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GROUNDWATER MASS TRANSPORT AND HETEROGENEOUS EQUILIBRIUM CHEMICAL REACTION——A KIND OF FREE BOUNDARY PROBLEM

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Abstract

This paper considers another kind of chemical reaction different from [1]. The difference is that the chemical reactions under consideration are assumed to be heterogeneous, fast, reversible and classical. As a typical example, the author reduces this problem to an one phase Stefan problem with noncommon data, and investigates the classical solution as well as the weak solution.

§1. Introduction

We consider the transport of reacting solutes in porous media (see [2]). The solution originally present within the porous medium contains a reacting cation M_1 and a reacting anion M_2, both of which are in equilibrium with a crystalline solid M_1M_2. The displacing solution contains an inert anion M_4 and the reacting cation M_1. The solute-transport-affecting reaction is sufficiently fast and reversible. It is represented by the chemical equation

$$\overline{M_1M_2} \rightleftharpoons M_1 + M_2. \tag{R4}$$

As the displacing solution does not contain M_2 and has M_1 with a concentration smaller than that in the original solution, percolation of the displacing solution through a column will cause a continuous shift of the chemical equilibrium (R4) to the right, bringing about dissolution of $\overline{M_1M_2}$.

It is known that because of such a dissolution process, eventually two zones must develop within the leached column: an upper zone I, from which the solid $\overline{M_1M_2}$ has been entirely leached out and a lower zone II, in which the solid is apresent. It can be proved by means of elementary chemical equilibrium considertions that the dissolution occurs at, and only at, the interface between zone I and

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II and the interface must move gradually and continuously downward (see Fig. 1).

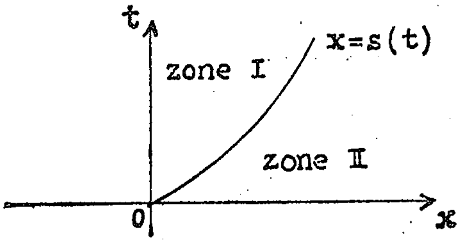

Fig. 1

We reduce this problem to an one phase Stefan problem with noncommon data. In section 2, we give the mathematical model and its reduction. The classical solution and the weak solution are investigated in section 3 and 4 respectively.

§ 2. Mathematical Model and Its Redution

Because of the differing nature of the chemical processes in two zones, the basic solute transport equations for the present case must be formulated for each zone seperately[2]. Before doing this, we define

$$\text{zone I: } \Omega_1=\{x\in(0,\ s(t)),\ t\in(0,\ T)\};$$
$$\text{zone II: } \Omega_2=\{x\in(s(t),\ l),\ t\in(0,\ T)\},$$
$$\Omega=\{x\in(0,\ l),\ t\in(0,\ T)\};$$
$$\Gamma_1=\{0\}\times(0,\ T);\ \Gamma_2=\{l\}\times(0,\ T);\ \Gamma_s=\{s(t)\}\times(0,\ T);$$
$$\Gamma_0=(0,\ l)\times\{0\};\ L=D\frac{\partial^2}{\partial x^2}-Q\frac{\partial}{\partial x};\ q=-D\frac{\partial}{\partial x}+Q,$$

where l and T are positive constants, D and Q are positive constants representing the coefficients of longitudinal dispersion and x direction volume flux of water respectively, $s(t)$ is the interface with $s(0)=0$. Then we have following system[2]

$$\theta\frac{\partial c_4}{\partial t}=Lc_4 \quad \text{in } \Omega, \tag{2.1}$$
$$c_4=c_{4B}=0 \quad \text{on } \Gamma_0, \tag{2.2}$$
$$qc_4=Qc_{4F} \quad \text{on } \Gamma_1, \tag{2.3}$$
$$\frac{\partial c_4}{\partial x}=0 \quad \text{on } \Gamma_2, \tag{2.4}$$

zone I:

$$\theta\frac{\partial c_4^I}{\partial t}=Lc_4^I \quad \text{in } \Omega_1, \tag{2.5}$$
$$qc_4^I=\theta c_{4F}^I \quad \text{on } \Gamma_1, \tag{2.6}$$
$$\bar c_{12}^I=0 \quad \text{in } \Omega_1, \tag{2.7}$$

zone II:

$$\theta\frac{\partial c_4^{II}}{\partial t}+\rho\frac{\partial \bar c_{12}^{II}}{\partial t}=Lc_4^{II} \quad \text{in } \Omega_2, \tag{2.8}$$
$$c_4^{II}=c_{4B} \text{ and } \bar c_{12}^{II}=\bar c_B \quad \text{on } \Gamma_0, \tag{2.9}$$
$$\frac{\partial c_4^{II}}{\partial x}=0 \quad \text{on } \Gamma_2, \tag{2.10}$$

$$K_{12}=c_1^{II}c_2^{II} \quad \text{in } \Omega_2, \tag{2.11}$$

where the superscripts I and II indicate the zones involved. For $i=1$ and $i=2$, the functions $c_i^I(x, t)$ and $c_i^{II}(x, t)$ are solutions of two different sets of eqnations and hence they are two different functions.

c_i's and $\bar{c}_{12}$ are the concentrations (moles per unit volume of water) of the species M_i ($i=1, 2, 4$) and $\overline{M_1M_2}$ respectively.

ρ is the porous medium's bulk density.

θ is the volumetric water content.

$\bar{c}_B$, ρ, θ, K_{12}, c_{iB} and c_{iF}^I ($i=1, 2, 4$) are positive constants and satisfy

$$c_{1F}^I=c_{4F},\ c_{2F}^I=0,\ c_{1B}^{II}=c_{2B}^{II}=\sqrt{K_{12}}.\ c_{4B}=0, \tag{2.12}$$

Clearly, additional boundary information at the moving interface $x=s(t)$ is needed. In the present case, that is

$$c_i^I(s(t)-, t)=c_i^{II}(s(t)+, t), \tag{2.13}$$

$$(\theta(c_i^{II}-c_i^I)+\rho\bar{c}_{12}^{II})\frac{ds}{dt}=qc_i^{II}-qc_i^I \quad \text{on } \Gamma_s, \tag{2.13'}$$

(2.13)′ is equivalent to the following

$$\rho\ \bar{c}_{12}^{II}(s(t)+, t)\ \frac{ds}{dt}=D\left(\frac{\partial c_i^I(s(t)-, t)}{\partial x}-\frac{\partial c_i^{II}(s(t)+, t)}{\partial x}\right). \tag{2.14}$$

Thus, the problem (2.5)—(2.14) (called PI) is a kind of free bounbary problem, i. e., it can be reduced to a one phase Stefan problem with noncommon data. In fact, setting

$$v=c_1^m-c_2^m, \quad m=\text{I, II},$$

we have

$$\theta\ \frac{\partial v}{\partial t}=Lv \quad \text{in } \Omega_1\cup\Omega_2, \tag{2.15}$$

$$v=0 \quad \text{on } \Gamma_0, \tag{2.16}$$

$$qv=Qc_{1F}^I \quad \text{on } \Gamma_1, \tag{2.17}$$

$$\frac{\partial v}{\partial x}=0 \quad \text{on } \Gamma_2, \tag{2.18}$$

$$v(s(t)-, t)=v(s(t)+, t), \tag{2.19}$$

$$\frac{\partial v(s(t)-, t)}{\partial x}=\frac{\partial v(s(t)+, t)}{\partial x}. \tag{2.20}$$

Comparing (2.1)—(2.4) with (2.15)—(2.20), we get (see [3])

$$v\equiv c_4 \quad \text{in } \Omega.$$

From this and (2.8), (2.11), we have

$$c_1^{II}=\frac{1}{2}(c_4+(c_4^2+4K_{12})^{1/2}), \tag{2.21}$$

$$c_2^{II}=\frac{1}{2}(-c_4+(c_4^2+4K_{12})^{1/2}), \tag{2.22}$$

$$\frac{\partial \bar{c}_{12}^{II}}{\partial t}=\frac{2DK}{\rho}\cdot\frac{(\partial c_4/\partial x)^2}{(c_4^2+4K_{12})^{3/2}}. \tag{2.23}$$

We can extend the difinitions of c_4^{II} and c_{12}^{II} to zone I because of (2.21)—(2.23) and (2.9). Then setting

$$u_i = c_i^{II} - c_i^{I} \quad \text{in } \Omega_1,$$

we reduce the original problem to an equivalent one as follows.

$$Du_{xx} - Qu_x - \theta u_t = g(x, t) \quad \text{in } \Omega_1, \tag{2.24}$$

$$-Du_x + Qu = h(t) \quad \text{on } \Gamma_1, \tag{2.25}$$

$$u(s(t), t) = 0, \tag{2.26}$$

$$-\lambda(s(t), t)\dot{s} = Du_x(s(t), t), \tag{2.27}$$

$$s(0) = 0, \tag{2.28}$$

where

$$g(x, t) = \rho c_{12t}^{II} = \frac{2DK(\partial c_4/\partial t)^2}{(c_4^2 + 4K_{12})^{3/2}} \geqslant 0, \tag{2.29}$$

$$h(t) = \frac{Q}{2}\left(\frac{4K_{12} + c_{1F}c_4(0, t)}{(c_4^2(0, t) + 4K_{12})^{1/2}} - c_{1F}\right), \tag{2.30}$$

$$\lambda(x, t) = \rho \bar{c}_{12}^{II}(x, t) = \rho \bar{c}_B + \int_0^t g(x, t)\,dt \geqslant \rho \bar{c}_B > 0, \tag{2.31}$$

$$u = u_1 = u_2. \tag{2.32}$$

§ 3. Classical Solution

In this section, we investigate the classical solution of PI.

Definition 3.1. *The classical solution of PI is a triplet* $\{T, u, s\}$ *such that* $s(t)$ *belongs to* $C[0, T] \cap C^1(0, T)$, $u_x \in c(\Omega_1 \cup \Gamma_1 \cup \Gamma_s)$, *and* $u(x, t) \in C(\bar{\Omega}_1) \cap C^{2,1}(\Omega_1)$ *satisfying* (2.24)—(2.28).

Assumption A(*).

$$2\sqrt{K_{12}} - c_{1F} > 0.$$

We have

Theorem 1. *Under Assumption A, there exists a unique solution* (T^*, $s(t)$, $u(x, t)$) *to PI in* $\Omega_{1,T^*} = (0, s(t)) \times (0, T^*)$ *and the following cases, only these cases, are possible*

(a) $T^* = +\infty$,

(b) $T^* < +\infty$, *but* $\limsup\limits_{t \to T^*} u_x(s(t), t) = +\infty$,

(c) $T^* < +\infty$, *but* $\liminf\limits_{t \to T^*} s(t) = 0$ *or* $\limsup\limits_{t \to T^*} s(t) = l$.

The proof of this theorem depends on the following lemmas.

Lemma 3.2. *The PI has a unique solution* $c_4(x, t) \in c(\bar{\Omega}) \cap c^{2,1}(\Omega)$ *satisfying*

$$0 \leqslant c_4(x, t) \leqslant c_{1F}, \tag{3.1}$$

(*) The assumption A seems natural to ensure $s(t) > 0$ as $t > 0$ is small, then it yields that the data $h(t) \geqslant h(0) > 0$, and $g \geqslant 0$, hence, are noncommon.

$$-Qc_{1F}/D \leqslant \frac{\partial c_4(x, t)}{\partial x} \leqslant 0. \tag{3.2}$$

Proof This is a classical result, and by maximum principle theestimates (3.1) and (3.2) are obtained.

Attumption B. $s(t)$ is nondecreasing and $s(t) \in C^1[0, T]$.

Decomposing

$$u = v + W, \tag{3.3}$$

one can show that v and w satisfy the following systems respectively:

$$Dv_{xx} - Qv_x - \theta v_t = g \quad \text{in } \Omega_1, \tag{3.4}$$

$$-Dv_x + Qv = 0 \quad \text{on } \Gamma_1, \tag{3.5}$$

$$v(s(t), t) = 0 \quad \text{on } \Gamma_s, \tag{3.6}$$

and

$$Dw_{xx} - Qw_x - \theta w_t = 0 \quad \text{in } \Omega_1, \tag{3.7}$$

$$-Dw_x + Qw = h(t) \quad \text{on } \Gamma_1, \tag{3.8}$$

$$w(s(t), t) = 0 \quad \text{on } \Gamma_s. \tag{3.9}$$

Lemma 3. 3. *Assume A and B. Then the problem* (3.7)—(3.9) *has a unique solution* $w(x, t)$, *which and* $w_x(x,t)$ *are continuous on* $\overline{\Omega}_1 = [0, s(t)] \times [0, T]$. *Moreover, the following estimates hold:*

$$0 \leqslant w(x, t) \leqslant P_0(s(t) - x), \quad -P_0 \leqslant \frac{\partial w(s(t), t)}{\partial x} \leqslant 0, \tag{3.10}$$

$$|w_x| \leqslant P, \tag{3.11}$$

$$|w_{xx}| \leqslant R, \tag{3.12}$$

where P_0 *and* P *depend only on* D, Q *and* $\|h\|_{C'}$, R *depends only on* D, Q, $\|h\|_{C'}$ *and* $\dot{s}(t)$.

Proof It is easy to see that $h(t) \geqslant h(0) > 0$. Thus the proof is the same as in [4].

Analogously, we have

Lemma 3. 4. *The estimates* (3.10)—(3.12) *hold for* $-v$ *with* $\|h\|_{C'}$ replaced by $\|g_x\|_\infty$.

Remark 3. 5. Strictly speaking, we need the assumptions: $g \in c^{1,1/2}(\overline{\Omega}_1)$ and $h(t) \in c^1[0, T]$, but these depend on the smoothness of c_4 which is regarded as given functions. We assume that c_4 is sufficiently smooth as we need in this section.

As usual way, we have the integral representation for $s(t)$ and $u(x, t)$:

$$A(s(t), t) = \int_0^t h(t)\,dt - \theta \int_0^{(t)} u(\xi, t)\,d\xi, \tag{3.13}$$

where

$$A(s(t), t) = \int_0^{s(t)} \lambda(\xi, t)\,d\xi. \tag{3.14}$$

We define the operator $F: s(t) \to b(t)$ as follows

$$A(b(t), t) = \int_0^t h(t)\,dt - \theta \int_0^{s(t)} u(\xi, t)\,d\xi. \tag{3.15}$$

Namely,

$$\dot{b}(t)\lambda(b(t),\ t)=h(t)-\int_0^{s(t)}(Du_{\xi\xi}-Qu_\xi-g)d\xi-\int_0^{b(t)}\lambda_t(\xi,\ t)d\xi. \qquad (3.16)$$

Hence,

$$\dot{b}(t)\geqslant\{h(0)-[\|h'(t)\|_\infty+(DR+QP)K]t-\|g\|_\infty(K+\dot{b}(t))t\}/(\rho\bar{c}_B). \qquad (3.17)$$

That is

$$\dot{b}(t)>0$$

if $0\leqslant\dot{s}(t)\leqslant K$ and $t\in[0,\ \sigma]$ with sufficiently small $\sigma>0$. For all $t\in[0,\ \sigma]$, we find that $0\leqslant\dot{b}(t)\leqslant K$, and $F(s(t))=b(t)$ keeps the monotonicity and the same bound of $s(t)$. *Proof of Theorem 1* For $t\in[0,\ \sigma]$, we choose the subset

$$E=\{s(t)\in C^1[0,\ \sigma]\mid s(t)\nearrow,\ s(0)=0,\ 0\leqslant\dot{s}(t)\leqslant K\}$$

of Banach space $C[0,\ \sigma]$ and use Schauder fixed point theorem. We can show that F has a fixed point in E. The details of the proof can be found in [4]. According to Theorems 3 and 5 in [5], the solution can be extended to maximal domain $(0,\ s(t))\times(0,\ T^*)$. On the other hand the case (b), maybe, occurs (see [6]), but in our case, if (a) is not true, i. e., $T^*<+\infty$ and (c) do not occur, we have

$$0<\delta_1=\liminf_{t\to T^*}s(t)\leqslant\limsup_{t\to T^*}s(t)=\delta_2<1 \qquad (3.18)$$

for some δ_1 and δ_2. As done in [7] and [8], we get, for any $0<T_1<T<T^*$, the following inequality

$$\int_{T_1}^{T}dt\int_0^{s(t)}u_{xx}^2(x,\ t)+\int_0^{s(T)}u(x,\ T)\leqslant K_1+K_2\int_{T_1}^{T}u_x^3(s(t),\ t)dt, \qquad (3.19)$$

where K_i's $(i=1,\ 2,\ 2\cdots)$ are generic constants depending only on the data and δ_1, δ_2. If (b) is not true, we conclude that

$$\int_{T_1}^{T^*}dt\int_0^{s(t)}u_{xx}^2(x,\ t)dx+\int_0^{S(T^*)}u_x^2(x,\ T^*)dx\leqslant K_3. \qquad (3.20)$$

This means that the solution can be extended to a larger interval $[0,\ \bar{T}]$ with $\bar{T}>T^*$ (see [8]); the proof is completed.

Theorem 2. *If $\|g\|_\infty$ is sufficiently small, then the case (b) is impossible.*

Proof We observe that

$$u_x^2u_t=2((uu_xu_t)_x-(uu_x^2)_t/2-(uu_{xx}u_t))$$

and

$$\begin{aligned}0&=\int_{\Omega'}u_x^2(Du_{xx}-Qu_x-\theta u_t-g)=\int_{T_1}^{T_2}Du_x^3/3\Big|_0^{s(t)}-\int_{\Omega'}Qu_x^3+gu_x^2-\theta u_x^2u_t\\&=\int_{T_1}^{T_2}Du_x^3(s(t),\ t)/3-\int_{T_1}^{T_2}(Du_x^3/3-2\theta uu_xu_t)\big|_{x=0}+\theta\int_0^{S(t)}u(\xi,\ t)u_x^2(\xi,\ t)d\xi\Big|_{T_1}^{T_2}\\&\quad-\int_{\Omega'}(\theta u_x^3+gu_x^2)+\int_{\Omega'}uu_{xx}(Du_{xx}-Qu_x-g),\end{aligned} \qquad (3.21)$$

where

$$\Omega'=(0,\ s(t))\times(T_1,\ T_2),\quad 0<T_1<T_2<\infty.$$

Hence

$$\int_{T_1}^{T_2} u_x^3(s(t),\ t)\,dt \leqslant \int_{\Omega'} -Dvu_{xx}^2 + \varepsilon\int_{\Omega'} u_{xx}^2 + K_1. \tag{3.22}$$

Here, we have used

$$\int_{\Omega'} u_x^3 = \oint_{\partial\Omega'} uu_x^2 - \int_{\Omega'} 2uu_x u_{xx}, \quad w \geqslant 0,$$

(2.25), (3.10), (3.11) and Cauchy inequality.

By virtue of (3.19), (3.22) and noting $0 \leqslant -v \leqslant K_1\|g\|_\infty$, we have

$$\int_{T_1}^{T_2} u_x^3(s(t),\ t) \leqslant K_1. \tag{3.23}$$

So, the case (b) is impossible, because of (3.19) and (3.20).

§ 4. Weak Solution

In this section, we consider the weak solution of PI. First, we introduce the following notations:

$$V = \{v \in H^1(0,\ l)\,|\,v=0, \text{ a.e. on } \Gamma_2\},$$

$$W = \{v \in L^2(0,\ T;\ V) \cap H^1(0,\ T;\ L^2(0,\ l))\,|\,v=0, \text{ a. e. on } (0,\ l)\times\{T\}\},$$

$$H\{\xi\} = \begin{cases} 0, & \xi < 0, \\ (0,\ 1), & \xi = 0, \\ 1, & \xi > 0, \end{cases}$$

$$\tilde{H}(\xi) = \begin{cases} 0, & \xi \leqslant 0 \\ 1, & \xi > 0 \end{cases}, \quad \tilde{\tilde{H}}(\xi) = \begin{cases} 0, & \xi < 0, \\ 1, & \xi \geqslant 0. \end{cases}$$

Noting $g = \lambda_t$, we, naturally, assume the following conditions:

(A1) $\lambda \in L^2(\Omega)$, $\lambda_t \in L^2(\Omega)$; $\lambda(x,\ t) \geqslant \lambda(x,\ 0) = \rho\bar{c}_B > 0$ and $\lambda_t(x,\ t) \geqslant 0$ a. e. in Ω;

(A2) $h \in L^2(0,\ T)$, $h(t) \geqslant h(0) > 0$ a. e. in $(0,\ T)$;

(A3) $h \in H^1(0,\ T)$, $h(t) \geqslant h(0) > 0$ a. e. in $(0,\ T)$;

Definition 4. 1. *A function $u(x,\ t)$ defined a. e. in Ω, will be called a weak solution of PI if $u \in L^2(0,\ T;\ V)$ satisfies the following identity*

$$\iint_\Omega (Du_x - Qu)\phi_x - (\theta u + \lambda\varkappa)\phi_t = \int_0^T (t)\,\phi(0,\ t) \tag{4.1}$$

for all $\phi \in W$, where $\varkappa$ is measurable and $\varkappa \in H(u)$ a. e. in Ω.

Theorem 3. *Under Assumptions (A1) and (A2), there exists a unique weak solution of PI.*

Proof In the same way as in [9] we construct an approximating problem of PI as follows.

Let ε be a positive infinitesimal sequence such that

$$\{H_\varepsilon \in C^\infty(\mathbf{R})\},\ \{\lambda_\varepsilon \in C^\times(\bar{\Omega})\},\ \{h_\varepsilon \in C^\infty(0,\ T)\}$$

and $\{u_{0\varepsilon} \in C^\infty[0,\ l]\}$ satisfy

$$
\begin{aligned}
&H_\varepsilon(\xi)=0 \quad \text{if } \xi\leqslant 0,\\
&H_\varepsilon(\xi)=1 \quad \text{if } \xi>\varepsilon,\\
&0\leqslant H_\varepsilon'(\xi)\leqslant 2/\varepsilon,\\
&\forall \xi\in\mathbb{R},\ \lim_{\varepsilon\to 0} H_\varepsilon(\xi)=\tilde{H}(\xi);
\end{aligned}
\tag{4.2}
$$

$$\lambda_\varepsilon\geqslant\rho\bar{c}_B,\ \lambda_{\varepsilon t}\geqslant 0,\ \lambda_\varepsilon\to\lambda \text{ and } \lambda_{\varepsilon t}\to\lambda_t \text{ in } L^2(\Omega) \text{ strong;} \tag{4.3}$$

$$h_\varepsilon(t)\geqslant h(0)>0,\ h_\varepsilon\to h \text{ in } L^2(0,T) \text{ strong;} \tag{4.4}$$

$$u_{0\varepsilon}=0 \text{ and if } x>\varepsilon,\ -K\leqslant u_{0\varepsilon}'\leqslant 0,\ -Du_{0\varepsilon x}(0)+Qu_{0\varepsilon}(0)=h_\varepsilon(0). \tag{4.5}$$

We introduce the following approximating problem:

P_ε: to find $u_\varepsilon\in C^1(\Omega\cap\Gamma_1)\cap C^2(\Omega)$ such that

$$Du_{\varepsilon xx}-Qu_{\varepsilon x}-(\theta u_\varepsilon+\lambda_\varepsilon H_\varepsilon(u_\varepsilon))_t=0 \quad \text{in } \Omega, \tag{4.6}$$

$$u_\varepsilon=u_{0\varepsilon} \quad \text{on } \Gamma_0, \tag{4.7}$$

$$-Du_{\varepsilon x}+Qu_\varepsilon=h_\varepsilon \quad \text{on } \Gamma_1, \tag{4.8}$$

$$u_\varepsilon=0 \quad \text{on } \Gamma_2. \tag{4.9}$$

For P_ε, there exists uniquely one classical solution u_ε, and equation (4.7) yields

$$\int_0^1\int_0^t (Du_{\varepsilon x}-Qu_\varepsilon)\phi_x+(\theta u_\varepsilon+\lambda_\varepsilon H_\varepsilon(u_\varepsilon))_t\phi=\int_0^t h_\varepsilon(\tau)\phi(0,\tau) \tag{4.10}$$

for all $\phi\in L^2(0,T;V)$.

After taking $\phi=u_\varepsilon$ in (4.10), we obtain

$$\int_0^1\int_0^t u_{\varepsilon x}^2+\int_0^1 u_\varepsilon^2(x,t)dx+\int_0^1\lambda_{\varepsilon t}(x,t)\int_0^{u_\varepsilon(x,t)} H_\varepsilon'(\xi)\xi\, d\xi dx+\int_0^t u_\varepsilon^2(0,t)dt\leqslant K, \tag{4.11}$$

where K depends only on $\|h(t)\|_{L^2(0,T)}$, $\|\lambda_t\|_{L^2(\Omega)}$, $\|\lambda\|_{L^2(\Omega)}$.

Hence

$$\|u_\varepsilon\|_{L^\infty(0,T;L^2(0,1))\cap L^2(0,T;H^1(0,1))}\leqslant K \tag{4.12}$$

and

$$0\leqslant H_\varepsilon(u_\varepsilon)\leqslant 1,\quad \int_0^1\int_0^T H_\varepsilon(u_\varepsilon)dxdt\leqslant K. \tag{4.12'}$$

By (4.11) and (4.12), there exist u and $\varkappa$, such that, passing to subsequences if necessary, as $\varepsilon\to 0$,

$$u_\varepsilon\to u \text{ in } L^\infty(0,T;L^2(0,1)) \text{ weak star and in } L^2(0,T;H^1(0,l)) \text{ weak,} \tag{4.13}$$

$$H_\varepsilon(u_\varepsilon)\to\varkappa \text{ in } L^\infty(\Omega) \text{ weak star and in } L^2(\Omega) \text{ weak.} \tag{4.14}$$

From (4.10), one finds[9]

$$\iint_\Omega (Du_x-Qu)\phi_x-(Du+\lambda\varkappa)\phi_t=\int_0^T h(t)\phi(0,t)dt,\ \forall\phi\in H^1(\Omega). \tag{4.15}$$

In order to complete the proof of existence, it thus remains to show that $\varkappa\in H(u)$. This is done in [9], i. e.,

$$\tilde{H}(u)\leqslant\varkappa\leqslant\tilde{\tilde{H}}(u) \quad \text{a. e. in } \Omega. \tag{4.16}$$

Concerning the uniqueness, we suppose that u_1, $\varkappa_1$ and u_2, $\varkappa_2$ are two weak solutions. Set $w=u_1-u_2$ and

$$\phi=\begin{cases}\mathrm{Exp}\left(-\frac{Q}{D}x\right)\int_{t_1}^{t} w(x,\tau)d\tau, & 0\leqslant t\leqslant t_1,\\ 0, & t_1\leqslant t\leqslant T.\end{cases}$$

Then

$$\iint (Dw_x-Qw)\phi_x-\theta w\phi_t-\lambda(\varkappa_1(u_1)-\varkappa_2(u_2))\phi_t=0. \tag{4.17}$$

Applying (A1) and $(\varkappa_1(u_1)-\varkappa_2(u_2))(u_1-u_2)\geqslant 0$, we get

$$0=\frac{D}{2}\int_0^1 \mathrm{Exp}\left(\frac{Qx}{D}\right)\phi_x^2(x,0)dx+\theta\int_0^1\int_0^{t_1}\mathrm{Exp}\left(-\frac{Q}{D}x\right)\cdot w^2(x,t)dx\,dt$$
$$+\int_0^1\int_0^{t_1}\lambda(\varkappa_1(u_1)-\varkappa_2(u_2))(u_1-u_2)\mathrm{Exp}\left(-\frac{Q}{D}x\right)dx\,dt.$$

Hence, $w=0$ a. e. in Ω.

We proceed to consider the regularity and properties of the weak solution.

Theorem 4. *Under the assumptions (A1) and (A3), the weak solution u of PI belongs to $L^\infty(0,T;H(0,1))\cap H^1(0,T;L^2(0,1))\cap C^{1/2,1/4}(\bar{\Omega})$, and $u\geqslant 0$ in Ω.*

Proof The proof follows from (4.10) by taking $\phi=u_{xt}$, the maximum principle and imbedding theorem.

Lemma 4. 2. *The set $S=\{t\in[0,T]\,|\,u(0,t)>0\}$ is dense in $[0,T]$.*

Proof For any $t=t_0>0$, in (4.10) taking

$$\phi(x,t)=\begin{cases}(\delta-x)^2(t-t_0)^2(t_1-t)^2 & \text{in } (0,\delta)\times(t_0,t_1),\\ 0, & \text{otherwise},\end{cases}$$

we have

$$\int_{t_0}^{t_1} Du(0,t)\delta(t-t_0)^2(t_1-t)^2-\int_0^{\delta}\int_{t_0}^{t_1}(Du-2Qu(\delta-x))(t-t_0)^2(t_1-t)^2$$
$$-\int_0^{\delta}\int_{t_0}^{t_1}(\theta u+\lambda\varkappa)2(\delta-x)^2(t-t_0)(t_0+t_1-2t)(t_1-t)$$
$$=\int_{t_0}^{t_1}h(t)\delta^2(t_1-t)^2(t-t_0)^2. \tag{4.18}$$

Then let $\delta=\eta(t_1-t_0)\,(0<\eta\ll 1)$, (4.18) gives

$$\int_0^1 u(0,t_0+\alpha(t_1-t_0))\alpha^2(1-\alpha)^2d\alpha$$
$$\geqslant\eta(t_1-t_0)\int_0^1 h(t_0+\alpha(t_1-t_0))\alpha^2(1-\alpha)^2d\alpha-K\int_0^1\eta^2(1-\alpha)\alpha|2\alpha-1|d\alpha\cdot(t_1-t_0)$$
$$\geqslant h(0)(t_1-t_0)^2\eta/2>0. \tag{4.19}$$

So, there exists $\xi\in[t_0,t_1]$, such that $u(0,\xi)>0$. Here we have used the fact that η it sufficiently small and the continuity of $u\geqslant 0$ from Theorem 4. In sequel, we also assume (A1) and (A3) hold.

Thoerem 5. *If $h(0)\geqslant Ml\,(M=\max\limits_{\Omega}\lambda_t)$, then the weak solution $u(0,t)>0$, $\forall t\in(0,T)$.*

Proof Let.

$$z=u_s+w, \tag{4.20}$$

where

$$w=-M(l-x-D(1-\mathrm{EXP}(-Q(l-x)/D)/Q)/Q. \tag{4.21}$$

For any $t_0\in(0, T)$, a quick calculation in $\Omega\cap\{t>t_0\}$ gives

$$Dz_{xx}-Qz_x-(\theta+\lambda H'_s(u_s))z_t=\lambda_t H_s(u_s)-M\leqslant 0,$$

$$z(1, t)=0,$$

$$-Dz_x+Qz|_{x=0}=h(t)-Ml\geqslant h(0)-Ml\geqslant 0.$$

By maximum principle, the negative minimum value of z can not be reached in $\bar{\Omega}\cap\{t>t_0\}$, and there exists a point (x_s, t_0) satisfying

$$\min_{\Omega\cap\{t>t_0\}} z=z(x_s, t_0)<0, \tag{4.22}$$

i. e.,

$$z(x, t)\geqslant u_s(x_s, t_0)+w(x_s, t_0), \quad \forall t\in[t_0, T].$$

Hence,

$$u_s(0, t)\geqslant u_s(x_s, t_0)+M(x_s-(\mathrm{EXP}(\alpha(1-x_s))-\mathrm{EXP}(-l\alpha))/\alpha)/Q$$

$$\left(\alpha=\frac{Q}{D}\right). \tag{4.23}$$

Since $0\leqslant x_s\leqslant 1$, there exists x^* such that, passing to subsequences if necessary, as $s\to 0$,

$$x_s\to x^*.$$

If $x^*=0$, then $u(0, t)\geqslant u(0, t_0)$, and if $x^*\neq 0$, then

$$u(0, t)\geqslant M(x^*-(\mathrm{EXP}(-\alpha(1-x^*))-\mathrm{EXP}(-\alpha l))/\alpha)/(\alpha D)>0. \tag{4.24}$$

Applying lemma 4.2, we can choose $t_0\in(0, T)$ with $u(0, t_0)>0$. (4.24) then impl ies $u(0, t)>0$ $\forall t\in(0, T)$.

We conclude this paper by proving the following important property of weak solution.

Theorem 6. *If* $\operatorname{mes}\{x\in(0, 1)\,|\,\lambda(x, 0)>0\}>0$, *then*

$$\operatorname{mes}\{(x, t)\in\Omega\,|\,u(x, t)=0\}>0. \tag{4.25}$$

Proof Suppose $u(x, t)>0$ a. e. in Ω. Then u tatisfies (4.7)—(4.9) and (4.6) with $H_s(u_s)=1$. Thus

$$\iint_\Omega (Du_x-Qu)\phi_x-(\theta u+\lambda)\phi_t-\int_0^l \lambda(x, 0)\phi(x, 0)$$

$$=\int_0^T h(t)\phi(0, t)dt, \quad \forall\phi\in W. \tag{4.26}$$

Comparing (4.26) with (4.15) $(\varkappa=1)$, we obtain

$$\int_0^1 \lambda(x, 0)\phi(x, 0)dx=0, \tag{4.27}$$

and it yields

$$\lambda(x, 0)=0 \quad \text{a. e. in } (0, 1). \tag{4.27}$$

This contradiction leads to the completion of the proof.

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