Chin. Ann. of Math.
11B: 4 (1990), 478—490.

MULTIPLE PERIODIC SOLUTIONS OF ASYMPTOTICALLY LINEAR NONAUTONOMOUS HAMILTONIAN SYSTEMS**

WANG ZHIQIANG (王志强)*

Abstract

This paper studies the existence of periodic solutions for nonautonomous asymptotically linear Hamiltonian systems. By using the Z_p index theory some multiplicity results for nonautonomous systems are given, which generalize some results for autonomous systems due to Amman and Zenhder.

§1. Introduction

In this paper, we consider the existence of multiple periodic solutions of asymptotically linear Hamiltonian system

$$\begin{cases} \dot{z}(t) = JH_z(t, z(t)), \\ z(0) = z(T), \end{cases} \tag{1.1}$$

where $T>0$ is a real number and

$$z = (z_1, \cdots, z_{2n}) \in R^{2n}, \quad \dot{z}(t) = \frac{dz(t)}{dt},$$

$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic structure in R^{2n}, I_n denotes the identity matrix in R^n.

The following assumptions were usually used.

(1) $H \in C^2(R \times R^{2n}, R)$ $H_z(t, 0) = 0$, and

$$H(t+T, z) = H(t, z), \quad \forall z \in R^{2n}, t \in R. \tag{1.2}$$

(2) There is a $\beta > 0$ s. t.

$$-\beta \leqslant H_{zz}(t, z) \leqslant \beta, \quad \forall (t, z) \in R \times R^{2n}.$$

(3) There exists a symmetric and time-independent matrix

$$b_\infty \in L(R^{2n}), \quad \text{s. t.},$$

$$JH_z(t, z) = Jb_\infty z + o(\|z\|), \text{ as } \|z\| \to \infty, \quad \text{uniformly in } t.$$

Because of (3), (1.1) is usually called an asymptotically linear system. From

Manuscript received January 11, 1988.

* Department of Mathematics, Beijing University, Beijing, China.

** This research is supported in part by the National Postdoctoral Science Fund.

(1), θ is an equilibrium point. The aim is to find nontrivial T-periodic solutions. In [1, 2, 4, 5, 9] the existence of nontrivial solutions of (1.1) was given under some additional assumptions on b_0 and b_∞. Also in [2] some existence results of multiple solutions were obtained with the additional assumption that H is either even in z (i. e., $H(t, z)=H(t, -z)$) or of time independence (i. e., $H(t, z)=H(z)$). In [4, 5], an existence result of at least two nontrivial solutions for general H was obtained under some more restricted assumptions on b_0 and b_∞. In the following, we shall give some multiple existence results without the assumption of $H(t, z)=H(t, -z)$ or $H=H(z)$. We assume that there is an integer $p>1$ such that

$$H(t+T/p, z)=H(t, z), \quad \forall (t, z)\in R\times R^{2n}. \tag{1.3}$$

Obviously, (1.3) implies (1.2). Under this condition we can easilysee that (1.1) possesses a Z_p symmetry, i. e., z is a solution of (1.1), $\tilde{z}=Rz$ defined by

$$(Rz)(t)=z(t+T/p) \tag{1.4}$$

is also a solution.

Definition 1.1. *Two solutions z_1, z_2 are called geometrically different if $z_1\neq R^l z_2$ $\forall l\in z$, where $R^l z_2=R(R^{l-1}z_2)$.*

We shall seek geometrically different solutions of (1.1). Under some additional assumptions on b_0, b_∞ and the eigenvalues in the inteval $[-\beta, \beta]$ of the linearization of (1.1) we can give the multiple existence of geometrically different solutions. The method of solving the problem is to use a Z_p-index theory developed in [11, 12]. In [12] we use this index theory to give multiple solutions for a nonautonomous wave equation. In [7], a Z_p index theory was in dependently given to study subharmonic solutions, which is only a special case of our index theory.

Finally, our method can also be used to study second order Hamiltonian systems. Some results and further references in this field can be found in [3, 6, 10, 13].

§ 2. Main Results and Proofs

1. The index $i(b_0, b_\infty, \tau)$

We recall a definition of index $i(b_0, b_\infty, \tau)$ for two symmetric matrices $b_0, b_\infty\in L(R^{2n})$ and a positive number τ introduced in [1, 2].

If $b\in L(R^{2n})$ is symmetric and $\mu\geqslant 0$, we consider the quadratic form on $R^{2n}\times R^{2n}$, defined as

$$2\mu\langle Jx_1, x_2\rangle-\langle bx_1, x_1\rangle-\langle bx_2, x_2\rangle, \tag{2.1}$$

$(x_1, x_2)\in R^{2n}\times R^{2n}$. It is represented by the matrix $Q(\mu, b)\in L(R^{4n})$:

$$Q(\mu,b)=\mu\begin{pmatrix}0 & J^T\\ J & 0\end{pmatrix}-\begin{pmatrix}b & 0\\ 0 & b\end{pmatrix}. \tag{2.2}$$

We denote by $m^+(\cdot)$, $m^\circ(\cdot)$ and $m^-(\cdot)$ the positive, the zero and the negative Morse index of a quadratic form or of a matrix representing this form. Now assume the two matrices $b_0, b_\infty \in L(R^{2n})$ to be symmetric and let $\tau \geqslant 0$. Define two integers $i^\pm = i^\pm(b_0, b_\infty, \tau)$ as follows

$$i^\pm = \frac{1}{2}[m^\pm(Q(0, b_0)) - m^\pm(Q(0, b_\infty))\}$$
$$+\sum_{j=1}\{m^\pm(Q(j\tau, b_0)) - m^\pm(Q(j\tau, b_\infty))\}. \tag{2.3}$$

And set

$$i(b_0, b_\infty, \tau) = \max\{i^+, i^-\} \in Z. \tag{2.4}$$

2. A result with convex nonlinearity

Now, let $T>0$. H satisfies

$(H_1)_p$ $H \in C^2(R\times R^{2n}, R)$, $H_z(t, 0)=0$, and there is an integer $p>1$ such that

$$H(t+T/p, z) = H(t, z), \quad \forall (t, z) \in R\times R^{2n}. \tag{2.5}$$

From now on, we fix p and decompose it as

$$p = p_1^{r_1} \cdots p_s^{r_s},$$

where $p_1 < \cdots < p_s$ are prime factors of p and $r_j > 0$ integers.

For simplicity, we introduce some notations. Let N be the set of all nonnegative integers. $N^* = N\setminus\{0\}$. For $m, n \in N$, $\langle m, n\rangle$ denotes the greatest common divisor of m and n. $m\wedge n = \min\{m, n\}$. For $k, l \in Z$ if $\langle |k|, |l|\rangle = 1$, we call k relatively prime to l. For a real number $\tau \in R$, $[\tau] \triangleq \max\{a\in Z \mid a \leqslant \tau\}$. Now we can state our first result as follows.

Theorem 2.1. *Let H satisfy $(H_1)_p$ and*

(H_2) *$H(t, \cdot)$ is strictly convex for each $t \in [0, T]$;*

(H_3) *$\exists \beta > 0$ such that*

$$-\beta \leqslant H_{zz}(t, z) \leqslant \beta, \quad \forall (t, z) \in R\times R^{2n};$$

(H_4) $JH_z(t, z) = Jb_0 z + 0(\|z\|)$, $\|z\| \to 0$,

$$JH_z(t, z) = Jb_\infty z + 0(\|z\|), \quad \|z\| \to \infty$$

uniformly in t for two symmetric time independent matrices $b_0, b_\infty \in L(R^{2n})$.

(H_5) set $M_\beta = \left\{m \in N^* \,\middle|\, \frac{2m\pi}{T} \leqslant \beta\right\}$, *assume that there exists a ν, $1 \leqslant \nu \leqslant s$ such that $t_\nu^* \in N$ given by*

$$p_\nu^{t_\nu^*} = \max\{\langle p_\nu^{r_\nu}, m\rangle \mid m \in M_\beta\} \tag{2.7}$$

satisfies

$$\left[\frac{\beta T}{2\pi}\right] \cdot nt_\nu^* < r_\nu. \tag{2.8}$$

Then if $i = i\left(b_0, b_\infty, \frac{2\pi}{T}\right) > 0$, there are at least $i/2$ nonconstant geometrically different T-periodic solutions of the system $\dot z = JH_z(t, z)$, provided $\sigma(Jb_\infty) \cap i\frac{2\pi}{T}Z = \varnothing$.

Remark 2.1. Without loss of generality, we assume $\beta \geqslant \frac{2\pi}{T}$, and then $\left[\frac{\beta T}{2\pi}\right] \geqslant 1$. It follows that

$$r_\nu > t_\nu^*.$$

Proof Theorem 2.1 We follow the basic frame worked out in [1, 2] and [4], and pay attention to the appearance of Z_p symmetry. And then we shall use a Z_p index theory developed in [11, 12] to derive the result.

It is known that the operator A defined by $Au = -J\dot{u}$ with domain $\mathscr{D}(A) = \{u \in H^1(0, T; R^{2n}) \mid u(0) = u(T)\}$ is a self-adjoint operator and has a pure point spectrum $\sigma(A) = \frac{2\pi}{T} Z$. And every eigenvalue $\lambda \in \sigma(A)$ has multiplicity $2n$ and the eigenspace $E(\lambda) = \ker(\lambda - A)$ is spanned by the orthogonal basis

$$e^{-i\lambda t}\phi_j, \quad j = 1, 2, \cdots, n,$$

where we have identified R^{2n} as $\mathbf{C}^n$ and $\{\phi_j\}_{j=1}^n$ is the orthogonal basis of $\mathbf{C}^n$. In particular, ker $(A) = \mathbf{C}^n$, it consists of the constant functions.

Now, finding T-periodic solutions of the system $\dot{z} = JH_z(t, z)$ is equivalent to finding solutions of the following operator equation

$$Au = F(u), \quad u \in \mathscr{D}(A), \tag{2.9}$$

where $F(u) = H_z(t, u)$ is Nemytskii operator.

We introduce a Z_p-action on $\mathscr{D}(A) = \{u \in H^1(0, T; R^{2n}) \mid u(0) = u(T)\}$ as follows:

$$u \mapsto (Ru)(t) = u(t + T/p). \tag{2.10}$$

It is obvious that R is a linear action and R is isometric on the inner product of $L^2(0, T; R^{2n})$. By Assumption $(H_1)_p$ equation (2.10) is equivariant under this action, i. e.,

$$ARu = RAu, \quad F(Ru) = RF(u). \tag{2.11}$$

On eigenspace $E(\lambda)$, if $\lambda = \frac{2\pi}{T} m$, $m \in Z$, then

$$R(e^{-i\lambda t}\phi_j) = e^{-im2\pi/p} e^{-i\lambda t}\phi_j, \quad j = 1, 2, \cdots, n. \tag{2.12}$$

From the assumptions, F satisfies $F(\theta) = \theta$ and

$$-\beta\|u - v\|^2 \leqslant \langle F(u) - F(v), u - v\rangle \leqslant \beta\|u - v\|^2, \tag{2.13}$$

for any $u, v \in (0, T; R^{2n})$. The following bounded symmetric operators B_0, B_∞ on $L^2(0. T, R^{2n})$ were introduced in [1, 2]:

$$B_0 u(t) = b_0 u(t), \quad B_\infty u(t) = b_\infty u(t).$$

And the condition $\sigma(Jb_\infty) \cap \frac{i2\pi}{T} Z = \varnothing$ is equivalent to $0 \notin \sigma(A - B_\infty)$.

By the saddle point reduction method [1, 2, 4] and the estimate (2.13) for the nonlinearity F, one can reduce the problem of finding nontrivial solutions of the equation (2.9) to the problem of finding nontrivial critical orbits of a Z_p invariant function defined on the following finite dimensional Z_p invariant subspace $Y \subset PL^2$

$(0, T; R^{2n})$ where P is the projection

$$P=\int_{-\beta}^{\beta} dE_\lambda$$

onto the eigenspace of A belonging to the eigenvalues contained in $(-\beta, \beta)$; here E_λ is the spectral resolution of the operator A. We con assume that $\beta\notin\sigma(A)$. We have the following lemma.

Lemma 2. 1. *There are a function $f\in C^2(Y, R)$ and an injective C^1-map u: $Y\to L^2(0, T; R^{2n})$ satisfying $u(\theta)=\theta$ and $I_{mu}\subset\mathscr{D}(A)$ with the following properties:*

(f_1) *$f(\theta)=0$, $f'(\theta)=0$, and $y\in Y$ is a critical point of f if and only if $u(y)$ is a solution of the equation $Au=F(u)$. f is of the form:*

$$f(y)=\frac{1}{2}\langle Au(y), u(y)\rangle-\int_0^T H(t, u(y(t)))dt.$$

(f_2) *u is Z_p equivariant, i. e., $u(Ry)=Ru(y)$ for any $y\in Y$, and then f is Z_p invariant, i. e.,*

$$f(Ry)=f(y), \quad \forall y\in Y.$$

(f_3) *If $0\notin\sigma(A-B_\infty)$, then f satisfies the Palais-Smale condition.*

(f_4) *B_0 and B_∞ commute with the projection P, and there is a constant $\delta>0$ such that*

$$\frac{1}{2}\langle (A-B_\infty)y, y\rangle-\delta\leqslant f(y)\leqslant\frac{1}{2}\langle (A-B_\infty)y, y\rangle+\delta$$

for every $y\in Y$. Moreover

$$f''(\theta)=(A-B_0)|_Y.$$

Proof The proof of) (f_1), (f_3), (f_4) follows from [1, 2]. And (f_2) follows from the equivariance (2.11). From (2.11) we can prove with a similar argument in [12] that $u(y)$ is Z_p equivariant. And it follows immediately that f is Z_p invariant.

Obviously, the critical orbits of f correspond to the geometrically diferent solutions of (1.1). To find many critical orbits we need a Z_p index theory developed in [11] and [12]. We recall this theory with a slight modification to suffice our need.

Let the coordinates of $\mathbf{C}^a\times R^b$ be given by

$$u=(z_1, \cdots, z_a, z_{a+1}\cdots a_{a+b})$$

with z_{a+j} being real, $j=1, 2, \cdots, b$. A Z_p action on $\mathbf{C}^a\times R^b$ is introduced by

$$u\to Ru=(e^{im_1 2\pi/p}z_1, \cdots, e^{im_a 2\pi/p}z_a, z_{a+1}, \cdots, z_{a+b}), \tag{2.14}$$

where $m_j\neq 0$ are integers, $j=1, 2, \cdots, a$.

Recall that p is given by (2.6). And set

$$E_p=\{n\in N^* \mid \exists n_j\in N\ j=1, 2, \cdots, s, \text{ s. t. } n=p_1^{n_1}\cdots p_s^{n_s}\}. \tag{2.15}$$

For any $n\in E_p$ we have an index map $\sigma_n:\Sigma\to N\cup\{+\infty\}$ (cf. [11,12]), where

$$\Sigma=\{A\subset\mathbf{C}^a\times R^b \mid A \text{ is closed and } RA\subset A\}. \tag{2.16}$$

We denote the greatest comomn divisor and the smallest common multiple of $\{|m_j|\}_{j=1}^{a}$ by M and m respectively, and decompose them as

$$M=M'\cdot p_1^{l_1}\cdots p_s^{l_s}, \tag{2.17}$$

$$m=m'\cdot p_1^{t_1}\cdots p_s^{t_s}, \tag{2.18}$$

where M' and m' are relatively prime to p. And $l_j\in N$, $t_j\in N$. Now we fix $n=p_1^{t_1}\cdots p_s^{t_s}$, then $n\in E_p$.

Lemma 2.2. *Assume that there exists a ν, $1\leqslant\nu\leqslant s$, such that*

$$a(t_\nu-l_\nu)<r_\nu-l_\nu. \tag{2.19}$$

Then σ_n has the following properties:

(i) *If A_1, $A_2\in\Sigma$, and there is a Z_p equivariant map $\psi: A_1\to A_2$, then*

$$\sigma_n(A_1)\leqslant\sigma_n(A_2).$$

(ii) *If A_1, $A_2\in\Sigma$, $\sigma_n(A_1\cup A_2)\leqslant\sigma_n(A_1)+\sigma_n(A_2)$.*

(iii) *If $K\in\Sigma$ is compact, then there is a $\delta>0$ such that*

$$\sigma_n(N_\delta(K))=\sigma_n(K),$$

where $N_\delta(K)=\{u\in C^a\times R^b\,|\,\mathrm{dist}\,(u,\ K)\leqslant\delta\}$,

(iv) *If $K\in\Sigma$ and $K\cap R^b=\varnothing$, then*

$$\sigma_n(K)<+\infty.$$

(v) *For any $u\notin R^b$, $\sigma_n([u])=1$.*

(vi) *If $1<\sigma_n(A)<+\infty$, then A contains infinitly many orbits.*

(vii) *If $W\subset C^a\times R^b$ is a Z_p invariant subspace and $W\cap R^b=\{\theta\}$, then for any bounded invariant open neighbourhood $\Omega\subset W$ of the origin,*

$$\sigma_n(\partial\Omega)=\frac{1}{2}\dim_R W.$$

Proof (i)—(iii) is the same as in [11, 12], and (iv)—(vii) can be proved by an obvious modification of the proof in [12]. We omit it.

Now, the subspace $Y=PL^2(0,\ T;\ R^{2n})$ given previously is a Z_p invariant subspace, and

$$Y=\bigoplus_{|\lambda|\leqslant\beta}E(\lambda). \tag{2.20}$$

Since $\lambda=\frac{2\pi}{T}m$ for a certain $m\in Z$, and the action of Z_p on $E(\lambda)$ is given by (2.12), we know that $E(0)=ker(A)=R^{2n}$ belongs to the fixed points space for the action of Z_p. So we can identify Y as $C^a\times R^b$, $b=2n$, and $a=\dim_C\bigoplus_{0<|\lambda|\leqslant\beta}E(\lambda)$. By (2.12), for $\lambda=\frac{2\pi}{T}m$

$$R(e^{-i\lambda t}\phi_j)=e^{-im2\pi/p}e^{-i\lambda t}\phi_j,\quad j=1,\ 2,\ \cdots,\ n.$$

When we set

$$M=G.\ C.\ D.\ \text{of}\ \left\{m\in N^*\,\Big|\,\frac{2\pi}{T}m\leqslant\beta\right\},$$

it follows that $M=1$. And we set

$$m=S.C.M. \text{ of } \left\{m\in N^* \middle| \frac{2\pi}{T}m\leqslant\beta\right\}$$

and decompose it as

$$m=m'\cdot p_1^{t_1}\cdots p_s^{t_s}, \tag{2.21}$$

where m' is relatively prime to p.

Assumption (H_5) implies that

$$t_\nu=t_\nu^*<r_\nu$$

and from (2.8)

$$\left[\frac{\beta T}{2\pi}\right]\cdot nt_\nu<r_\nu.$$

We may note that

$$a\triangleq\dim_C \bigoplus_{0<|\lambda|<\beta} E(\lambda)=\left[\frac{\beta T}{2\pi}\right]\cdot n. \tag{2.22}$$

So we obtain (note $l_j\equiv 0$, $j=1, 2, \cdots, s$)

$$a(t_\nu-l_\nu)<r_\nu-l_\nu. \tag{2.23}$$

And then, when we fix $n=p_1^{t_1}\cdots p_s^{t_s}$ as in (2.21), σ_n has all the properties of Lemma 2.2 on the space $Y=PL^2(0, T; R^{2n})$.

Now, we need an existence theorem of many critical orbits for Z_p invariant functionals.

For $c\in R$, set

$$K_c=\{u\in Y \mid f(u)=c,\ f'(u)=0\},$$
$$f_c=\{u\in Y \mid f(u)\leqslant c\}.$$

For $k\in N^*$, define

$$A_k=\{A\subset\Sigma \mid \sigma_n(A)\geqslant k\}$$

and if $A_k\neq\varnothing$, define

$$c_k(f)=\inf_{A\in A_k}\sup_{y\in A} f(y). \tag{2.24}$$

Obviously, $c_1(f)\leqslant c_2(f)\leqslant\cdots$.

Lemma 2.3. [12] *For each $k\in N^*$, $c_k(f)$ is a critical value of f provided $-\infty<c_k(f)<+\infty$.*

Lemma 2.4.[12] *Assume that there are k, $l\in N^*$ such that*

$$-\infty<c\triangleq c_k(f)=c_{k+1}(f)=\cdots=c_{k+l}(f)<+\infty.$$

Then $\sigma_n(K_c)\geqslant l+1$. Moreover, if $\sigma_n(K_c)<+\infty$, K_c contains infinitely many orbits.

Lemma 2.5. *Assume f as above with $f(\theta)=0$. Assume that*

(1) *there exist a k-dimensional (real) Z_p invariant subspace $\tilde{Y}$ satisfying $E(0)\cap\tilde{Y}=\{\theta\}$ and a constant α such that*

$$f|_{Y^\perp}\geqslant\alpha,$$

(2) *there exists $A\in A_l$, $l>k$ and $A\cap E(0)=\varnothing$ such that*

$$\sup_{y\in A}f(y)<0.$$

Then $-\infty<c_j(f)<0$, *for* $j=k+1, \cdots, l$. *Moreover,* f *has at least* $l-k$ *critical orbits provided*

$$K_{c_j}\cap E(0)=\varnothing, \quad j=k+1, \cdots, l.$$

Proof It follows from (2) that $c_j<0$, for $1\leqslant j\leqslant l$. To prove $c_{k+1}>-\infty$ we firstly prove that if $A\subset\Sigma$ with $\sigma_n(A)\geqslant k+1$ and $A\cap E(0)=\varnothing$, then

$$A\cap\tilde{Y}^{\perp}=\varnothing.$$

If not, $A\cap\tilde{Y}^{\perp}=\varnothing$. set $y=(y_1, y_2)$, $y_1\in\tilde{Y}$, $y_2\in\tilde{Y}^{\perp}$. Then $\forall y\in A$, $y_1\neq 0$. Therefore when we set Q: $Y\to\tilde{Y}$, the orthogonal projection which is equivariant, we have

$$QA\subset\tilde{Y}\setminus\{\theta\}.$$

Define $\rho:QA\to\tilde{Y}$ by $\rho(y_1)=\frac{y_1}{\|y_1\|}$. Then ρ is also equivariant and

$$\rho(QA)\subset S=\{y_1\in\tilde{Y}\mid\|y_1\|=1\}.$$

Since $\tilde{Y}\cap E(0)=\{\theta\}$, from (i), (vii) of Lemma 2.2, we have

$$\sigma_n(A)\leqslant\sigma_n(\rho(QA))\leqslant\sigma_n(S)=k,$$

a contradiction.

It follows from this and (1) that

$$c_{k+1}\geqslant\alpha>-\infty.$$

Now, if $K_{c_j}\cap E(0)=\varnothing$, $j=k+1, \cdots, l$, by (iv) of Lemma 2.2,

$$\sigma_n(K_{c_j})<+\infty, \quad j=k+1, \cdots, l,$$

since K_{c_j} is compact from the (P. S.) condition.

If $c_{k+1}<\cdots<c_l$, then f has $l-k$ critical values, and then at least $l-k$ critical orbits. If some of $\{c_j\}_{j=k+1}^{l}$ are the same, from lemma 2.4, f has infinitely many critical orbits.

Remark 2.2. The above result is a Z_p version of the similar results in [2, 8] about Z_2 and S^1 symmetry. This result is also a generalization of Theorem 2.3 in [12].

Now, we apply Lemma 2.5 to finish the proof of Theorem 2.1. We first assume $i=i^+\left(b_0, b_\infty; \frac{2\pi}{T}\right)>0$ and apply Lemma 2.5 to $-f$. For $\lambda=j\frac{2\pi}{T}\in\sigma(A)$ with $j\geqslant 1$, there is an orthogonal basis (real) of $E(\lambda)$:

$$\cos\lambda t e_k+\sin\lambda t\, Je_k, \quad k=1, 2, \cdots, 2n, \tag{2.25}$$

where $\{e_k\}_{k=1}^{2n}$ is the standard basis of R^{2n}. And $\{\phi_j\}_{j=1}^{n}$, the complex basis, was obtained by

$$\phi_j=e_j+ie_{n+j}, \quad j=1, 2, \cdots, n.$$

From (2.25), we have

$$E(\lambda)\oplus E(-\lambda)=\{\cos\lambda t y_1+\sin\lambda t\, y_2, 0\leqslant t\leqslant T\mid y_1, y_2\in R^{2n}\}.$$

By a direct calculation, one can notice that the restriction of the operator $A-B_0$ (resp. $A-B_\infty$) onto the subspace $E(\lambda)\oplus E(-\lambda)$ defines a quadratic form which agrees with the quadratic form $Q(\mu, b)$ defined by (2.2) for $\mu=j\frac{2\pi}{T}$ and $b=b_0$ (resp. $b=b_\infty$). From Lemma 12.3 in [1] $\sigma(B0)=\sigma(b_0)$, $\sigma(B_\infty)=\sigma(b_\infty)$. And by assumption (H_3) $\sigma(b_0)\subset[-\beta, \beta]$, $\sigma(b_\infty)\subset[-\beta, \beta]$. So it follows from the definition of $i^\pm$ and the above observations that

$$m^\pm((A-B_0)|_Y)-m^\pm((A-B_\infty)|_Y)=i^\pm\left(b_0, b_\infty, \frac{2\pi}{T}\right).$$

Now, choose $\widetilde{Y}$ = the subspace on which $A-B_\infty$ is positively definite. By (iv) of Lemma 2.1, for $y\in\widetilde{Y}^\perp$,

$$\begin{aligned}-f(y)&\geqslant-\frac{1}{2}\langle(A-B'_\infty)y, y\rangle-\delta\\ &\geqslant\frac{\varepsilon}{2}\|y\|^2-\delta \quad \text{for some } \varepsilon>0\\ &\geqslant-\delta.\end{aligned}$$

From the convexness of H in z we know $b_0>0$, $b_\infty>0$. So it follows that $E(0)\cap\widetilde{Y}=\{\theta\}$. Again set W = the sub-space on which $A-B_0$ is positively definite. From (f_4) of Lemma 2.1, $-f'(\theta)=-(A-B_0)|_Y$. Then we may take $s>0$ small enough so that

$$-f(y)<0 \quad \forall y\in S_s,$$

where $S_s=\{y\in W\mid \|y\|=s\}$.

By $b_0>0$, $E(0)\cap W=\{\theta\}$. By virtue of (vii) of Lemma 2.2,

$$\sigma_n(S_s)=\frac{1}{2}\dim_R W.$$

Now, applying Lemma 2.5, we conclude that

$$-\infty<c_j(-f)<0, \text{ for } \frac{1}{2}\dim_R\widetilde{Y}\leqslant j\leqslant\frac{1}{2}\dim_R W.$$

We need to show $K_{c_j}\cap E(0)=\varnothing$. Let $y_0\in E(0)\cap K_{c_j}(-f)$, then $y_0\in\operatorname{Ker}(A)$. By virtue of Lemma 7 in [2] we know $u(y_0)=y_0$ and $u(y_0)$ is a constant periodic solution, hence a zero point of $H_z(t, z)$. But since H is strictly convex in z we obtain $y_0=0$, hence $u(y_0)=0$, and $f(y_0)=0$. This contradicts $c_j<0$. Therefore $K_{c_j}(-f)\cap E(0)=\varnothing$. Applying Lemma 2.5 again, we prove that $-f$ has at least

$$\begin{aligned}\frac{1}{2}\dim_R W-\frac{1}{2}\dim_R\widetilde{Y}&=\frac{1}{2}[m^+((A-B_0)|_Y)-m^+((A-B_\infty)|_Y)]\\ &=\frac{1}{2}i^+\left(b_0, b_\infty, \frac{2\pi}{T}\right)\end{aligned}$$

critical orbits. By (i) of Lemma 2.1, we obtain at least $\frac{1}{2}i^+\left(b_0, b_\infty, \frac{2\pi}{T}\right)$ geometrically different nonconstant T-periodic solutions of the system $\dot{z}=JH_z(t, z)$.

For the case of $i^-\left(b_0, b_\infty, \frac{2\pi}{T}\right)>0$, we may apply the dual version of Lemma 2.5 to $-f$. We omit the details.

Remark 2.3. Condition (2.8) can be satisfied for the case that there is a ν, $1\leqslant \nu\leqslant s$, s. t.

$$\langle p_\nu, m\rangle=1, \quad \forall m\in M_\beta.$$

In fact, in this case $t_\nu^*=0$.

Corollary 2.1. *Let H satisfy (H_2), (H_3), (H_4). Then there is a $P>0$ such that if H satisfies, in addition,*

$(H_1)_p$ $H\in C^1(R\times R^{2n}, R)$ $H_z(t,0)=0$ *and for a* $p>P$,

$$H(t+T/p, z)=H(t, z), \quad \forall (t, z)\in R\times R^{2n},$$

there are at least $i/2$ nonconstant geometrically different T-periodic solutions of the system $\dot z=JH_z(t, z)$.

Proof Set $M=S.C.M.$ of $\{m\in M_\beta\}$ and write it as $M=q_1^{\theta_1}\cdots q_l^{\theta_l}$, where $q_1<\cdots<q_l$ are prime factors and $\theta_j>0$, $j=1, 2, \cdots, l$. For $p\in N^*$, if there exists a factor $\tilde p\,|\,p$ such that $\langle\tilde p, M\rangle=1$ and H satisfies $(H_1)_p$, then by the above remark we can apply Theorem 2.1 to reach the conclusion. Otherwise, we may assume $p=q_1^{r_1}\cdots q_l^{r_l}$. Set

$$t_j=\left[\frac{\beta T}{2\pi}\right]\cdot n\cdot\theta_j$$

and

$$P=q_1^{t_1}\cdots q_l^{t_l}.$$

Then if $p>P$ with the form $q_1^{r_1}\cdots q_l^{r_l}$, there exists a ν, $1\leqslant\nu\leqslant l$, such that

$$\left[\frac{\beta T}{2\pi}\right]\cdot n\cdot\theta_\nu=t_\nu<r_\nu.$$

Again using Theorem 2.1, we obtain the result.

3. A result without convexness assumption

The next result we will give does not need the convexness of the nonlinearity H.

Theorem 2.2. *Assume that H satisfies*

$(H_1)_p$ $H\in C^2(R\times R^{2n}, R$, $H_z(t,\theta)=0$ *and for a* $p\in N^*$, $p>1$ *with form* (2.6),

$$H\left(t-\frac{T}{p}, z\right)=H(t, z), \quad \forall (t, z)\in R\times R^{2n}.$$

(H_2) $\exists$ *constants* $\alpha<\beta$ *satisfying* $\alpha\cdot\beta>0$ *such that*

$$\alpha\leqslant H_{zz}(t, z)\leqslant\beta, \quad \forall (t, z)\in R\times R^{2n}$$

without loss of generality assume α, $\beta\notin\sigma(A)$.

(H_3) $\exists$ $k\in Z$ *and* $s>0$ *such that*

$$\frac{k\pi}{T}+s\leqslant\frac{H(t, z)}{\|z\|^2}\leqslant\frac{(k+1)\pi}{T}-s, \quad \|z\|\to\infty$$

uniformly in $t\in[0, T]$.

(H_4) *Set* $M_{[\alpha,\beta]}=\left\{m\in Z\;\middle|\;\frac{\alpha T}{2\pi}<m<\frac{\beta T}{2\pi}\right\}$. *Assume that there is a* ν, $1\leqslant\nu\leqslant s$, *such that* t_ν^*, $l_\nu^*\in N$ *given by*

$$p_\nu^{t_\nu^*}=\max\{\langle p_\nu^{r_\nu}, m\rangle\,|\,m\in M_{[\alpha,\beta]}\},$$

$$p_\nu^{l_\nu^*}=\min\{\langle p_\nu^{r_\nu}, m\rangle \mid m\in M_{[\alpha,\beta]}\}$$

satisfy

$$\left(\left[\frac{\beta T}{2\pi}\right]-\left[\frac{\alpha T}{2\pi}\right]\right)\cdot n\cdot (t_\nu^*-l_\nu^*)<r_\nu-l_\nu^*.$$

Then if $H_{zz}(t,\theta)<\frac{2\pi l}{T}$, *for an* $l\in Z$, $l\leqslant k$ (*or* $H_{zz}(t,\theta)>\frac{2\pi l}{T}$, $l\geqslant k+1$) *there are at least* $(k-l+1)n$ (*or* $(l-k)n$) *geometrically different nonconstant* T*-periodic solutions of the system* $\dot z=JH(t,z)$.

Proof The idea is essentially the same as Theorem 2.1. Put

$$P=\int_\alpha^\beta dE_\lambda,\quad Y=PL^2(0,T;R^{2n}),$$

then similar to Lemma 2.1, we have

Lemma 2.6. *There are a function* $f\in C^2(Y,R)$ *and an injective* C^1*-map* $u\colon Y\to L^2(0,T;R^{2n})$ *satisfying* $u(\theta)=\theta$ *and* $\mathrm{Im}\,u\subset\mathscr{D}(A)$ *with the following properties:*

(f_1) $f(\theta)=0$, $f'(\theta)=0$ *and* $y\in Y$ *is a critical point of* f *if and only if* $u(y)$ *is a* T*-periodic solution of* $\dot z=JH_z(t,z)$. f *is of the form*

$$f(y)=\frac{1}{2}\langle Au(y), u(y)\rangle-\int_0^T H(t,u(y(t)))dt.$$

(f_2) u *is* Z_p *equivariant, i.e.,* $u(Ry)=Ru(y)$ *for any* $y\in Y$, *and then* f *is* Z_p *invariant, i. e.,*

$$f(Ry)=f(y),\quad \forall y\in Y.$$

(f_3) f *satisfies* (*P. S*) *condition.*

(f_4) $\exists$ *a constant* $\delta>0$ *such that for any* $y\in Y$,

$$\frac{1}{2}\left\langle\left(A-\frac{2(k+1)\pi}{T}+2\varepsilon\right)y, y\right\rangle-\delta\leqslant f(y)$$
$$\leqslant\frac{1}{2}\left\langle\left(A-\frac{2k\pi}{T}-2\varepsilon\right)y, y\right\rangle+\delta.$$

(f_5) *If* $H_{zz}(t,\theta)<\frac{2l\pi}{T}$ $\left(\text{or } H_{zz}(t,\theta)>\frac{2l\pi}{T}\right)$, *then*

$$f''(\theta)>\left(A-\frac{2l\pi}{T}\right)\Big|_Y\left(\text{or } f''(\theta)<\left(A-\frac{2l\pi}{T}\right)\Big|_Y\right).$$

Now, $Y=\bigoplus_{\alpha<\lambda<\beta}E(\lambda)$. *Obviously* (*assume* $\beta>\alpha>0$),

$$\dim Y=2n\left(\left[\frac{\beta T}{2\pi}\right]-\left[\frac{\alpha T}{2\pi}\right]\right).$$

Since the Z_p action R on $E(\lambda)$ is given by $\left(\text{if } \lambda=m\frac{2\pi}{T},\ m\in Z\right)$

$$R(e^{-i\lambda t}\phi_j)=e^{-2im\pi/p}e^{-i\lambda t}\phi,\quad j=1,2,\cdots,n$$

when we write

$$M=G.C.D. \text{ of } \{|m|\mid m\in M_{[\alpha,\beta]}\},$$
$$m=S.C.M. \text{ of } \{|m|\mid m\in M_{[\alpha,\beta]}\}$$

and decompose them as

$$M=M'\cdot p_1^{l_1}\cdots p_s^{l_s}, \quad m=m'\cdot p_1^{t_1}\cdots p_s^{t_s}.$$

where M' and m' are relatively prime to p, by assumption (H_4),

$$l_\nu=l_\nu^*, \quad t_\nu=t_\nu^*$$

and

$$l_\nu\leqslant t_\nu<r_\nu.$$

So, if we fix $n=p_1^{t_1}\cdots p_s^{t_s}$, by Lemma 2.2, σ_n has all the properties in Lemma 2.2. In this situation, $\mathrm{Fix}_{Z_p}\cap Y=\{\theta\}$. Assume in the assumption $l\leqslant k$. Set

$$\widetilde{Y}=\mathrm{span}\left\{e^{-i\lambda t}\phi_j,\ j=1,\ 2,\ \cdots,\ n\,\Big|\,\lambda\in\left[\frac{(k+1)2\pi}{T},\ \beta\right]\cap\sigma(A)\right\}.$$

Then by (f_4) of Lemma 2.6,

$$-f(y)\,|_{\widetilde{Y}^\perp}\geqslant-\delta>-\infty.$$

And set

$$W=\mathrm{span}\left\{e^{-i\lambda t}\phi_j,\ j=1,\ 2,\cdots,\ n\,|\,\lambda\in\left[\frac{2l\pi}{T},\ \beta\right]\cap\sigma(A)\right\},$$

$$S_\varepsilon=\{y\in W\,|\,\|y\|=\varepsilon\}.$$

Then by (f_5) of Lemma 2.6, for $\varepsilon>0$ small enough

$$-f(y)<0. \quad \forall y\in S_\varepsilon.$$

Applying Lemma 2.5, $-f$ has at least

$$\sigma_n(S_\varepsilon)-\frac{1}{2}\dim_R\widetilde{Y}=\frac{1}{2}\,(\dim_R W-\dim_R\widetilde{Y})=\frac{1}{2}\dim_R\bigoplus_{\frac{2l\pi}{T}\leqslant\lambda\leqslant\frac{2k\pi}{T}}E(\lambda)$$

$$=(k-l+1)n$$

critical orbits which are not zero. And they correspond to the geometrically different nonconstant T-periodic solutions of $\dot{z}=JH_z(t,\ z)$.

Remark 2.4. A result similar to Corollary 2.1 can be obtained.

4. Second order Hamiltonian systems

The same method as above can be used to study the following second order differential equations.

$$\begin{cases}-\ddot{x}=F_x(t,\ x), & x\in R^n,\ n\geqslant1,\\ x(0)=x(2\pi), \\ \dot{x}(0)=\dot{x}(2\pi),\end{cases} \tag{2.26}$$

where $F\in C^2(R\times R^n,\ R)$, $F(t+2\pi,\ x)=F(t,\ x)$, $\forall(t,\ x)\in R\times R^n$.

Theorem 2.3. *Assume that F satisfies*

$(F_1)_p$ *For a certain $p\in N^*$, $p>1$ with form* (2.6),

$$F\left(t+\frac{2\pi}{p},\ x\right)=F(t,\ x), \quad \forall(t,\ x)\in R\times R^n.$$

(F_2) *$\exists$ constants $0<\alpha<\beta$ such that*

$$\alpha\leqslant F_{xx}(t,\ x)\leqslant\beta \quad \forall(t,\ x)\in R\times R^n.$$

(F_3) *$\exists\ k\in N^*$ and $\varepsilon>0$ such that*

$$\frac{1}{2}k^2+\varepsilon\leqslant\frac{F(t,\ x)}{\|x\|^2}\leqslant\frac{1}{2}(k+1)^2-\varepsilon, \quad \|x\|\to\infty.$$

(F_4) *Set* $M_{[\alpha,\beta]}=\{m\in N^*|\alpha\leqslant m^2\leqslant\beta\}$ (*without loss of generality, assume* $\alpha\neq m^2$, $\beta\neq m^2$, $\forall m\in N$. *Assume is a* ν, $1\leqslant\nu\leqslant s$, *such that* t_ν^*, l_ν^* *given by*

$$p_\nu^{t_\nu^*}=\max\{\langle p_\nu^{r_\nu}, m\rangle|m\in M_{[\alpha,\beta]}\},$$

$$p_\nu^{l_\nu^*}=\min\{\langle p_\nu^{r_\nu}, m\rangle|m\in M_{[\alpha,\beta]}\}$$

satisfy

$$([\beta]-[\alpha])\cdot n\cdot(t_\nu^*-l_\nu^*)<r_\nu-l_\nu^*.$$

Then if there is $l\in N^*$ *such that* $F_{xx}(t,\theta)<l^2\leqslant k^2$ (*or* $F_{xx}(t,\theta)>l^2\geqslant(k+1)^2$), *there are at least* $(k-l+1)n$ (or $(l-k)n$) *nonconstant geometricallry different* 2π*-periodic solutions of the system* (2.26).

The proof of this result is quite similar to the proof of Theorem 2.2. We do not give it here in order to shorten the paper.

References

[1] Amann, H. & Zehnder, E., Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations, *Ann. Scuola Norm. Sup. Pisa*, **7**(1980), 539—603.

[2] Amann, H. & Zehnder, E., Periodic solutions of asymptotically linear Hamiltonian systems. *Manuscripta Math.*, **32**, (1980), 149—189.

[3] Benci, V., A geometrical index for the group S^1 and some applications to the study of periodic solutions of O. D. E., *Comm. Pure & Appl. Math.*, **34**(1981), 393—432.

[4] Chang, K. C., Solutions of asymptotically linear operator equations via Morse theory, *Comm. Pure & Appli. Math.*, **34**(1891), 693—712.

[5] Chang, K. C., Morse theory and its applications, Lecture on Seminare de Math., Superieures Univ. de Montreal (1983).

[6] Li, S. J. & Liu, J. Q., Nontrivial critical points for asymptotically quadratic functions (preprint).

[7] Michalek, R. & Tarantello, G., Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems (preprint).

[8] Rabinowitz, P. H., Some minimax theorems and applications to nonlinear partial differential equations, Nonlinear Analysis, (1981), 161—177.

[9] Rabinowitz, P. H., Periodic solutions of Hamiltonian systems: a survey, *SIAM J. Math. Anal.*, **13** (1982), 343—352.

[10] Wang, Z. Q., Multiple solutions for indefinite functionals and applications to asymptotically linear problems (to appear).

[11] Wang, Z. Q., A Z_p index theory (to appear).

[12] Wang, Z. Q., Multiple periodic solutions for a class of nonlinear nonautonomous wave equations (to appear).

[13] Coti Zelati, V., perturbations of secnod order Hamiltonian systems via Morse theory, *Boll. U. M. I. Analysis Fanziouale e Appl. Serie* **VI, IV-C**, 1(1985), 307—322.