4, it follows that (2) has a $4w$-periodic solution, where $w=(4k+1)^{-1}$. Now we have $y(t)=x(t-w)=x(t-(4k+1)w)=x(t-1)$, so $x'(t)=-f(x(t), x(t-1))$.

Corollary 1. *If $|\alpha-\beta|>2\pi$, (1) has at least one non-trivial periodic solution.*

Corollary 2. *If $|\alpha-\beta|>2n\pi$ (n is a natural number), (1) has at least n non-trivial periodic solutions.*

Proof It is easy to see that in (α, β) or (β, α) there exist at least n points: $2k_1\pi+\pi/2, \cdots, 2k_n\pi+\pi/2$, where $k_1, \cdots, k_n$ are different integers, that is, there exist at least n different integers that satiify (5). With the aid of Theorem 1, we can complete the proof.

Corollary 3. *If $\alpha=\infty$, $\beta\neq\infty$ (or $\beta=\infty$. $\alpha\neq\infty$), then (1) has infinite non-trivial periodic solutions.*

Remmak. When $k=1$ and $f(y, x)$ is independent of y, Theorem 1 becomes Theorem 1.1 in [4] by Kaplan and Yorke.

Example 1. The equation

$$x'(t)=-\frac{13x^3(t-1)}{(2+\cos x(t))+x^2(t-1)}$$

has at least two non-trivial periodic solutions.

Proof Here

$$f(y, x)=\frac{13x^3}{(2+\cos y)+x^2}.$$

It is easy to see that conditions 1°, 2° are satisfied. And

$$|f(y, x)|\leqslant\frac{13|x|^3}{|x|^2}=13|x|,$$

so condition 3° is satisfied.

Since

$$|3f(y, x)|=\frac{3}{2}\frac{13|x|^3}{\left(1+\frac{\cos y}{2}\right)+\frac{x^2}{2}}\geqslant\frac{3}{2}\frac{13|x|^3}{\frac{3}{2}+\frac{3}{2}x^2}=\frac{13|x|^3}{1+x^2}$$

and

$$\int_0^{+\infty}\frac{13|x|^3}{1+x^2}\,dx=+\infty,$$

condition 4° is satisfied. And

$$\frac{13|x|^3}{1+x^2}\geqslant\frac{13|x|^3}{(2+\cos y)+x^2}=|f(y, x)|,$$

so condition 5° is satisfied. It is obvious that $\alpha=0$ and $\beta=13$, and $2(2\pi)=12.57<13=|\alpha-\beta|$. From Corollary 2, the conclusion of this example can be proved.

Now, we consider the equation with several time lags

$$x'(t)=F(x(t), x(t-1), \cdots, x(t-n)), \tag{6}$$

where $F(x_1, x_2, \cdots, x_{n+1})$ is continuous on R^{n+1}.

Let $f(y, x)=-F(y, x, -y, -x, y, x, -y, -x, \cdots)$. Suppose that $f(y, x)$

satisfies conditions 1°—5°.

Theorem 2. *If*

1) $\lim\limits_{x\to 0}\dfrac{f(y,x)}{x}=\alpha,\quad \lim\limits_{x\to\infty}\dfrac{f(y,x)}{x}=\beta,$

2) $\min(\alpha,\ \beta)<\dfrac{\pi}{2}<\max(\alpha,\ \beta),$

then (6) *has a non-trivial periodic solution with period* 4.

Proof Construct the coupled system

$$\begin{cases} x'=-f(x,\ y), \\ y'=f(y,\ x). \end{cases} \tag{7}$$

From Lemmas 3 and 4, there is a non-trivial periodic solution $(x(t),\ y(t))$ of (7) with period 4, and from the proof of Lemma 3, we have $x(t-1)=y(t)$, $x(t-2)=-x(t)$, $x(t-3)=-y(t)$, $x(t-4)=x(t)$, $x(t-5)=x(t-1)=y(t)$, $x(t-6)=x(t-2)=-x(t)$, $\cdots$, Therefore,

$$\begin{aligned} x'(t)&=-f(x(t),\ y(t)) \\ &=F(x(t),\ y(t),\ -x(t),\ -y(t),\ x(t),\ y(t),\ \cdots) \\ &=F(x(t),\ x(t-1),\ x(t-2),\ x(t-3),\ x(t-4),\ x(t-5),\ \cdots). \end{aligned}$$

Thus, $x=x(t)$ is a non-trivial periodic solution of (6) with period 4.

Example 2. The equation

$$x'(t)=\frac{3x(t-1)x(t-3)x(t-5)}{(2+\cos(x(t)x(t-2)x(t-4)))+x^2(t-1)}$$

has a non-trivial periodic solution.

Proof Here

$$f(y,\ x)=\frac{3x^3}{(2+\cos y^3)+x^2}.$$

Similar to the proof of example 1, it is easy to prove that $f(y,x)$ satisfies conditions 1°—5°, and $\alpha=0$, $\beta=3$. From Theorem 2, this example is true.

Corollary. *Suppose that $f(x)$ is continuous odd and satisfies $xf(x)>0$ for $x\neq 0$. Suppose that*

$$\alpha=\lim_{x\to 0} f(x)/x,\quad \beta=\lim_{x\to\infty} f(x)/x$$

exist (allowing either to be 0 *or* ∞). *Suppose*

$$F(x)=\int_0^x f(s)\,ds\to+\infty \text{ as } x\to\infty.$$

Suppose that

$$\min(\alpha,\ \beta)<\pi/2<\max(\alpha,\ \beta).$$

Then both the equations

$$x'(t)=-(f(x(t))+f(x(t-1))+f(x(t-2))+\cdots+f(x(t-(4n-1)))) \tag{8}$$

and

$$x'(t)=-(f(x(t-1))+f(x(t-2))+f(x(t-3))+\cdots+f(x(t-(4n+1)))) \tag{9}$$

have non-trivial periodic solutions.

Proof For equations (8) and (9), we have $f(y, x)=f(x)$, so this corollary is immediately from Theorem 2.

By our method, we can consider not only the retarded type equations, but also the advanced type equations and mixed type equations. For example, we can consider the equation

$$x'(t)=F(x(t), x(t-1), \cdots, x(t-n), x(t+1), \cdots, x(t+m)), \tag{10}$$

and give some theorems which are like the ones we obtained above, but here we will omit them. It is very interesting that even thouth we do not know exactly what is the solution of initial value problem of (10), we can still find its non-trivial periodic solutions.

References

[1] Grafton, R. B., A periodicity theorem for autonomous functional differential equations, *J. Differential Equations*, **6** (1969), 87—109.

[2] Nussbaum, R. D., Periodic solutions of some nonlinear autonomous functional differential equations. *J. Differeotial Equations*, **14** (1973), 360—394.

[3] Grafton, R. B., Periodic solutions of certain Lienard equations with delay, *J. Differential Equations*, **11** (1972), 519—527.

[4] Kaplan, J. L. & Yorke, J. A., Ordinary differential equations which yield periodic solutions of differential delay Equations, *J. Math. Anal. Appl.*, **48** (1974), 317—324.

[5] Wen Lizhi, *Science Bulletin*, **31**: 22 (1986), 1756—1757.

[6] Wen Lizhi, Xia Huaxing, *Scientia Sinica, Series A*, **9** (1987), 909—916.