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PERIODIC SOLUTIONS AND ALMOST PERIODIC SOLUTIONS OF THE SCALAR ORDINARY DIFFERENTIAL EQUATION

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Abstract

This paper develops a method which enables us to study the number, existence and stability of periodic solutions and almost periodic solutions of the scalar ordinary differential equation. Some applications of the method are also given.

§1. Introduction

In this paper we study the non-autonomous differential equation

$$dx/dt=f(t, x), \tag{1.1}$$

where $f(t, x)\in C^1(R\times R; R)$, a scalar function, almost periodic in t uniformly for x in compact sets or periodic in t with period T. Additional hypotheses on f are given in Sections 3 and 4.

We now develop a method which is similar to the Liapunov function method in the stability and the Dulac function method in the qualitative theory of plane autonomous differential systems[1]. The method enables us to study the number, existence and stability of periodic solutions and almost periodic solutions of equation (1.1) if we can find a suitable function $B(t, x)$. Some applications of the method are also given.

§2. Preliminaries

For the scalar equation

$$dx/dt=f(t, x),\ f(t, x)\in C^1(R\times R; R), \tag{2.1}$$

and a given function $B(t, x)\in C^1(R\times \times R; R)$, we define a function

$$\Delta=f\partial B/\partial x+\partial B/\partial t-B\partial f/\partial x, \tag{2.2}$$

and sets

$$H=\{(t, x)\,|\,B(t, x)=0,\ (t, x)\in R^2\},$$
$$W=\{(t, x)\,|\,\Delta(t, x)=0,\ (t, x)\in R^2\}.$$

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It is obvious that equation (2.1) can be considered as an autonomous system in the (t, x)-plane:

$$dx/ds=f(t, x),\ dt/ds=1. \tag{2.3}$$

We state our main hypotheses

H1: $\Delta\geqslant 0$ in the (t, x)-plane;

H2: The sets H and W have no two-dimensional subsets, $H\cap W$ has no accumulation points in plane and H has no the isolated points.

Under the above hypotheses we prove Lemmas 1—3 and Theorem 1.

Lemma 1. *Let $I\subset H$ be a segment without the multiple points. Then function B has different signs on the two sides of I (in a neighborhood of I).*

Proof If B has the same sign on the two sides of I, then B takes maximal or minimal values at all points of I. So $\partial B/\partial x=\partial B/\partial t=0$ on I and (2.2) means $\Delta=0$ on I, i. e., $I\subset H\cap W$. It contradicts H2 and the proof is complete.

Lemma 2. *The set of the multiple points of the curves in H has no accumulation points in plane.*

Proof At the multiple points of the curves in H we have $B=0$, $\partial B/\partial x=\partial B/\partial t=0$ and so $\Delta=0$ ((2.2)). Hence the set of the multiple points is contained in $H\cap W$ and has no accumulation points in plane (H2).

Lemma 3. *The curves in H have no the multiple points.*

Proof If A is a multiple point of the curves in H, then there is a neighborhood U of A such that no other multiple points in U and the curves in H passing A divide U into several parts. Function B does not change its sign in each part. Lemma 1 means B has different signs in two adjacent parts.

On H we have ((2.2))

$$dB/ds|_{(2.3)}=f\partial B/\partial x+\partial B/\partial t=\Delta\geqslant 0.$$

Noticing that $H\cap W$ has no accumulation points and the lemma in § 3 of [1] (that lemma holds if we take a segment I, the sides (in a neighborhood) of I instead of simple closed curve, the interior and exterior in the lemma respectively). We know that in U the trajectories meeting $H\cap U-\{A\}$ of (2.3) go from the parts in which B is negative into the parts in which B is positive. It shows that A is a singular point of (2.3). The fact that (2.3) has no singular points completes the proof.

Theorem 1. *The curves in H are disjointed, and they divide the (t, x)-plane into several parts. B has different signs in two adjacent parts.*

Proof The conclusion follows immediately from Lemmas 1 and 3.

Corollary 1. *There are no closed curves in H.*

Proof The conclusion follows from Theorem 1 and the fact that (2.3) has no singular points.

Corollary 2. *Every trajectory of* (2.3) *meets a curve in H at most once.*

§ 3. Almost Periodic Solutions

Consider the scalar system

$$dx/dt = f(t, x) \tag{3.1}$$

where $f(t, x) \in C^1(R\times R;\ R)$, a scalar function, almost periodic in t uniformly for x in compact sets.

Definition 1. *For system* (3.1) *if there is a function $B(t, x) \in C^1(R\times R;\ R)$ whichis almost periodic in t uniformly for x in compact sets and satisfies the following conditions:*

(1) *hypotheses H1 and H2 hold,*

(2) *the curves in H defined by $B(t, x)=0$ are almost periodic in t,*

then system (3.1) *is called a P-system with function B.*

Definition 2. *Let* (3.1) *be a P-system with function B. A solution $x(t)$ of* (3.1) *defined on $(-\infty, +\infty)$ is far from zero of B if there exists a constant $\mu>0$ such that*

$$|B(t, x(t))| \geqslant \mu > 0 \text{ for } t \in (-\infty, +\infty).$$

Theorem 2. *Let* (3.1) *be a P-system with function B and satisfy the following conditions:*

(1) *H has n almost periodic curves $x=\psi_i(t)$, $i=1, 2, \cdots, n$;*

(2) *all bounded solutions of* (3.1) *defined on $(-\infty, +\infty)$ which can not meet H are far from zero of B;*

(3) *Δ is almost periodic in t uniformly for x in compact sets, and there exist constants t_0 and $\lambda>0$ such that*

$$\Delta(t_0, x) \geqslant \lambda > 0 \text{ for } x \in (-\infty, +\infty).$$

Then (3.1) *has at most $n+1$ almost periodic solutions and the almost periodic solutions in the region where $B>0$ ($B<0$) are asymptotically stable (unstable).*

In the proof of Theorem 2 we need the following lemmas, which can be obtained directly from the definition of almost periodic function.

Lemma 4. *For almost periodic function $\phi(t)$, it is impossible that $\phi(c)=0$, $\phi(t)<0$ for $t<c$ and $\phi(t)>0$ for $t>c$.*

Lemma 5. *For almost periodic function $\phi(t)\geqslant 0$ ($\not\equiv 0$), it is impossible that $\lim\limits_{t\to+\infty}\phi(t)=0$.*

Proof of Theorem 2. We first prove that every bounded solution in the region where $B>0$ ($B<0$) is asymptotically stable (unstable).

Let $x=\phi(t)$ be a bounded solution of (3.1) defined on $(-\infty, +\infty)$ and lie in the region where $B>0$, i. e., $B(t, \phi(t))>0$ for $t\in(-\infty, +\infty)$. Then we have

((2.2))

$$\partial f/\partial x=-\Delta/B+(f\partial B/\partial x+\partial B/\partial t)/B \text{ for } x=\phi(t),\ t\in(-\infty,\ +\infty).$$

$$\int_0^t (\partial f/\partial x)(t,\ \phi(t))\,dt=-\int_0^t (\Delta/B)(t,\ \phi(t))dt+\ln B(t,\ \phi(t))-\ln B(0,\ \phi(0)). \tag{3.2}$$

The condition (2) of Theorem 2 and the boundedness of $x=\phi(t)$ implies that $\ln B(t,\ \phi(t))$ is bounded on $(-\infty,\ +\infty)$.

Since $B(t,\ \phi(t))$ is bounded on $(-\infty,\ +\infty)$, there exists an $M>0$ such that $0<B(t,\ \phi(t))\leqslant M$ on $(-\infty,\ +\infty)$. So we have

$$\int_0^t (\Delta/B)(t,\ \phi(t))dt\geqslant(1/M)\int_0^t \Delta(t,\ \phi(t))dt \text{ for } t\geqslant 0. \tag{3.3}$$

We now prove $\lim\limits_{t\to+\infty}\Delta(t,\ \phi(t))\neq 0$. Suppose that is not true, there is an $N>0$ such that

$$\Delta(t,\ \phi(t))<\lambda/4 \text{ for } t>N. \tag{3.4}$$

Since $\Delta(t,\ x)$ is almost periodic in t uniformly for x in compact sets, there exists a $\tau>N-t_0$ such that

$$|\Delta(t+\tau,\ \phi(t+\tau))-\Delta(t,\ \phi(t+\tau))|\leqslant\lambda/4 \text{ for } t\in(-\infty,\ +\infty),$$

especially

$$|\Delta(t_0+\tau,\ \phi(t_0+\tau))-\Delta(t_0,\ \phi(t_0+\tau))|\leqslant\lambda/4.$$

Noticing that $t_0+\tau>N$ and (3.4), we have

$$\Delta(t_0,\ \phi(t_0+\tau))\leqslant\lambda/2.$$

It contradicts the condition (3) of Theorem 2. Hence

$$\lim_{t\to+\infty}\Delta(t,\ \phi(t))\neq 0$$

and $\int_0^{+\infty}\Delta(t,\ \phi(t))dt=+\infty$ ($\Delta(t,\ x)$ is uniformly continuous on R for x in compact sets.) From (3.3), (3.2) and the boundedness of $\ln B(t,\ \phi(t))$, we conclude that

$$\int_0^{+\infty}\partial f/\partial x(t,\ \phi(t))dt=-\infty. \tag{3.5}$$

Let $y=x-\phi(t)$, equation (3.1) becomes

$$dy/dt=\partial f/\partial x(t,\ \phi(t))y+\text{h. o. t.} \tag{3.6}$$

(3.5) means that the trivial solution $y=0$ of (3.6) and hence the solution $x=\phi(t)$ of (3.1) are asymptotically stable.

Similarly, we can prove that every bounded solution in the region where $B<0$ is asymptotically unstable.

Next we prove that every almost periodic solution of (3.1) can not meet the curves in H.

If $x=\phi(t)$ is an almost periodic solution of (3.1) and meets a curve $x=\psi_{i_0}(t)$, $1\leqslant i_0\leqslant n$, at $(c,\ \phi(c))$ in the $(t,\ x)$-plane, then $(c,\ \phi(c))$ is the unique intersection point of the two curves (Corollary 2 of Theorem 1) and $F(t)=\pm(\psi_{i_0}(t)-\phi(t))$ are almost periodic. It contradicts Lemma 4. So every almost periodic solution of (3.1) can not meet the curves in H.

The curves $x=\psi_i(t)$, $i=1, 2, \cdots, n$, divide the (t, x)-plane into $(n+1)$ strips. We now show that in each strip (3.1) has at most one almost periodic solution.

If $x=\phi_1(t)$ and $x=\phi_2(t)$ are the almost periodic solutions of (3.1) in a strip D and $B>0$ in the interior D. Then the two curves do not meet each other in the (t, x)-plane. Without loss of generality, we suppose $\phi_1(t)>\phi_2(t)$. In the (t, x)-plane all the solutions through $\{0\}\times[\phi_2(0), \phi_1(0)]$ are bounded on $(-\infty, +\infty)$ and far from zero of B. Hence for these solutions, (3.5) holds, i. e., they are asymptotically stable, and so

$$\lim_{t\to+\infty}(\phi_2(t)-\phi_2(t))=0. \tag{3.7}$$

It contradicts Lemma 5. Therefore (3.1) has at most one almost periodic solution in the strip D. If $B<0$ in the interior of D the proof is similar. The proof of Theorem 2 is complete.

We now give some applications of Theorem 2. For simplicity let $AP(R)=\{f\,|\,f$ is a real almost periodic function on $(-\infty, +\infty)\}$.

Example 1 Consider the Riccati equation

$$dx/dt=x^2+g(t)x+f(t),\ g,\ f\in AP(R). \tag{3.8}$$

Let $B(t, x)=-x-g(t)$. So we have

$$\Delta(t, x)=(x+g(t))^2-(f(t)+\dot{g}(t)),\ \text{where } \dot{g}=dg/dt.$$

If there is a constant α such that $f+\dot{g}\leqslant\alpha<0$, then (3.8) is a P-system and H has unique curve $x=-g$. In addition, if $\dot{g}\in AP(R)$, then $\Delta(t, x)$ is almost periodic in t uniformly for x in compact sets and the condition (3) of Theorem 2 is satisfied. We now have to check the condition (2) of Theorem 2.

We choose $\varepsilon_0>0$ so small that

$$\mp\varepsilon g+\varepsilon^2+f+\dot{g}<\alpha/2,\ 0\leqslant\varepsilon\leqslant\varepsilon_0.$$

In the set $\{(t, x)\,|\,-g(t)+\varepsilon_0\leqslant x,\ t\in(-\infty, +\infty)\}$ (resp. $\{t, x)\,|\,-g(t)-\varepsilon_0\geqslant x$, $t\in(-\infty, +\infty)\}$) the function $B(t, x)=-(x+g(t))$ takes its maximal value $-\varepsilon_0$ (resp. minimal value ε_0) at $x=-g(t)+\varepsilon_0$ (resp. $x=-g(t)-\varepsilon_0$).

Noticing that on the curves $x=-g(t)\pm\varepsilon$, $0\leqslant\varepsilon\leqslant\varepsilon_0$,

$$dB/dt\,|_{(3.8)}=\pm\varepsilon g-\varepsilon^2-f-\dot{g}>-\alpha/2,$$

we know that every solution of (3.8) which meet

$$x=-g(t)\pm\varepsilon_0$$

must meet $x=-g(t)$. Hence all the solutions which can not meet H are in the region where $|B|\geqslant\varepsilon_0>0$ and we come to the following conclusion:

If $f+\dot{g}\leqslant\alpha<0$ and $\dot{g}\in AP(R)$, then (3.8) has at most two almost periodic solutions.

If we take $B(t, x)=x$, then $\Delta=-x^2+f(t)$. Similarly, we can prove that if there is a constant α such that $f(t)\leqslant\alpha<0$, then (3.8) has at most two almost periodic solutions.

Example 2 Consider

$$dx/dt=x+(x^2-\sin^2 t-3)c(t),\ c(t)\in AP(R). \tag{3.9}$$

We take $B(t, x)=x^2-\sin^2 t-3$, then

$$\Delta(t, x)=x^2+\sin^2 t+3-\sin 2t\geqslant 2. \tag{3.10}$$

H has two curves $x=\pm(\sin^2 t+3)^{1/2}$. We need to check the condition (2) of Theorem 2 only. But we can do it just as Example 1. So (3.9) has at most three almost periodic solutions.

Example 3 Consider

$$dx/dt=a(t)x^3+b(t)x^2+c(t), \tag{3.11}$$

where $a(t)$, $b(t)$, $c(t)\in AP(R)$, $b(t)>0$ and $c(t)\geqslant\alpha>0$, α is a constant.

We take $B(t, x)=x^3$, then $\Delta=x^2(bx^2+3c)$.

Choose $\varepsilon_0>0$ so small that $\pm a\varepsilon^3+b\varepsilon^2+c\geqslant\alpha/2$, $0\leqslant\varepsilon\leqslant\varepsilon_0$. Then on $x=\pm\varepsilon$, $0\leqslant\varepsilon\leqslant\varepsilon_0$,

$$dB/dt|_{(3.11)}=3\varepsilon^2(\pm a\varepsilon^3+b\varepsilon^2+c)\geqslant\varepsilon^2\alpha 3/2.$$

Hence all the solutions which can not meet $x=0$ are in the region where $|B|\geqslant\varepsilon_0^3>0$, $\Delta\geqslant 3\alpha\varepsilon_0^2$, and (3.11) has at most one almost periodic solution in $\{(t, x)|x>\varepsilon_0,\ t\in(-\infty, +\infty)\}$ or $\{(t, x)|x<-\varepsilon_0,\ t\in(-\infty, +\infty)\}$ (Using Theorem 2 in those sets). On the other hand, $dx/dt|_{x=0}=c(t)\geqslant\alpha>0$ means (3.11) has no almost periodic solution which meets $x=0$ (Lemma 4). Therefore (3.11) has at most two almost periodic solutions.

Theorem 3. *If* (3.1) *satisfies the following conditions:*

(1) *$\partial f/\partial x$ is almost periodic in t uniformly for x in compact sets and $f\in C^2(R\times R; R)$,*

(2) *all the curves defined by $\partial f/\partial x(t, x)=0$ are almost periodic in t and $x=\psi_i(t)$, $i=1, 2, \cdots, n$, are such curves,*

(3) *$f(t, \psi_i(t))\partial^2 f/\partial x^2(t, \psi_i(t))+\partial^2 f/\partial x\partial t(t, \psi_i(t))\geqslant 0$ (resp. $\leqslant 0$), $i=1, 2, \cdots, n$, and the equality holds in a set which has no accumulation points in R,*

then (3.1) *has at most $n+1$ almost periodic solutions.*

Proof We now take $B=\partial f/\partial x$ (resp. $B=-\partial f/\partial x$). The curves $x=\psi_i(t)$, $i=1, 2, \cdots, n$, divide the (t, x)-plane into $n+1$ strips. The condition (3) of Theorem 3 implies $\Delta\geqslant 0$ on $x=\psi_i$, $i=1, 2, \cdots, n$, i. e., on $x=\psi_i(t)$, we have

$$dB/dt|_{(3.1)}=\Delta\geqslant 0.$$

Hence all the conclusions in Section 2 are true. As in the proof of Theorem 2 we assert that every almost periodic solution of (3.1) can not meet the curves $x=\psi_i(t)$, $i=1, 2, \cdots, n$.

Suppose $x=\phi_1(t)$ and $x=\phi_2(t)$ are the almost periodic solutions of (3.1) in a strip D and $\partial f/\partial x>0$ in the interior of D. We may assume $\phi_1(t)>\phi_2(t)$. So we have

$$G(t)=f(t,\ \phi_1(t))-f(t,\ \phi_2(t))>0$$

and

$$\lim_{T\to+\infty}(1/T)\int_0^T G(t)dt>0\ (G(t)\in AP(R)\text{ and Theorem 3.8 in [2]}),$$

i. e.,

$$\int_0^{+\infty}G(t)dt=+\infty.$$

On the other hand, we have

$$\phi_1(t)-\phi_2(t)=\phi_1(0)-\phi_2(0)+\int_0^t G(t)dt.$$

It contradicts the boundedness of $\phi_1(t)-\phi_2(t)$. Hence (3.1) has at most one almost periodic solution in the strip D. If $\partial f/\partial x<0$ in the interior of D, the proof is similar. Therefore (3.1) has at most $n+1$ almost periodic solutions. The proof of Theorem 3 is complete.

If we consider

$$dx/dt=g(x)+p(t),\ g\in C^2(R;\ R)\text{ and }p(t)\in AP(R),\tag{3.12}$$

then the conclusion of Theorem 3 can be strengthened.

Corollary. *If $dg/dx=0$ has n solutions $x=a_i$, $i=1, 2, \cdots, n$, and $(g(a_i)+p(t))d^2g/dx^2(a_i)\geqslant 0$ (resp. $\leqslant 0$), $i=1, 2, \cdots, n$, where the equality only holds on a set which has no accumulation points on R, then (3.12) has r almost periodic solutions, where $n-1\leqslant r\leqslant n+1$ for $n\geqslant 1$ and $0\leqslant r\leqslant 1$ for $n=0$.*

Proof From Theorem 3 we know that (3.12) has at most $(n+1)$ almost periodic solutions. We now show that (3.12) has at least $n-1$ almost periodic solutions for $n\geqslant 1$.

The n lines $x=a_i$, $i=1, 2, \cdots, n$, divide the (t, x)-plane into $n+1$ strips and $n-1$ strips $a_i\leqslant x\leqslant a_{i+1}$, $i=1, 2, \cdots, n-1$, (here we assume $a_1<a_2<\cdots<a_n$.) are bounded for x. On the line $x=a_i$,

$$d(dg/dx)/dt\,|_{(3.12)}=(g(a_i)+p(t))d^2g/dx^2(a_i)\geqslant 0\ (\text{resp. }\leqslant 0).$$

It means that any solution of (3.12) meeting the line $x=a_i$ or $x=a_{i+1}$ goes into the strip $a_i<x<a_{i+1}$ from the outside of the strip as t increases (resp. decreases) if $dg/dx>0$ in the strip. Noticing that $g(x)$ is strictly monotonous in all the strips, we assert that (3.12) has at least one almost periodic solution in each strip $a_i<x<a_{i+1}$, $i=1, 2, \cdots, n-1$. (Theorem 12.9 in [2]). It completes the proof of Corollary.

§ 4. Periodic Solutions

Consider the scalar equation

$$dx/dt = f(t, x), \tag{4.1}$$

where $f(t, x) \in C^1(R \times R: R)$, a scalar function, periodic in t with period T.

Equivalently, we have the system

$$dx/ds = f(t, x), \; dt/ds = 1, \; (t, x) \in S^1 \times R, \tag{4.2}$$

where $S^1 \times R$ is a cylinder obtained from $[0, T] \times R$ by identifying points $(0, x)$ and (T, x), $x \in R$, or more conveniently the cylinder is viewed as the entire (t, x)-plane in which the points (t_1, x_1) and (t_2, x_2) are considered identical if and only if $(t_1 - t_2)/T$ is an integer and $x_1 = x_2$.

It is obvious that (4.1) has periodic solution $x = x(t)$ with period T if and only if the cylinder differential equations (4.2) has periodic solution $(s, x(s))$ with period T.

It is known that (4.1) has no nonconstant periodic solution with period other than T.

Definition 3. *For equation (4.1), if there is a function $B(t, x) \in C^1(R \times R; R)$, which is periodic in t with period T and satisfies the following conditions:*

(1) hypotheses H1 and H2 hold;

(2) the curves defined by $B(t, x) = 0$ are periodic in t with period T,

then equation (4.1) (or (4.2)) is called a P-system with function B.

Theorem 4. *If (4.1) is a P-system with function B and satis-fies the following conditions:*

(1) (4.1) has no periodic solutions in W;

(2) set H has n periodic curves $x = \psi_i(t)$, $i = 1, 2, \cdots, n$, then (4.1) has r periodic solutions, where $n-1 \leqslant r \leqslant n+1$ for $n \geqslant 1$ and $0 \leqslant r \leqslant 1$ for $n = 0$. Moreover, the periodic solution in the region where $B > 0$ (resp. $B < 0$) are stable (resp. unstable).

Proof From Theorem 1 and its Corollary 2, we know that the curves $x = \psi_i$ $i = 1, 2, \cdots, n$, are disjointed and the periodic solutions of (4.1) can not meet them. The curves $x = \psi_i(t)$, $i = 1, 2, \cdots, n$, divide the cylinder $S^1 \times R$ into n+1 regions, and the $n-1$ regions of them are $D_i = \{(t, x) \mid \psi_i(t) < x < \psi_{i+1}(t), \; t \in [0, T]\}$, $i = 1, 2, \cdots, n-1$. (here we assume $\psi_1 < \psi_2 < \cdots < \psi_n$).

On $x = \psi_i(t)$, $i = 1, 2, \cdots, n$,

$$dB/ds|_{(4.2)} = \Delta \geqslant 0.$$

It means that every trajectory of (4.2) meeting the curve $x = \psi_i(t)$ or $x = \psi_{i+1}(t)$, $t \in [0, T]$, goes into the region D_i as s increases if $B > 0$ in D_i (or decreases if $B < 0$ in D_i).

If $B > 0$ in D_i, then every trajectory $\phi(t; 0, \xi)$ of (4.2) starting at $(0, \xi) \in A_i = \{0\} \times [\psi_i(0), \psi_{i+1}(0)]$ must meet $B_i = \{T\} \times \{\psi_i(T), \psi_{i+1}(T)] = \{T\} \times [\psi_i(0), \psi_{i+1}(0)]$ at $(T, \phi(T; 0, \xi))$. Hence we can define a continuous mapping $P: A_i \to B_i$, $(0, \xi) \to (T, \phi(T; 0, \xi))$. Noticing that A_i and B_i are identical, we assert that there exists at least one point $(0, \xi_0) \in A_i$ such that $\xi_0 = \phi(T; 0, \xi_0)$ (Brouwer fixed point theorem), i. e., (4.2) has at least one periodic trajectory in D_i. If $B<0$ in D_i, then the proof is similar. Hence (4.2) has at least $n-1$ periodic trajectories and (4.1) has at least $n-1$ periodic solutions.

We now show that (4.2) has at most one periodic trajectory in each of the $(n+1)$ regions.

In each of the $(n+1)$ regions $B \neq 0$, hence we have ((2.2))

$$\partial f/\partial x = -\Delta/B + (f\partial B/\partial x + \partial B/\partial t)/B.$$

If $x=\phi(t)$ is a periodic trajectory of (4.2), then

$$\int_0^T \partial f/\partial x(t, \phi(t))dt = -\int_0^T (\Delta/B)(t, \phi(t))dt. \tag{4.3}$$

On the other hand, by means of Poincaré mapping we assert that if

$$\int_0^T \partial f/\partial x(t, \phi(t))dt < 0 \text{ (resp. } >0),$$

then $x=\phi(t)$ is stable (resp. unstable) ([1]). Hence (4.3) and the condition (1) of Theorem 4 imply that the periodic trajectories in the regions where $B>0$ (resp. $B<0$) are stable (resp. unstable).

Noticing that in the cylinder two adjacent periodic trajectories of (4.2) have different stability, we know that (4.2) has at most one periodic trajectory in each of the $n+1$ regions. The proof is complete.

In Examples 1–3, if we assume that $f(t, x)$ is periodic in t, then we can obtain the number, existence and stability of periodic solutions by Theorem 4.

Example 4. Consider

$$dx/dt = 1 - x^3 + x\sin t. \tag{4.4}$$

We take $B(t, x) = \exp(-\cos t)$, then $\Delta = 3x^2\exp(-\cos t)$. By Theorem 4, equation (4.4) has at most one stable periodic solution.

On the other hand

$$dx/dt|_{x=0} = 1 \text{ and } dx/dt|_{x=2} = 0;$$

hence every solution of (4.4) meeting the set

$$\{(t, x) | x=0, t \in [0, 2\pi]\} \text{ or } \{(t, x) | x=2, t \in [0, 2\pi]\}$$

goes into $\{(t, x) | 0 \leqslant x \leqslant 2, t \in [0, 2\pi]\}$ as t increases. As in the proof of Theorem 3 we know that (4.4) has one periodic solution. Therefore (4.4) has a unique stable periodic solution.

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