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AN ELASTOPLASTIC PROBLEM WITH FRICTIONS

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Abstract

This poper discusses on elastoplastic problem with frictions given on a part of the boundary. Assuming that friction follows simplified Coulom's law, the author gives an explicit relaxation of energy and proves that the solution exists.

§1. Introduction

We consider a friction problem in the geometrically linear elastoplasticity (Hencky's plasticity) theory. The general mathematical formulation of Coulomb's law was given in [7] while the general theory of Hencky's plasticity was given in [17]. We combine the two theories here to establish an appropriate variational formulation and prove the existence result by "relaxing" the problem reasonably.

It is now well known that the strain solutions of the Hencky's plasticity problems are not in the Sobolev spaces $W^{1,p}$ in the sense that their strain tensor components are bounded Hadon measures instead of L^p functions (cf. [17]). A consequence of this result is that the solutions we obtained by minimization of energy are, as a matter of fact, not the solutions of the original problem in the following sense: Firstly, the functions that minimize the potential of energy can no more be proved to satisfy the Euler-Lagrange equations without further regularity results being obtained. Secondly, if any boundary condition of Dirichlet type $u_{\Gamma_1}=u_0$ is imposed in the original problem, we can not expect it to be satisfied by the minimizer (for details of this, see [17] and the following sections) and this is justified to be physically reasonable. The first difficulty is ignored at the moment beaause of the physical nature of the problem. The second was mathematically explained in [17] by an argument of "relaxation of energy" which was motivated by the following philosophy: the boundary condition is NOT preserved but the difference between the expected boundary condition and the actual boundary displacement should minimize the "energy" in some sense. We will see the mathematical representation of this in the following. In the friction problems, the

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same thing happens: due to plasticity, we can not "hold" the material tightly on the boundary to satisfy the displacement boundary condition imposed, so neither should we expect that we can "hold" the material on the boundary to satisfy Comlomb's law. Therefore, the existence result should also be established on some kind of relaxation principal. It is not easy to guess by direct observations what the relaxation principal is.

However, it is well known from physics that plasticity can be understood to mean the layers of molecules of the material satisfy a kind of friction law similar to that of Coulomb. According to a personal communication between the author and N. Q. Son, the friction effects in plasticity can therefore be visualized physically as follows: on the boundary of contact, we stick another piece of material which obeys the same Hencky's plasticity law together with another internal firction law which can be expressed mathematically as the Coulomb's law between its layers of molecules. Considering the material formed by the two pieces as a whole, we can get a relaxation of energy and establish the existence result. This paper is, as a matter of fact, a mathematical realization of the above physical arguments.

It should be pointed out that the results of this article have an interesting connection with the recent paper by E. De Giorgi et al (cf. [4]) though the settings of the problems are different. This work was partially supported by a Chinese national post-doctorial research grant when I worked in Fudan University, Shanghai, PRC. I would like to express my indebtedness to my collegues in the Institute of Mathematics, Fudan University for their continuous encouragement and invaluable help.

§ 2. Mathematical Formulation of the Problem

To establish the mathematical formulation of Coulomb's law in linearized elasticity-plasticity problems, we follow formally the friction theory developed in [7] and [8] and the elasticity-plasticity theory developed in [17]: Find $u\in H^1(\Omega; R^3)$, $\sigma\in L^2(\Omega; E)$ such that

$$e(u)=A\sigma+\lambda \text{ in } \Omega, \tag{1}$$

$$\operatorname{div}\sigma+f(x)=0 \text{ in } \Omega, \tag{2}$$

$$\lambda: (\phi-\sigma)\leqslant 0 \text{ a. e. in } \Omega \text{ for all } \phi\in L^2(\Omega: E),$$
$$\text{such that } |\phi^D(x)|\leqslant k \text{ a. e.,} \tag{3}$$

$$u|_{\Gamma_1}=u_0, \tag{4}$$

$$\sigma n|_{\Gamma_2}=g, \tag{5}$$

$$(\sigma n)_n|_{\Gamma_3}=F_n, \tag{6}$$

$$|(\sigma n)_\tau|<l \Rightarrow u_\tau=0 \text{ on } \Gamma_3, \tag{7}$$

$$|(\sigma n)_\tau|=l \Rightarrow \exists q\geqslant 0 \text{ such that } u_\tau=-q(\sigma n)_\tau \text{ on } \Gamma_3, \tag{8}$$

where Ω is a bounded, open, C^2 subset of R^3; E is $R^{3\times 3}_{sym}$ and $E^D=\{\xi\in E:\ \xi_{11}+\xi_{22}+\xi_{33}=0\}$; e is the linearized deformation operator from $H^1(\Omega;\ R^3)$ into $L^2(\Omega;\ E)$ such that for any $u\in H^1(\Omega;\ R^3)$,

$$e_{ij}(u)=\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)/2 \text{ for } i,\ j=1,\ 2,\ 3;$$

A is a linear operator from E into E such that for any $\sigma\in E$,

$$A\sigma=b_1\sigma^D+b_2 \operatorname{tr}\sigma I$$

with $b_1,\ b_2>0$, $\operatorname{tr}\sigma=\sigma_{11}+\sigma_{22}+\sigma_{33}$, I=identity in E and $\sigma^D=\sigma-\frac{1}{3}\operatorname{tr}\sigma I$; $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ are C^1 open subsets of Γ, $\overline{\Gamma}_1\cup\overline{\Gamma}_2\cup\overline{\Gamma}_3=\Gamma$, $\Gamma_i\cap\Gamma_j$ is empty for all $i\neq j$; l and k are positive physical constants; u_0 is the trace of an H^1 function; $g\in L^\infty(\Gamma_2;\ R^3)$; F_n is in $W^{1/p,p/(p-1)}(\Gamma_3)$ for some $p\in(1,\ 3/2)$; $f\in L^\infty(\Omega;\ R^3)$.

If we let Γ_3 be empty, then the problem becomes an isotropic homogeneous elasto-plastic problem which has been studied in [2] and [17]. To simplify the arguments in the following, we suppose that Ω is connected. We will write simply $W^{s,p}(\Omega;\ R^3)$ as $W^{s,p}(\Omega)$ in the following when there is no confusion.

§ 3. Variational Formulation and Limit Analysis Problems

As the expressions (1)-(8) are very complicated, it is not easy to see how to prove the existence of a solution by using that formulation. Duvaut, Lions, and more succesfully, Temam, had proposed to solve the plasticity problem by using variational minimization method. We follow the same method and transform (1)-(8) into a pair of formally equivalent variational problems:

$$\mathscr{P}\text{: In}\ f\left\{\int_\Omega \psi(e(u))dx+\int_{\Gamma_3} l|u_\mathscr{T}|\,d\Gamma-\int_\Omega f(x)u(x)dx-\int_{\Gamma_2} u\cdot g\,d\Gamma-\int_{\Gamma_3} F_n(u\cdot n)d\Gamma\right\} \tag{9}$$

subject to $u\in A=\{u\in H^1(\Omega),\ u|_{\Gamma_1}=u_0\}$ and

$$\mathscr{P}^*\text{: Sup}\left\{-\int_\Omega A\sigma:\sigma\,dx+\int_{\Gamma_1}(\sigma\cdot n)u_0 d\Gamma\right\} \tag{10}$$

subject to $\sigma\in A^*=\{\sigma\in L^2(\Omega;\ E),\ |\sigma^D(x)|\leqslant k$ a. e., $\operatorname{div}\sigma(x)+f(x)=0,\ \sigma n|_{\Gamma_2}=g,\ (\sigma n)_n|_{\Gamma_3}=F_n,\ |(\sigma n)_\mathscr{T}|_{\Gamma_3}|\leqslant l.\}$ with

$$\psi(\xi)=\text{Sup}\{\xi:\eta-A\eta:\eta;\ \eta\in E,\ |\eta^D|\leqslant k\} \tag{11}$$

for all $\xi\in E$ and we have the following

Proposition 3. 1. *If u is a solution of $\mathscr{P}$, σ a solution of $\mathscr{P}^*$, then $(u,\ \sigma)$ satisfies the relations* (1)-(8) *and vice versa.*

The proof of this proposition follows from the results established in [8]. We just have to notice that in this proposition we have assumed that $u \in H^1$ in advance. It is in this sense we meant that $\mathscr{P}$ and $\mathscr{P}^*$ were formally equivalent to (1)-(8) at the beginning of this section.

Propositon 3. 2. *The function ψ defined in* (11) *satisfies the following properties:*

$$c_1(\operatorname{tr}\xi)^2+c_3(|\xi^D|-1)\leqslant\psi(\xi)\leqslant c_2(\operatorname{tr}\xi)^2+c_4(|\xi^D|+1), \tag{12}$$

$$\psi(\xi)\geqslant 0,\ \psi(0)=0, \tag{13}$$

with c_1, c_2, c_3, c_4 being positive constants (*cf.* [17]).

Proposition 3. 3. $\operatorname{Inf}\mathscr{P}=\sup\mathscr{P}^*$ (*cf.* [9]).

It follows from Proposition 3.2 that ψ has only linear growth at infinity with respect to $\xi \in E^D$. So in general, we do not have any result ensuring that $\operatorname{Inf}\mathscr{P} > -\infty$. Therefore, a condition which can guarantee that the functional is bounded below is desirable. In [17], one such condition called the limit analysis hypothesis was given via the following problem:

$$\mathscr{P}LA\text{: }\operatorname{Inf}\left\{\int_\Omega \psi_\infty(e^D(u))dx+\int_{\Gamma_3} l|u_{\mathscr{T}}|d\Gamma\right\} \tag{14}$$

subject to $u\in AL=\left\{u\in H^1(\Omega)\ \ u|_{\Gamma_1}=0,\ \int_\Omega f(x)u(x)+\int_{\Gamma_2} g\cdot u\,d\Gamma+\int_{\Gamma_3} F_n(u\cdot n)\,d\Gamma=1,\ \operatorname{div}u=0.\right\}$ with $\psi_\infty(\xi^D)=\lim\limits_{t\to\infty}\psi(t\xi^D)/t$ for any $\xi\in E$. Its dual problem is

$$\mathscr{P}LA^*\text{: }\operatorname{Sup}\{\lambda\} \tag{15}$$

subject to $\sigma\in AL^*=\{\sigma\in L^2(\Omega;\ E);\ |\sigma^D(x)|\leqslant k$ a. e.; $\operatorname{div}\sigma+\lambda f(x)=0;\ \sigma n|_{\Gamma_2}=\lambda g;\ (\sigma n)_n|_{\Gamma_3}=\lambda F_n;\ |(\sigma n)_{\mathscr{T}}|\leqslant l$ on $\Gamma_3\}$ and we have, similar to Proposition 3.3, that

$$\operatorname{Inf}\mathscr{P}LA=\operatorname{Sup}\mathscr{P}LA^*. \tag{16}$$

The limit analysis will be given in detail later for technical reasons.

§ 4. Function Spaces and Limit Analysis Hypothe-sis

As we have noticed in the previous section, the function of potential of energy has only linear growth at infinity with respect to some part of its variables. We can anticipate from this fact that any estimate concerning the minimizing sequence $\{u_m\}$ obtained from the energy can not be expected to be better than

$$|e^D(u_m)|_{L^1(\Omega;\ E)}+|\operatorname{div}u_m|^2_{L^2(\Omega)}\leqslant\text{const.}$$

in general. This kind of estimate does not guarantee, in general, the boundedness and neither the weak convergence of the minimizing sequence in any Sobolev space of type $W^{1,p}$. Therefore, we have to introduce new function spaces. Most results of this section come from [16].

Definitions 4. 1. *Let $M(\Omega)$ denote the space of bounded Radon measures and we define*

$$BD(\Omega)=\{u\in L^1(\Omega),\ e(u)\in M(\Omega;\ E)\},$$

$$U(\Omega)=\{u\in BD(\Omega),\ \operatorname{div}u\in L^2(\Omega)\}$$

with the following norms

$$|\mu|_{M(\Omega)}=\operatorname{Sup}\left\{\int_\Omega\phi\mu;\phi\in C_0(\Omega),\ |\phi(x)|\leqslant 1\right\},$$

$$|u|_{BD(\Omega)}=|u|_{L^1}+|e(u)|_{M(\Omega;E)},$$

$$|u|_{U(\Omega)}=|u|_{BD(\Omega)}+|\operatorname{div}u|_{L^2(\Omega)}.$$

Definitions 4. 2. *The "weak" topology of*

i) $BD(\Omega)$ *is defined as:* $u_n\rightharpoonup u$ *in* $BD(\Omega)$ *if and only if* $u_n\to u$ *in* $L^1(\Omega)$ *and* $e(u_n)\to e(u)$ *in* $M(\Omega;E)$ *weak-star.*

ii) $U(\Omega)$ *is defined as:* $u_n\rightharpoonup u$ *in* $U(\Omega)$ *if and only if* $u_n\rightharpoonup u$ *in* $BD(\Omega)$ *and* $\operatorname{div}u_n\rightharpoonup\operatorname{div}u$ *in* $L^2(\Omega)$ *(in usual* L^p *spaces, "$\rightharpoonup$" denotes weak convergence).*

From the above definitions, we can draw the following consequences: $U(\Omega)$ is embedded continuously in $BD(\Omega)$; the embedding $BD(\Omega)\hookrightarrow L^p(\Omega)$ is continuous for any $p\in[1,3/2]$ and is compact for any $p\in[1,3/2)$; from any bounded sequence of $BD(\Omega)$ or $U(\Omega)$, we can extract a subsequence which converges weakly in $BD(\Omega)$ or $U(\Omega)$ respectivly.

Definitions 4.3. *For any finite number of convex functions* $\{\psi_i\}_{i=1}^m$ *which satisfy the estimates of type* (12) *and* (13), *we define an intermediate topology on* $U(\Omega)$ *given by the following distance*

$$d|u,v)=|u-v|_{L^1(\Omega)}+\left|\int_\Omega(|e^D(u)|-|e^D(v)|)\right|+\left[\int_\Omega(\operatorname{div}(u-v))^2dx\right]^{1/2}$$
$$+\sum_{i=1}^m\left|\int_\Omega[\psi_i(e(u))-\psi_i(e(v))]\right|. \tag{17}$$

We have also the following results: For any $u\in BD(\Omega)$, the trace operator

$$u\in BD(\Omega)\mapsto u|_\Gamma\in L^1(\partial\Omega) \tag{18}$$

is well defined and is continuous with respect to the topology defined by (17). $H^1(\Omega)$ is dense in $U(\Omega)$ with respect to the topology defined by (17). We just point out here that the definiti on of $\psi_i(e(u))$ in (17) as a bounded measure for any $u\in U(\Omega)$ is given in [6].

Now, as all the proper function spaces have been defined, we give explicitly the limit analysis hypothesis and make clear its mathematical significance.

Hypothesis 4. 4. $\operatorname{Inf}\mathscr{P}LA>1$.

Proposition 4. 5. *Under Hypothesis* 4.4, *the problem* $\mathscr{P}^*$ *admits a unique solution and any minimizing sequence of* $\mathscr{P}$ *will stay in a bounded set of* $U(\Omega)$.

Proof The fact that $\mathscr{P}^*$ admits a unique solution under Hypothesis 4.4 is easy to see by applying the theory of convex analysis on L^2 and by noticing that the underlying function set is nonempty (see also [9]). As for the second conclusion, when meas $(\Gamma_1)>0$, it was provedin [17]. We do not discuss the case meas $(\Gamma_1)=0$ with Γ_1 nonempty for there is no proper trace theorem in this case.

What left therefore is the case where Γ_1 is empty. By Hypothesis 4.4, we have

$$\int_{\Gamma_3} l|u_{\mathscr{T}}|d\Gamma \geqslant \operatorname{Inf} \mathscr{P}LA \cdot \left| \int_{\Omega} f(x)u(x)dx + \int_{\Gamma_2} g\cdot u\, d\Gamma + \int_{\Gamma_3} F_n(u\cdot n)d\Gamma \right| \quad (19)$$

For all $u \in \mathscr{R} = \{a \wedge x + b:\ a,\ b \in R^*\}$. As $\mathscr{R}$ is a finite dimensional space, it follows from [8] that

$$\int_{\Gamma_3} l|u_{\mathscr{T}}| - \left| \int_{\Omega} f(x)u(x)dx + \int_{\Gamma_2} g\cdot u\, d\Gamma + \int_{\Gamma_3} F_n(u\cdot n)d\Gamma \right| \geqslant c|u|_{L^1(\Omega)} \quad (20)$$

for all $u \in \mathscr{R}$ and c is some positive constant. It is derived in [17] that if $\{u_m\}$ is a minimizing sequence, then Hypothesis 4.4 implies that

$$\int_{\Omega} |e^D(u_m)| + \int_{\Omega} |\operatorname{div} u_m|^2 dx + \int_{\Gamma_3} l|u_{\mathscr{T}}|d\Gamma \leqslant \text{const}. \quad (21)$$

Now the conclusion that $|u_m|_{U(\Omega)} \leqslant$ const follows directly from (20) and (21).

§5. Relaxation

In the preceding section, we showed that under Hypothesis 4.4, the minimizing sequence of $\mathscr{P}$ is bounded in $U(\Omega)$ and, therefore, contains a subsequence which converges in $U(\Omega)$ weakly. We want to show that the limit function obtained is a solution of the problem. To show this, we have two things to explain and verify. The first is that the weak convergence in $U(\Omega)$ does not guarantee that the boundary condition can be maintained. It is also physically reasonable to have "solutions" that do not preserve the displacement boundary condition imposed in advance. So we have to find a compromise to say that the limit function is a solution in some sense. The second is that, following that compromise, we have to justify that the functional considered is l. s. c. with respect to the convergence available.

The first thing has been studied in [17]. We recall the main facts here in adapting to our situation. The second problem will be treated in the nextsection for our particular problem. The boundary condition is relaxed to $u\cdot n|_{\Gamma_1} = u_0\cdot n$ and the problem is relaxed to

$$\mathscr{PR}: \operatorname{Inf}\left\{ \int_{\Omega} \psi(e(u)) + \int_{\Gamma_1} \psi_\infty(\mathscr{T}^D(u_{0\mathscr{T}} - u_{\mathscr{T}}))d\Gamma + \int_{\Gamma_3} l|u_{\mathscr{T}}|d\Gamma \right.$$
$$\left. - \int_{\Omega} f(x)u(x)dx - \int_{\Gamma_2} u\cdot g\, d\Gamma - \int_{\Gamma_3} F_n(u\cdot n)d\Gamma \right\} \quad (22)$$

subject to $u \in AR = \{u \in H^1(\Omega),\ u\cdot n|_1 = u_0\cdot n]$, where $\mathscr{T}$ is the linear operator from $L^1(\partial\Omega)$ into $L^1(\partial\Omega; E)$ defined by $\mathscr{T}(u)_{ij} = (u_i n_j + u_j n_i)/2$. We know that $\mathscr{PR}$ is a relaxation of $\mathscr{P}$ in the following sense:

i) $\operatorname{Inf}\mathscr{PR} = \operatorname{Inf}\mathscr{P}$,

ii) any minimizing sequence of $\mathscr{P}$ is also a minimizing sequence of $\mathscr{PR}$.

We note now that the third integral in the exprssion (22) has no relation with the potential of energy while the relaxation of the displacement boundary condition brings us a term linked directly with ψ. The philosophy in proving the existence results of plasticity problems suggests that we relax it further under the restriction that the minimum value of energy is not affected.

Let us define

$$C=\{\xi\in E:\ \forall n\in R^3,\ |n|=1,\ |\xi n-(n^T\xi n)n|\leqslant l\}. \tag{23}$$

Proposition 5.1. *$C=RI+C^D$ where C^D is a bounded convex subset of E^D containing a neighbourhood of* 0.

Proof Noting that $\xi n-(n^T\xi n)n=\xi^D n-n^T\xi^D n$, we conclude easily that $C=RI+C^D$ with $C^D\subset E^D$. To show that C^D is bounded, let ξ be in C^D, λ_1, λ_2, λ_3 be its threeeignvalues. It is easy to see that

$$\sum_{i=1}^{3}\lambda_i=0.$$

It follows from (23) that

$$\max\{|\lambda_i-\lambda_j|\}\leqslant 2l.$$

Therefore $|\lambda_i|\leqslant 4l/3$ for $i=1,2,3$ is clear. So C^D is bounded. The fact that C^D is a convex set is easy to see. Finally, C^D contains a neighborhood of 0 follows from the fact that $|\xi n-(n^T\xi n)n|\leqslant|\xi n|\leqslant|\xi|$.

In the following, we denote $\psi_1(\xi)=\mathrm{Sup}\{\xi:\ \eta-A\eta:\ \eta;\ \eta\in C\}$ and $\tilde\psi(\xi)=$ the convexification of $\min\{\psi_1(\xi),\ \psi(\xi)\}$ and we have

Proposition 5.2. *$\tilde\psi(\xi)$ satisfies the estimates* (12) *and* (13).

Proof Take

$$\tilde C=\{\xi\in E,\ |\xi^D|\leqslant k\}\cap C,\ \hat C=\{\sigma\in L^2(\Gamma_3;\ E^D),\ \sigma(x)\in\tilde C\ \text{a. e.}\}. \tag{24}$$

It is then easy to see that $\tilde\psi(\xi)\geqslant\sup\{\xi:\ \eta-A\eta:\ \eta;\ \eta\in\tilde C\}$. It is also easy to verify that $\tilde\psi(\xi)\leqslant\psi(\xi)$. So the estimates follow directly.

Proposition 5.3. *For any $u\in H^1(\Omega)$, we have*

$$\int_{\Gamma_3}\psi_{1\infty}(-\mathscr{T}^D(u_{\mathscr{T}}))\,d\Gamma=\int_{\Gamma_3}|u_{\mathscr{T}}|\,d\Gamma. \tag{25}$$

Proof Following the same method of proof as in [11], we get

$$\int_{\Gamma_3}\psi_{1\infty}(-\mathscr{T}^D(u_{\mathscr{T}}))=\mathrm{Sup}\int_{\Gamma_3}-\sigma:\ \mathscr{T}^D(u_{\mathscr{T}})d\Gamma$$

subject to $\sigma\in L^2(\Gamma_3;\ E^D)$ and $\sigma\in C$ a. e.. So it is easy to see that (25) holds. We want $\tilde\psi(-\mathscr{T}^D(u_{\mathscr{T}}))$ to be the relaxation of $l|u_{\mathscr{T}}|$ in the expression of potential of energy and so we try to give its explicit expression.

Proposition 5.4. *For any $u\in H^1(\Omega)$, we have*

$$\int_{\Gamma_3}\tilde\psi_\infty(-\mathscr{T}^D(u_{\mathscr{T}}))d\Gamma=\mathrm{Sup}\int_{\Gamma_3}-\sigma:\ \mathscr{T}^D(u_{\mathscr{T}})d\Gamma \tag{26}$$

subject to $\sigma\in\hat C$ (see (24)).

Proof To show (26), we just have to show that

$$\tilde{\psi}_\infty(\xi)=\operatorname{Sup}\{\sigma:\xi\}$$

subject to $\sigma\in\tilde{C}$ for all $\xi\in E^D$. By the definition of $\tilde{\psi}$, it is easy to verify that $\tilde{\psi}_\infty(\xi)\geqslant\operatorname{Sup}\{\sigma:\xi;\ \sigma\in\tilde{C}\}$ for all $\xi\in E^D$. To show the inverse inequality, by [9], we just have to prove that if we note $\psi_2(\xi)=\min(\psi_1(\xi),\ \psi(\xi))$, then

$$\psi_2^{**}(\xi)=\operatorname{Sup}\{\xi:\sigma-A\sigma:\sigma:\sigma\in\tilde{C}\}. \tag{27}$$

Since $\tilde{\psi}=\psi_2^{**}$ by the definition, we get the desired result. As a matter of fact,

$$\begin{aligned}\psi_2^{*}(\sigma)&=\operatorname{Sup}\{\xi:\sigma-\psi_2(\xi);\ \xi\in E\}\\&=\operatorname{Sup}\{\max(\xi:\sigma-\psi_1(\xi),\ \xi:\sigma-\psi(\xi));\ \xi\in E\}\\&\leqslant\max(\operatorname{Sup}\{\xi:\sigma-\psi_1(\xi);\ \xi\in E\},\ \operatorname{Sup}\{\xi:\sigma-\psi(\xi);\ \xi\in E\})\\&=\begin{cases}A\sigma:\sigma \text{ if } \sigma\in\tilde{C},\\+\infty \text{ if not.}\end{cases}\end{aligned}$$

It is then easy to see that (27) is true.

We can now give the relaxed problem of $\mathscr{PR}$ and show that it does not influence the minimum value of energy.

$$\mathscr{QR}:\ \operatorname{Inf}\Big\{\int_\Omega\psi(e(u))+\int_{\Gamma_1}\psi_\infty(\mathscr{T}^D(u_{0\mathscr{T}}-u_{\mathscr{T}}))d\Gamma+\int_{\Gamma_3}\tilde{\psi}_\infty(-\mathscr{T}^D(u_{\mathscr{T}}))d\Gamma$$
$$-\int_\Omega f(x)u(x)dx-\int_{\Gamma_2}g\cdot u\,d\Gamma-\int_{\Gamma_3}F_n(u\cdot n)d\Gamma\Big\} \tag{28}$$

subject to $u\in AR$. When we compare $\mathscr{QR}$ with $\mathscr{PR}$, it is easy to see that any addmissible function of $\mathscr{PR}$ is one of $\mathscr{QR}$ and $\operatorname{Inf}\mathscr{PR}\geqslant\operatorname{Inf}\mathscr{QR}$. So we have to show that

$$\operatorname{Inf}\mathscr{PR}=\operatorname{Inf}\mathscr{QR}. \tag{29}$$

By the generalized duality (cf. [11]), for any u which is $\mathscr{PR}$ admissible, σ which is $\mathscr{P}^*$ admissible, we have

$$\int_\Omega\psi(e(u))+\int_{\Gamma_1}\psi_\infty(\mathscr{T}^D(u_{0\mathscr{T}}-u_{\mathscr{T}}))\,d\Gamma+\int_{\Gamma_3}\tilde{\psi}_\infty(-\mathscr{T}^D(u_{\mathscr{T}}))d\Gamma-\int_\Omega f(x)u(x)dx$$
$$-\int_{\Gamma_2}g\cdot u\,d\Gamma-\int_{\Gamma_3}F_n(u\cdot n)d\Gamma\geqslant-\int_\Omega A\sigma:\sigma\,dx+\int_{\Gamma_1}(\sigma n)\cdot u_0\,d\Gamma,$$

which implies that $\operatorname{Inf}\mathscr{QR}\geqslant\operatorname{Sup}\mathscr{P}^*=\operatorname{Inf}\mathscr{PR}$. So (29) holds.

As we have already observed that H^1 is not the proper space that we want to work on, we now extend the definition domain of the problem $\mathscr{QR}$ so that the function space in which the minimizing sequence is bounded coincides with the one on which the problem is defined.

$$\mathscr{Q}:\ \operatorname{Inf}\Big\{\int_\Omega\psi(e(u))+\int_{\Gamma_1}\psi_\infty(\mathscr{T}^D(u_{0\mathscr{T}}))+\int_{\Gamma_3}\tilde{\psi}_\infty(-\mathscr{T}^D(u_{\mathscr{T}}))$$
$$-\int_\Omega f(x)u(x)dx-\int_{\Gamma_2}g\cdot u\,d\Gamma-\int_{\Gamma_3}F_n(u\cdot n)d\Gamma\Big\}$$

subject to $u\in\{u\in U(\Omega),\ u\cdot n|_{\Gamma_1}=u_0\cdot n\}$. The relation between $\mathscr{Q}$ and $\mathscr{P}$ is that $\operatorname{Inf}\mathscr{P}=\operatorname{Inf}\mathscr{Q}$ and any admissible function of $\mathscr{P}$ is one of $\mathscr{Q}$.

§ 6. Existence Theorem in Strain Problem

In this section, we follow the main ideas of [17] to show that for any sequence $\{u_m\}\subset U(\Omega)$ which converges weakly in $U(\Omega)$ to u, with each u_m being $\mathscr{Q}$ admissible, we have

$$\varliminf_{m\to\infty}\Big\{\int_\Omega\psi(e(u_m))+\int_{\Gamma_1}\psi_\infty(\mathscr{T}^D(u_{0\mathscr{T}}-u_{m\mathscr{T}}))\,d\Gamma+\int_{\Gamma_3}\tilde{\psi}_\infty(-\mathscr{T}^D(u_{m\mathscr{T}}))\,d\Gamma$$
$$-\int_\Gamma f(x)u_m(x)\,dx-\int_{\Gamma_2}g\cdot u_m\,d\Gamma-\int_{\Gamma_3}F_n(u_m\cdot n)\,d\Gamma\Big\}$$
$$\geqslant\int_\Omega\psi(e(u))+\int_{\Gamma_1}\psi_\infty(\mathscr{T}^D(u_{0\mathscr{T}}-u_{\mathscr{T}}))\,d\Gamma+\int_{\Gamma_3}\tilde{\psi}_\infty(-\mathscr{T}^D(u_{\mathscr{T}}))\,d\Gamma$$
$$-\int_\Omega f(x)u(x)\,dx-\int_{\Gamma_2}g\cdot u\,d\Gamma-\int_{\Gamma_3}F_n(u\cdot n)\,d\Gamma. \tag{30}$$

Theorem 6.1. *If (30) is true, then under Hypothesis 4.4, the problem $\mathscr{Q}$ admits at least one solution.*

Proof For any minimizing sequence $\{u_m\}$, we can suppose that it converges weakly in $U(\Omega)$ to a function u. It is easy to see that $u\cdot n|_{\Gamma_0}=u_0\cdot n$ and by (30) we know that u is a solution of $\mathscr{Q}$.

Now we give a proof of (30). We need, at first, more results concerning convex functional of measures.

Lemma 6.2. *If ψ_i: $E^D\to R^+\cup\{0\}$ for $i=1, 2$ such that*

i) $c_i(|p|-1)\leqslant\psi_i(p)\leqslant c_i'(|p|+1)$ *with* c_i, $c_i'>0$, $p\in E^D$, $i=1, 2$;

ii) ψ_i *is convex;*

iii) $\psi_2(p)\leqslant\psi_1(p)$;

then, if Ω is a bounded regular open subset of R^3 with boundary Γ; Γ_0 is a regular closed subset of Γ and if $\{\mu_n\}\subset M(R^3;\ E^D)$ such that

i) supp $\mu_n\subseteq\bar{\Omega}$,

ii) $\mu_n\to\mu$ *in* $M(R^3;\ E^D)$ *weak-star,*

we have

$$\varliminf_{n\to\infty}\int_\Omega\psi_1(\mu_n)+\int_{\Gamma_0}\psi_{2\infty}(\mu_n|_{\Gamma_0}\geqslant\int_\Omega\psi(\mu)+\int_{\Gamma_0}\psi_{2\infty}(\mu|_{\Gamma_0}),$$

with $\mu|_{\Gamma_0}$ being the part of measure of μ supported by Γ_0 for any $\mu\in M(R^3,\ E^D)$.

Proof We notify first that for any $\mu\in M(R^3;\ E^D)$, $\psi(\mu)\in M(R^3)$ (cf. [6]). Sec-ondly, since $\mu_n\to\mu$ in $M(R^3;E^D)$ weak-satr, we know that supp$\mu\subseteq\bar{\Omega}$. Finally, if we put $\Gamma_\delta=\{x\in R^3,\ \mathrm{dist}(x,\ \Gamma_0)<\delta\}$, we get the following estimate:

$$\int_\Omega\psi_1(\mu_n)+\int_{\Gamma_0}\psi_{2\infty}(\mu_n|_{\Gamma_0})\geqslant\int_{\Omega\setminus\bar{\Gamma}_\delta}\psi_1(\mu_n)+\int_{\Gamma_\delta}\psi_2(\mu_n).$$

By the standard l. s. c. property of convex functional of measures (cf. [6]), we have

$$\varliminf_{n\to\infty}\left\{\int_\Omega \psi_1(\mu_n)+\int_{\Gamma_0}\psi_{2\infty}(\mu_n|_\circ)\right\}\geqslant\int_{\Omega\setminus\bar{\Gamma}_\delta}\psi_2(\mu). \tag{31}$$

By letting $\delta\to0$ in the right hand side of (31), we get the desired inequality.

Now we are ready to prove (30). The main idea follows from that in [17] except the friction term. So we give only an outline of the proof.

Theorem 6. 3. (30) *holds under the hypotheses given at the beginning of the section.*

Proof Define a function $\beta\in C^\infty(R)$ such that

$$\beta(t)=\begin{cases}0 \text{ if } |t|\leqslant1,\\ 1 \text{ if } |t|\geqslant2,\end{cases}$$

$$0\leqslant\delta(t)\leqslant1 \text{ for all } t\in R.$$

We put $\Phi_{\delta1}(x)=\beta(\operatorname{dist}(x,\ \Gamma_2\cup\Gamma_3)/\delta)$ and $\Phi_{\delta2}(x)=\beta(\operatorname{dist}(x,\ \Gamma_1\cup\Gamma_2)/\delta)$ with $\delta>0$. It is easy to know that $\Phi_{\delta i}$ are regular functions (at least C^1). We also know that under our setting of the problem, $\psi(\xi)=c(\operatorname{tr}\xi)^2+\psi^D(\xi^D)$ for all $\xi\in E$ where c is a positive constant. ψ^D is a function defined on E^D satisfying the conditions of Lemma 6.2 and $\psi^D(\xi)\geqslant0$ for any $\xi\in E^D$.

Suppose that σ is any admissible function of Problem $\mathscr{P}^*$, $\{u_m\}$ is as in the beginning of the section. Then we have

$$\begin{aligned}
&\int_\Omega\psi(e(u_m))+\int_{\Gamma_1}\psi_\infty(\mathscr{T}^D(u_{0\mathscr{T}}-u_{m\mathscr{T}}))\,d\Gamma+\int_{\Gamma_3}\tilde{\psi}_\infty(\mathscr{T}^D(u_{m\mathscr{T}}))d\Gamma\\
&\quad-\int_\Omega f(x)u_m(x)\,dx-\int_{\Gamma_2}g\cdot u_m\,d\Gamma-\int_{\Gamma_3}F_n(u_m\cdot n)d\Gamma\\
&=\int_\Omega\left[c(\operatorname{div}u_m)^2-\frac{1}{3}(\operatorname{tr}\sigma)(\operatorname{div}u_m)\right]dx+\int_{\Gamma_1}u_{0\mathscr{T}}(\sigma n)_{\mathscr{T}}d\Gamma\\
&\quad+\int_\Omega\psi^D(e^D(u_m))+\int_{\Gamma_3}\tilde{\psi}_\infty(\mathscr{T}^D(-u_{m\mathscr{T}}))d\Gamma\\
&\quad+\int_{\Gamma_1}\psi_\infty(\mathscr{T}^D(u_{0\mathscr{T}}-u_{m\mathscr{T}}))\,d\Gamma-\int_\Omega\sigma^D\colon e^D(u_m)\\
&\quad-\int_{\Gamma_0}(u_{0\mathscr{T}}-u_{m\mathscr{T}})(\sigma n)_{\mathscr{T}}+\int_{\Gamma_3}u_{m\mathscr{T}}(\sigma n)_{\mathscr{T}}.
\end{aligned}\tag{32}$$

We know that the first term on the right hand side of (32) is l. s. c. since $\operatorname{div}u_m$ converges to $\operatorname{div}u$ in $L^2(\Omega)$ weakly. So we study only the rest part of the right hand side of (32):

$$\begin{aligned}
\text{rest}\geqslant&\int_{R^3}\psi^D(\Phi_{\delta1}e^D(\tilde{u}_m))-\int_{R^3\setminus\bar{\Omega}}\psi^D(\Phi_{\delta1}e^D(u_0))-\int_\Omega\sigma^D\colon e^D(u_m)\Phi_{\delta1}\\
&-\int_{\Gamma_1}\Phi_{\delta1}(u_{0\mathscr{T}}-u_{m\mathscr{T}})(\sigma n)_{\mathscr{T}}-\int_{\Gamma_1}(1-\Phi_{\delta1})(u_{0\mathscr{T}}-u_{m\mathscr{T}})(\sigma n)_{\mathscr{T}}\\
&-\int_\Omega(1-\Phi_{\delta1})\sigma^D\colon e^D(u_m)+\int_\Omega\psi^D((1-\Phi_{\delta1})e^D(u_m))\\
&-\int_{\Gamma_3}(1-\Phi_{\delta1})u_{m\mathscr{T}}(\sigma n)_{\mathscr{T}}+\int_{\Gamma_3}\tilde{\psi}^D_\infty(\mathscr{T}^D(-u_{m\mathscr{T}})(1-\Phi_{\delta1}))
\end{aligned}$$

$$+\int_{\Gamma_1}\psi^D_\infty((1-\psi_{\delta1})\mathscr{T}^D(u_{0\mathscr{T}}-u_{m\mathscr{T}}))$$

$$\geqslant I_1+\int_\Omega\psi^D((1-\Phi_{\delta1})\Phi_{\delta2}e^D(\hat{u}_m))+\int_{\Gamma_8}\tilde{\psi}_\infty((1-\Phi_{\delta1})\Phi_{\delta2}\mathscr{T}^D(-u_{m\mathscr{T}}))$$

$$+\int_\Omega\psi^D((1-\Phi_{\delta1})(1-\Phi_{\delta2})e^D(\hat{u}_m))+\int_{\Gamma_8}\tilde{\psi}_\infty((1-\Phi_{\delta1})(1-\Phi_{\delta2})\mathscr{T}^D(-u_{m\mathscr{T}}))$$

$$-\int_\Omega\Phi_{\delta2}(1-\Phi_{\delta1})\sigma^D\colon e^D(u_m)-\int_{\Gamma_8}\Phi_{\delta2}(1-\Phi_{\delta1})u_{m\mathscr{T}}(\sigma n)_{\mathscr{T}}$$

$$-\int_{\Gamma_1}\Phi_{\delta2}(1-\Phi_{\delta1})(u_{0\mathscr{T}}-u_{m\mathscr{T}})(\sigma n)_{\mathscr{T}}-\int_\Omega(1-\Phi_{\delta2})(1-\Phi_{\delta1})\sigma^D\colon e^D(u_m)$$

$$-\int_{\Gamma_8}(1-\Phi_{\delta2})(1-\Phi_{\delta1})u_{m\mathscr{T}}(\sigma n)_{\mathscr{T}}-\int_{\Gamma_1}(1-\Phi_{\delta2})(1-\Phi_{\delta1})(u_{0\mathscr{T}}-u_{m\mathscr{T}})(\sigma n)_{\mathscr{T}}$$

$$+\int_{\Gamma_1}\psi^D_\infty((1-\Phi_{\delta1})\Phi_{\delta2}\mathscr{T}^D(u_{0\mathscr{T}}-u_{m\mathscr{T}}))$$

$$+\int_{\Gamma_1}\psi^D_\infty((1-\Phi_{\delta1})(1-\Phi_{\delta2})\mathscr{T}^D(u_{0\mathscr{T}}-u_{m\mathscr{T}})) \qquad (33)$$

with

$$I_1=\int_{R^3}\psi^D(\Phi_{\delta1}e^D(\tilde{u}_m))-\int_{R^3\setminus\bar{\Omega}}\psi^D(\Phi_{\delta1}e^D(u_0))-\int_\Omega\sigma^D\colon e^D(u_m)\Phi_{\delta1}$$
$$-\int_{\Gamma_1}\Phi_{\delta1}(u_{0\mathscr{T}}-u_{m\mathscr{T}})(\sigma n)_{\mathscr{T}},$$

where

$$\tilde{u}_m=\begin{cases}u_m & x\in\Omega,\\ u_0 & x\in R^3\setminus\bar{\Omega},\end{cases}$$

and

$$\hat{u}_m=\begin{cases}u_m & x\in\Omega,\\ 0 & x\in R^3\setminus\bar{\Omega}.\end{cases}$$

Now the l. s. c. property in the last expression in (33) follows from [17] except that of the term

$$\int_\Omega\psi^D((1-\Phi_{\delta1})\Phi_{\delta2}e^D(u_m))+\int_{\Gamma_8}\tilde{\psi}_\infty((1-\Phi_{\delta1})\Phi_{\delta2}\mathscr{T}^D(-u_{m\mathscr{T}}))$$

which is proved in Lemma 6.2.

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