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CHROMATIC ENUMERATION FORROOTED OUTERPLANAR MAPS**

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Abstract

Let $G(m, s, t; \lambda)$ be the number of ways of λ-coloring all the rooted nonseparable outerplanar maps which are simple and have the edge number m, the valency s of the root-face, and the valency t of the root-vertex. The chromatic enumerating function

$$g(x, y, z; \lambda) = \sum_{m\geq 1, s\geq 2, t\geq 2} G(m, s, t; \lambda) x^m y^s z^t$$

is determined. Meanwhile, a number of explicit formulae for enumerating this kindof maps in general case and in bipartite case are provided.

§ 1. Introduction

On chromatic enumeration, the first paper which was published in 1973 by W. T. Tutte[1] is for rooted planar triangulations. Ten years later, the author generalized the theory into that for rooted nonseparable planar maps [2]. Very recently, the author also found the functional equation for the chromatic enumeration of rooted cubic planar maps[3]. However, up to now, no explicit expression has ever been attained of any chromatic enumerating function except only for the one of rooted plane trees, which is an easy case to some extent. This paper presents an explicit expression of the chromatic enumerating function for rooted outerplanar maps. In consequence, a number of explicit formulae forenumerating outerplanar maps in general and in bipartite cases are also revealed.

For a set of maps $\mathscr{M}$, rooted of course, we write

$$g_{\mathscr{M}}(x, y, z; \lambda) = \sum_{M\in\mathscr{M}} P(M; \lambda) x^{m(M)} y^{s(M)} z^{t(M)} \tag{1.1}$$

which is said to be chro-enu-function (chromatic enumerating function) of $\mathscr{M}$, or the chromatic sum function of $\mathscr{M}$ in Tutte's terminology, where

$P(M; \lambda)$: the chromatic polynomial of M;

$m(M)$: the edge number of M;

$s(M)$: the valency of the root-face of M;

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$t(M)$: the valency of the root-vertex of M.

Let $\mathscr{A}$ be the set of all rooted nonseparable outerplanar maps without multi-edge. Here, we notice that the vertex map and the loop map are not included in $\mathscr{A}$. The main purpose of this paper is to determine

$$\begin{cases} g_{\mathscr{A}}^{(\lambda)}=g_{\mathscr{A}}(x, y, z; \lambda); & f_{\mathscr{A}}^{(\lambda)}=g_{\mathscr{A}}(x, 1, z; \lambda); \\ d_{\mathscr{A}}^{(\lambda)}=g_{\mathscr{A}}(x, y, 1; \lambda); & h_{\mathscr{A}}^{(\lambda)}=g_{\mathscr{A}}(x, 1, 1; \lambda). \end{cases} \tag{1.2}$$

Now, we introduce two well known formulae on chromatic polynomials of maps for further use. The first one is

$$P(A; \lambda)=P(A-R; \lambda)-P(A\cdot R; \lambda) \tag{1.3}$$

for any $A\in\mathscr{A}$, where R is the root-edge of A, $A-R$ and $A\cdot R$ stand for the resultant maps of deleting and contracting R from A respectively. And, the second is

$$P(A_1\cup A_2; \lambda)=\frac{1}{\lambda(\lambda-1)\cdots(\lambda-i+1)}P(A_1; \lambda)P(A_2; \lambda) \tag{1.4}$$

provided that $A_1\cup A_2=K_i$, the corresponding graph is the complete graph of order i, for A_1, $A_2\in\mathscr{A}$, and $i\geqslant 1$.

§ 2. Functional Equations

For $A\in\mathscr{A}$, Let F_0 and F_1 be the root-face and the non-root-face which is incident to the root-edge R respectively. And, let $B(F)$ denote the boundary of a face F.

Lemma 2. 1. *For $A\in\mathscr{A}$, if A is not the link map, then $A-R$ has at least one cut-vertex, and moreover all the vertices except for exactly the two ends of R on $B(F_1)$ are cut-vertices of $A-R$.*

Proof The first statement is true because of no multi-edges. The second is from the outerplanarity that all the vertices on $B(F_1)$ are also on $B(F_0)$.

According to Lemma 2.1, we may see that $\mathscr{A}$ can be partitioned into the form as follows

$$\mathscr{A}=\mathscr{A}_0+\sum_{k\geqslant 2}\mathscr{A}_k, \tag{2.1}$$

where $\mathscr{A}_0$ consists of only one map, the link map; and $\mathscr{A}_k$ is the subset of all the maps A such that $A-R$ have k blocks, the maximal submaps of $A-R$ which belong to $\mathscr{A}$. For $A\in\mathscr{A}_k$, let $A_0, \cdots, A_{k-1}$ be the blocks in the order of the occurrences of the edges when one moves from the root-vertex to the non-root-end of R along $B(F_1)\cap B(F_0(A-R))$. From Lemma 2.1, $A_i\cap B(F_1)=e_i$ which is an edge and is treated as the root-edge of A_i for $0\leqslant i\leqslant k-1$. Now, we define an operation on two maps M and N as $M*N=M\cup N$ provided that $M\cap N=\{v\}$, where v is the non-root-end of the root-edge of M and is also the root-vertex of N with the properties:

(i) The root-edge of M is chosen to be the root-edge of $M*N$;

(ii) The root-edge of N is the succeeding edge of the root-edge of $M*N$ on $B(F_0(M*N))$.

Further, we denote

$$\bigvee_{0\leqslant i\leqslant k} M_i = M_0 * M_1 * \cdots * M_k. \tag{2.2}$$

In the form (2.2), the operation is done according to the natural order from the left hand side to the right because the operation is neither commutative nor associative. For a set of maps $\mathscr{M}$, let $|\mathscr{M}|$ be the cardinality of $\mathscr{M}$.

Lemma 2. 2. *For $k\geqslant 2$, we have*

$$|A_k| = |\{A \mid A = \bigvee_{0\leqslant i\leqslant k-1} A_i,\ A_i\in\mathscr{A},\ 0\leqslant i\leqslant k-1\}|. \tag{2.3}$$

Proof Let $\mathscr{A}^{*k} = \{A \mid A = \bigvee_{0\leqslant i\leqslant k-1} A_i,\ A_i\in\mathscr{A},\ 0\leqslant i\leqslant k-1\}$. For $A\in\mathscr{A}_k$, from Lemma 2.1, we have $A-R\in\mathscr{A}^{*k}$ by showing that the root-edge of $A-R$ is chosen to be the edge which is on $B(F_1)$ and is incident to the same root-vertex of A. Moreover, we may see that A is well defined from $A-R$ by the inversion. Further, as any $A^*\in\mathscr{A}^{*k}$ can be viewed to be $A-R$, $A\in\mathscr{A}_k$ can be defined by means of adding a new edge R in the root-face of A^*. Thus, a 1-to-1 correspondence between $\mathscr{A}_k$ and $\mathscr{A}^{*k}$ is found. This leads to the Lemma.

Lemma 2. 3. *For $\mathscr{A}_0$, we have*

$$g^{(\lambda)}_{\mathscr{A}_0} = \lambda(\lambda-1)xy^2z. \tag{2.4}$$

Proof By considering that $\mathscr{A}_0$ consists only of the link map (the link map has one edge, the valency of the root-face being 2, and the valency of the root-vertex being 1) and that the chromatic polynomial of the link map is $\lambda(\lambda-1)$, the lemma follows.

For the sake of convenience, let

$$\mathscr{A}_{(0)} = \sum_{k\geqslant 2}\mathscr{A}_k = \mathscr{A} - \mathscr{A}_0. \tag{2.5}$$

And accordingly, let

$$g^{(-)}_{\mathscr{A}_{(0)}} = g^{(-)}_{\mathscr{A}_{(0)}}(x,\ y,\ z;\ \lambda) = \sum_{A\in\mathscr{A}_{(0)}} P(A-R;\ \lambda)x^{m(A)}y^{s(A)}z^{t(A)}; \tag{2.6}$$

$$g^{(\cdot)}_{\mathscr{A}_{(0)}} = g^{(\cdot)}_{\mathscr{A}_{(0)}}(x,\ y,\ z;\ \lambda) = \sum_{A\in\mathscr{A}_{(0)}} P(A\cdot R;\ \lambda)x^{m(A)}y^{s(A)}z^{t(A)}. \tag{2.7}$$

Then, from (1.3), we have

$$g^{(\lambda)}_{\mathscr{A}} = g^{(\lambda)}_{\mathscr{A}_{(0)}} + g^{(-)}_{\mathscr{A}_{(0)}} - g^{(\cdot)}_{\mathscr{A}_{(0)}}. \tag{2.8}$$

Lemma 2. 4. *For $g^{(-)}_{\mathscr{A}_{(0)}}$, we have*

$$g^{(-)}_{\mathscr{A}_{(0)}} = xz\left(\frac{d^{(\lambda)}_{\mathscr{A}}g^{(\lambda)}_{\mathscr{A}}}{\lambda y - d^{(\lambda)}_{\mathscr{A}}}\right). \tag{2.9}$$

Proof By using Lemma 2.2 and the formula (1.4) for $i=1$, we obtain

$$g^{(-)}_{\mathscr{A}_{(0)}} = \sum_{k\geqslant 2}\frac{xyz}{\lambda^{k-1}}\sum_{A_0\in\mathscr{A}} P(A_0;\ \lambda)x^{m(A_0)}y^{s(A_0)-1}z^{t(A_0)}\prod_{i=1}^{k-1}\Big(\sum_{A_i\in\mathscr{A}} P(A_i;\ \lambda)x^{m(A_i)}y^{s(A_i)-1}\Big)$$

$$= \sum_{k\geqslant 2}\frac{xyz}{\lambda^{k-1}}\,y^{-1}g^{(\lambda)}_{\mathscr{A}}(y^{k-1}d^{\lambda}_{\mathscr{A}})^{k-1} = \frac{xyz}{y}\sum_{k\geqslant 2}\left(\frac{1}{\lambda y}d^{(\lambda)}_{\mathscr{A}}\right)^{k-1}g^{(\lambda)}_{\mathscr{A}} = xz\left(\frac{d^{(\lambda)}_{\mathscr{A}}}{\lambda y - d^{(\lambda)}_{\mathscr{A}}}\right)g^{(\lambda)}_{\mathscr{A}}.$$

The lemma is proved.

In order to determine $g^{(\cdot)}_{\mathscr{A}_{(0)}}$, we have to decomposite $\mathscr{A}_{(0)}$ further. Let $\mathscr{A}^{(0)}_{(2)}=\sum_{k\geq 3}\mathscr{A}_k$, then we have

$$\mathscr{A}_{(0)}=\mathscr{A}_2+\mathscr{A}^{(0)}_{(2)}. \tag{2.10}$$

For two maps A_1 and A_2, let $A_1\langle|\rangle A_2$ stand for the resultant map of identifying the root-edge of A_2 with the one of A_1 such that $B(F_0(A_1\langle|\rangle A_2))=(B(F_0(A_1))-R_1)\cup(B(F_0(A_2))-R_2)$. The root-edge of $A_1\langle|\rangle A_2$ is chosen to be the edge incident to the root-vertex with the opposite direction to the root-edge of A_1 on $B(F_0(A_1))$.

Lemma 2. 5. *Let $\mathscr{A}^{(R)}_2=\{A\cdot R\,|\,A\in\mathscr{A}_2\}$, and $\mathscr{A}^{()2}=\{A\,|\,A=A_1\langle|\rangle A_2,\ A_1, A_2\in\mathscr{A}\}$. Then, we have*

$$|\mathscr{A}^{(R)}_2|=|\mathscr{A}^{()2}|. \tag{2.11}$$

Proof For $A\in\mathscr{A}^{(R)}_2$, it is easy to see that the map A' which is obtained by identifying the two muti-edges corresponding to the two non-root-edges on the non-root-face boundary $B(F_1)$ of a map in $\mathscr{A}^{(R)}_2$ into one is in $\mathscr{A}^{()2}$. Conversely, for $A=A_1\langle|\rangle A_2\in\mathscr{A}^{()2}$, the unique map A' which is obtainned by splitting the edge corresponding to the root-edges of A_1 and A_2 into two multi-edges is a member of $\mathscr{A}^{(R)}_2$. Therefore, there is a 1-to-1 correspondence between $\mathscr{A}^{(R)}_2$ and $\mathscr{A}^{()2}$. This implies the lemma.

Lemma 2. 6. *For $\mathscr{A}_2$, we have*

$$g^{(\cdot)}_{\mathscr{A}_2}=\frac{xz}{\lambda(\lambda-1)y}\,g^{(\lambda)}_{\mathscr{A}}d^{(\lambda)}_{\mathscr{A}}. \tag{2.12}$$

Proof By employing Lemma 2.5 and the formula (1.3) for $i=2$, we have

$$\begin{aligned}g^{(\cdot)}_{\mathscr{A}_2}&=\sum_{A\in\mathscr{A}_2}P(A\cdot R;\ \lambda)x^{m(A)}y^{s(A)}z^{t(A)}\\&=xyz\Big(\sum_{B\in\mathscr{A}}P(B;\ \lambda)x^{m(B)}y^{s(B)-1}z^{t(B)}\Big)\frac{x}{\lambda(\lambda-1)}\Big(\sum_{D\in\mathscr{A}}P(D;\ \lambda)x^{m(D)-1}y^{s(D)-1}\Big)\\&=\frac{xz}{\lambda(\lambda-1)y}\,g^{(\lambda)}_{\mathscr{A}}d^{(\lambda)}_{\mathscr{A}}.\end{aligned}$$

The lemma is proved.

Lemma 2. 7. *Let $\mathscr{A}^{(0)(R)}_{(2)}=\{A\cdot R\,|\,A\in\mathscr{A}^{(0)}_{(2)}\}$, then we have*

$$\mathscr{A}^{(0)(R)}_{(2)}=\mathscr{A}. \tag{2.13}$$

Proof First, it is easy to check that the set on the left hand side of (2.13) is a subset of the set on the right hand side. The reasons are that for $A\in\mathscr{A}^{(0)}_{(2)}$, $A\cdot R$ is without multi-edges from $B(F_1)$ having at least four edges, and that $A\cdot R$ is of course a non-separable outerplanar map with the rooting rule as defined before. Moreover, for $A\in\mathscr{A}$, we may also treat $A=A'\cdot R'$, where A' is obtained by splitting the root-vertex of A into v_1 and v_2 with the edge $R'=\langle v_1, v_2\rangle$ as the root-edge of A' in the usual way so that both the valencies of v_1 and v_2 are at least 2 in A'. From Lemma 2.1, we have $A'\in\mathscr{A}^{(1)}_{(2)}$. This implies that the set on the right hand side of

(2.13) is also a subset of the set on the other side. The lemma is proved.

For $A \in \mathscr{A}$, let v be the root-vertex, $(e_0, e_1, \cdots, e_{m(A)-1})$ be the rotation of the edges incident to v such that $e_0 = R$, e_1, the edge which is non-rooted on $B(F_0)$, then $e_2, \cdots, e_{m(A)-1}$ with the order of the occurrences when one moves around v in the determined way. We denote the map obtained by splitting v into v_1 and v_2 with a new edge $\langle v_1, v_2 \rangle$ which is chosen to be the root-edge such that $e_1, \cdots, e_i$ are incident to the root-virtex v_1 by $\delta_i(A)$ for $i = 1, 2, \cdots, m(A)-1$.

Lemma 2. 8. *For $\mathscr{A}_{(2)}^{(0)}$, we have*

$$\mathscr{A}_{(2)}^{(0)} = \sum_{A \in \mathscr{A}} \{\delta_1(A), \delta_2(A), \cdots, \delta_{m(A)-1}(A)\}. \tag{2.14}$$

Proof For $A \in \mathscr{A}_{(2)}^{(0)}$, it is easily seen that A is also a member of $\mathscr{A}'$ which denotes the set on the right hand side of (2.14) from Lemma 2.7. Conversely, by noticing that neither multi-edge nor separable vertex appears in splitting and that the valency of the non-root-face incident to the root-edge is at least four in the resultant map of splitting a map $A \in \mathscr{A}$, we see that all the maps in $\mathscr{A}'$ are in $\mathscr{A}_{(2)}^{(0)}$. Therefore, the lemma is true.

Lemma 2. 9. *For $\mathscr{A}_{(2)}^{(0)}$, we have*

$$g_{\mathscr{A}_{(2)}^{(0)}}^{(\cdot)} = \frac{xyz}{1-z}(zd_{\mathscr{A}}^{(\lambda)} - g_{\mathscr{A}}^{(\lambda)}). \tag{2.15}$$

Proof According to Lemmas 2.8-9, we have

$$g_{\mathscr{A}_{(2)}^{(0)}}^{(\cdot)} = \sum_{A \in \mathscr{A}_{(2)}^{(0)}} P(A \cdot R; \lambda) x^{m(A)} y^{s(A)} z^{t(A)} = xyz \sum_{A \in \mathscr{A}} P(A; \lambda)(z + z^2 + \cdots + z^{t(A)-1}) x^{m(A)} y^{s(A)}$$

$$= xyz^2 \sum_{A \in \mathscr{A}} P(A; \lambda) \left(\frac{1 - z^{t(A)-1}}{1-z} \right) x^{m(A)} y^{s(A)} = \frac{xyz}{1-z}(zd_{\mathscr{A}}^{\lambda} - g_{\mathscr{A}}^{(\lambda)}).$$

This is whatthe lemma means.

Lemma 2. 10. *The function $g_{\mathscr{A}}^{(\lambda)}$ with $d_{\mathscr{A}}^{(\lambda)}$ satisfies the following functional equation:*

$$g_{\mathscr{A}}^{(\lambda)} = \lambda(\lambda-1)xy^2z - \frac{xyz^2}{1-z} d_{\mathscr{A}}^{(\lambda)} + xz \left(\frac{d_{\mathscr{A}}^{(\lambda)}}{\lambda y - d_{\mathscr{A}}^{(\lambda)}} - \frac{d_{\mathscr{A}}^{(\lambda)}}{\lambda(\lambda-1)y} + \frac{y}{1-z} \right) g_{\mathscr{A}}^{(\lambda)}. \tag{2.16}$$

Proof By using(2.8) and (2.10), from Lemmas 2.3, 2.4, 2.6 and 2.9, the theorem can be derived directly.

§ 3. Solution of the Equation

In this section, we investigate the functional equation (2.16) which looks to be a linear equation of $g_{\mathscr{A}}^{(\lambda)}$ but with another function $d_{\mathscr{A}}^{(\lambda)}$ which is still unknown. However, $d_{\mathscr{A}}^{(\lambda)}$ is related to $g_{\mathscr{A}}^{(\lambda)}$. Therefore, it can not be solved directly. Now, we see that Equation (2.16) can be expressed in the following form

$$\left(1-xz\left(\frac{d_{\mathscr{A}}^{(\lambda)}}{\lambda y-d_{\mathscr{A}}^{(\lambda)}}-\frac{d^{(\lambda)}}{\lambda(\lambda-1)y}+\frac{y}{1-z}\right)\right)g_{\mathscr{A}}^{(\lambda)}=\lambda(\lambda-1)xy^2z-\frac{xyz^2}{1-z}d_{\mathscr{A}}^{(\lambda)}. \quad (3.1)$$

In (3.1), if there is $z=\xi$ as a function of x, y in the function space with $\{1, x, y, \cdots, x^iy^j, \cdots\}$ as the basis such that the both sides become zero simultaneously, we may then find the expressions of x and $d_{\mathscr{A}}^{(\lambda)}$ with ξ as a parameter and further determine $d_{\mathscr{A}}^{(\lambda)}$ in the function space. In order to do this, we have to solve the following equations simultaneously:

$$\begin{cases}1-x\xi\left(\dfrac{d_{\mathscr{A}}^{(\lambda)}}{\lambda y-d_{\mathscr{A}}^{(\lambda)}}+\dfrac{1}{\lambda(\lambda-1)y}d_{\mathscr{A}}^{(\lambda)}-\dfrac{xy}{1-\xi}\right)=0;\\ \lambda(\lambda-1)xy^2\xi-\dfrac{xy^2}{1-\xi}d_{\mathscr{A}}^{(\lambda)}=0;\end{cases} \quad (3.2)$$

which are said to be the characteristic equations of equation (2.16), or in short, the ch-equations.

From the second in (3.2), $d_{\mathscr{A}}^{(\lambda)}$ is determined as a function of ξ. Then, by substituting it into the first, we find x as a function of ξ. In consequence, we have

$$\begin{cases}d_{\mathscr{A}}^{(\lambda)}=\lambda(\lambda-1)y\dfrac{1-\xi}{\xi},\\ x=(\xi-1)\left(-y\xi+\dfrac{\lambda-1-\xi}{\lambda-1-\lambda\xi}(1-\xi)^2\right)^{-1}.\end{cases} \quad (3.3)$$

For convenience, let us write

$$D_{\xi=1}^k=\left(\frac{d^k}{d\xi^k}\right)_{\xi=1} \quad (3.4)$$

for $k\geqslant0$, And, let

$$\begin{cases}\alpha(\xi;\lambda)=\beta(\xi,\lambda)\gamma(\xi,\lambda)^{-1};\\ \beta(\xi,\lambda)=\lambda-1-\xi;\\ \gamma(\xi,\lambda)=\lambda-1-\lambda\xi.\end{cases} \quad (3.5)$$

For $p, q, r\geqslant0$, we introduce a combinatorial number

$$\begin{aligned}A_q^p(r;\lambda)&=(-1)^{q+r}D_{\xi=1}^r\left(\frac{\beta(\xi;\lambda)^p}{\gamma(\xi;\lambda)^q}\right)\\ &=\sum_{j=0}^r\lambda^{r-j}(\lambda-1)^{q-p}(\lambda-2)^{p-j}\frac{p!(q+r-j-1)!r!}{(p-j)!(q-1)!j!(r-j)!}.\end{aligned} \quad (3.6)$$

In particular,

$$A_k^k(r;\lambda)=\sum_{j=0}^r\lambda^{r-j}(\lambda-2)^{k-j}\frac{r!k!(k+r-j-1)!}{j!(r-j)!(k-j)!(k-1)!}. \quad (3.7)$$

More specifically, for $\lambda=2$, we have

$$A_k^r(r;2)=\begin{cases}2^{r-k}\dfrac{r!(r-1)!}{(r-k)!(k-1)!}, & 0\leqslant k\leqslant r;\\ 0, & \text{otherwise.}\end{cases} \quad (3.8)$$

Then from (3.3), by employing the Lagrangian inversion, we obtain

$$d_{\mathscr{A}}^{\lambda}=\lambda(\lambda-1)y\sum_{n\geqslant1}\frac{x^n}{n!}D_{\xi=1}^{n-1}\frac{d}{d\xi}\left(\frac{1-\xi}{\xi}\right)(-y\xi+(1-\xi)^2\alpha(\xi;\lambda))^n. \quad (3.9)$$

By the binomial expansion and

$$\frac{d}{d\xi}\left(\frac{1-\xi}{\xi}\right)=-\frac{1}{\xi^2},\tag{3.10}$$

we may find

$$\begin{aligned}d_{\mathscr{A}}^{\lambda}&=\lambda(\lambda-1)y\sum_{n\geq1}\frac{x^n}{n!}\sum_{j=0}^{n}(-1)^{j+1}\binom{n}{j}y^jD_{\xi=1}^{n-1}(\xi^{j-2}(1-\xi)^{2n-2j}\alpha(\xi;\lambda)^{n-j})\\&=\lambda(\lambda-1)y\Big\{xy+\sum_{n\geq2}x^n\sum_{j=\lceil\frac{n+1}{2}\rceil}^{n}\frac{(-1)^{j+1}y^j}{j!(n-j)!}\\&\quad\times\frac{(n-1)!}{(2j-n-1)!}D_{\xi=1}^{2j-n-1}(\xi^{j-2}\alpha(\xi;\lambda)^{n-j})\Big\}.\end{aligned}$$

On account of (3.7), we have

$$\begin{aligned}d_{\mathscr{A}}^{(\lambda)}&=\lambda(\lambda-1)y\Big\{xy+\sum_{m\geq2}\sum_{s=\lceil\frac{m+1}{2}\rceil}^{m}\sum_{j=0}^{2s-m-1}(-1)^j\frac{(m-1)!}{s!(m-s)!j!}\\&\quad\times\frac{(s-2)!}{(2s-m-j-1)!(s-j-2)!}A_{m-s}^{m-s}(2s-m-j-1;\lambda)y^sx^m\Big\}.\end{aligned}\tag{3.11}$$

Now, we may find an explicit expression of the general chro-enu-functon $g_{\mathscr{A}}^{(\lambda)}$ without much difficulty by using the Lagrangian inversion to the form:

$$g_{\mathscr{A}}^{(\lambda)}=\frac{\lambda(\lambda-1)xy^3z\dfrac{\xi-z}{\xi}}{1-z\left(1+xy-x\dfrac{1-\xi}{\xi}\alpha(\xi;\lambda)+xz\dfrac{1-\xi}{\xi}\alpha(\xi;\lambda)\right)}\tag{3.12}$$

which is obtained by substituting (3.3) into (3.1) together with (3.5). However, some complicated calculation would be involved to occupy unnecessary space of this paper if we did it further. Of course, it seems to be a promising way to find a fair simple explicit expression of $g_{\mathscr{A}}^{(\lambda)}$, which has to be left to the reader, by the Lagrangian inversion with three variables x, y, and z at a time based on (3.3) also.

In what follows, we investigate several special cases which are more important for us to estimate the coloring average for this kind of maps thereafter.

First, let us discuss the equation satisfied by $f_{\mathscr{A}}^{(\lambda)}$ and $h_{\mathscr{A}}^{(\lambda)}$ which can be simply obtained by setting $y=1$ in (2.16). That is

$$f_{\mathscr{A}}^{(\lambda)}=\lambda(\lambda-1)xz-\frac{xz^2}{1-z}h_{\mathscr{A}}^{(\lambda)}+xz\left(\frac{h_{\mathscr{A}}^{(\lambda)}}{\lambda-h_{\mathscr{A}}^{(\lambda)}}-\frac{h_{\mathscr{A}}^{(\lambda)}}{\lambda(\lambda-1)}+\frac{1}{1-z}\right)f_{\mathscr{A}}^{(\lambda)}.\tag{3.13}$$

Then, we may also find the parametric expressions of $f_{\mathscr{A}}^{(\lambda)}$ and $h_{\mathscr{A}}^{(\lambda)}$ in the same way as follows:

$$\begin{cases}h_{\mathscr{A}}^{(\lambda)}=\lambda(\lambda-1)\dfrac{1-\xi}{\xi};\\ \xi-1=x\left(-\xi+\dfrac{\lambda-1-\xi}{\lambda-1-\lambda\xi}(1-\xi)^2\right),\end{cases}\tag{3.14}$$

from which, we derive

$$h_{\mathscr{A}}^{(\lambda)}=\lambda(\lambda-1)x+\sum_{m\geq2}H_m(\lambda)x^m,\tag{3.15}$$

where

$$H_m(\lambda)=\sum_{s=\lceil\frac{m+1}{2}\rceil}^{m}\sum_{j=0}^{2s-m-1}(-1)^j\frac{\lambda(\lambda-1)(m-1)!}{s!(m-s)!(s-j-2)!}\times\frac{(s-2)!}{(2s-m-j-1)!}A_{m-s}^{m-s}(2s-m-j-1;\lambda),\tag{3.16}$$

which is the case of (3.11) when $y=1$.

As for $f_{\mathscr{A}}^{(\lambda)}$, we may similarly substitute (3.14) into

$$f_{\mathscr{A}}^{(\lambda)}=\frac{\lambda(\lambda-1)xz(1-\xi^{-1}z)}{1-z-xz\left(1-\frac{1-\xi}{\xi}\frac{(\lambda-1)-\xi}{(\lambda-1)-\lambda\xi}(1-z)\right)}\tag{3.17}$$

to evaluate an explicit formula by the Lagrangian inversion with one or two variables.

Further, we turn out the special cases of $h_{\mathscr{A}}^{(\lambda)}$ when $\lambda=2$, or 3, which mean to determine $H_m(2)$, or $H_m(3)$ for $m\geqslant2$. From (3.8) and (3.16), we have

$$H_m(2)=\sum_{i=0}^{\lfloor\frac{m-1}{3}\rfloor}\binom{m-1}{i}\sum_{j=0}^{m-1-3i}(-1)^j2^{m-3i-j}\binom{m-i-2}{j}\binom{m-2i-j-2}{i-1},\tag{3.18}$$

by the reason which will be shown in the next section,

$$=2\sum_{\substack{i=1\\ m-3i\equiv1(\mathrm{mod}\,2)}}^{\lfloor\frac{m-1}{3}\rfloor}\frac{1}{m}\binom{m}{i}\binom{i+\frac{m-1-3i}{2}-1}{i-1},\tag{3.19}$$

which is with all the terms positive. From (3.7) and (3.16), we have

$$H_m(3)=\sum_{s=\lceil\frac{m+1}{2}\rceil}^{m}\sum_{j=0}^{2s-m-1}(-1)^j\frac{6(m-1)!(s-2)!}{s!(m-s-1)!(s-j-2)!j!}\times\sum_{i=0}^{2s-m-j-1}\frac{3^{2s-m-j-i-1}(s-j-i-2)!}{i!(2s-m-j-i-1)!(m-s-i)!}.\tag{3.20}$$

§ 4. The Case: $\lambda=2$

Because the vertex map does not belong to $\mathscr{A}$, the case: $\lambda=2$ of $h_{\mathscr{A}}^{(\lambda)}$, or more generally, $f_{\mathscr{A}}^{(\lambda)}$ and $g_{\mathscr{A}}^{(\lambda)}$ are in fact corresponding to the case of enumerating rooted nonseparable simple bipartite outerplanar maps. More precisely, let $\mathscr{B}$ be the set of all the maps of this kind and let

$$g_{\mathscr{B}}=g_{\mathscr{B}}(x,y,z)=\sum_{B\in\mathscr{B}}x^{m(B)}y^{s(B)}z^{t(B)},\tag{4.1}$$

similarly, $f_{\mathscr{B}}=g_{\mathscr{B}}(x,1,z)$, $d_{\mathscr{B}}=g_{\mathscr{B}}(x,y,1)$, and $h_{\mathscr{B}}=g_{\mathscr{B}}(x,1,1)$. Then, we have

$$2g_{\mathscr{B}}=g_{\mathscr{A}}^{(2)};\ 2f_{\mathscr{B}}=f_{\mathscr{A}}^{(2)};\ 2d_{\mathscr{B}}=d_{\mathscr{A}}^{(2)};\ 2h_{\mathscr{B}}=h_{\mathscr{A}}^{(2)}\tag{4.2}$$

whenever noticing that any bipartite map here is uniquely 2-colorable.

First, let us partition $\mathscr{B}$ into two parts as

$$\mathscr{B}=\mathscr{B}_0+\mathscr{B}_1,\tag{4.3}$$

where $\mathscr{B}_0$ consists only of one map, the link map.

Lemma 4. 1. *Let* $\mathscr{B}^{(R)}=\{B-R\,|\,B\in\mathscr{B}_1\}$. *Then, we have*

$$\mathscr{B}^{(R)}=\{B\,|\,B=\bigvee_{0\leqslant i\leqslant 2k}B_i,\ B_i\in\mathscr{B},\ 0\leqslant i\leqslant 2k,\ k\geqslant 1\}. \tag{4.4}$$

Proof Similarly to the proof of Lemma 2.2 except for what related to the bipartiteness.

Theorem 4. 2. *For* $\mathscr{B}$, *we have*

$$g_{\mathscr{B}}=xy^2z+xz\left(\frac{d_{\mathscr{B}}^2}{y^2-d_{\mathscr{B}}^2}\right)g_{\mathscr{B}}. \tag{4.5}$$

Proof From (4.3), we have

$$g_{\mathscr{B}}=g_{\mathscr{B}_0}+g_{\mathscr{B}_1}. \tag{4.6}$$

It is easy to see that

$$g_{\mathscr{B}_0}=xy^2z. \tag{4.7}$$

From Lemma 4.1, we derive

$$\begin{aligned} g_{\mathscr{B}_1}&=\sum_{B\in\mathscr{B}_1}x^{m(B)}y^{s(B)}z^{t(B)}\\ &=\sum_{k\geqslant 1}zyx\prod_{i=1}^{2k}\Big(\sum_{B_i\in\mathscr{B}}x^{m(B_i)-1}y^{s(B_i)-1}\Big)\times\Big(\sum_{B_0\in\mathscr{B}}x^{m(B_0)}y^{s(B_0)-1}z^{t(B_0)}\Big)\\ &=xz\left(\frac{d_{\mathscr{B}}^2}{y^2-d_{\mathscr{B}}^2}\right)g_{\mathscr{B}}. \end{aligned} \tag{4.8}$$

By substituting (4.7) and (4.8) into (4.6), the theorem is obtained.

Theorem 4. 3. *we have*

$$d_{\mathscr{B}}=\sum_{m\geqslant 1}x^m\sum_{\substack{s\geqslant\lceil\frac{2m+4}{3}\rceil\\ s\equiv 0(\mathrm{mod}\,2)}}^{m-1}\frac{(m-1)!}{(m-s+1)!\,(s-1)!}\binom{\frac{s}{2}-2}{m-s}y^s. \tag{4.9}$$

Proof From Theorem 4.2, by setting $z=1$ in Equation (4.5), we see

$$d_{\mathscr{B}}=x\left(y^2+\frac{d_{\mathscr{B}}^3}{y^2-d_{\mathscr{B}}^2}\right). \tag{4.10}$$

Then, by using the Lagrangian inversion, the theorem is derived.

Lemma 4. 4. *We have*

$$\left(\frac{d_{\mathscr{B}}^2}{y^2-d_{\mathscr{B}}^2}\right)^i=\sum_{n\geqslant 2i}x^n\sum_{\substack{j=0\\ n\equiv j(\mathrm{mod}2)}}^{\lfloor\frac{n-2i}{3}\rfloor}\frac{2i(n-1)!}{j!\,(n-j)!}\binom{\frac{n-j}{2}}{i+j}y^{n-j}. \tag{4.11}$$

Proof According to (4.10), by the Lagrangian inversion, the lemma can be derived from showing that

$$\frac{d}{d\xi}\left(\frac{\xi^2}{y^2-\xi^2}\right)^i=2iy^2\left(\frac{\xi}{y^2-\xi^2}\right)\left(\frac{\xi^2}{y^2-\xi^2}\right)^{i-1}, \tag{4.12}$$

$$D_{\xi=0}^{n-2i-3j}(y^2-\xi^2)^{-(i+j+1)}=\begin{cases}(n-2i-3j)!\binom{\frac{n-j}{2}}{i+j}y^{-2(1+\frac{n-j}{2})}, & \text{if } n\equiv j(\mathrm{mod}2);\\ 0, & \text{otherwise.}\end{cases} \tag{4.13}$$

Theorem 4. 5. *We have*

$$g_{\mathscr{B}}=xy^2z+\sum_{m\geqslant 2}\sum_{t=2}^{\lfloor\frac{m+2}{s}\rfloor}\sum_{\substack{s=\lceil\frac{2m+4}{t}\rceil\\ s\equiv 0(\mathrm{mod}\,2)}}^{m-t+2}\frac{2(t-1)(m-t-1)!}{(m-t-s+2)!(s-2)!}\binom{\frac{s}{2}-1}{m-s+1}y^s z^t x^m. \tag{4.14}$$

Proof From (4.5), by expansion, we find

$$g_{\mathscr{B}}=xy^2z\sum_{i\geqslant 0}(xz)^i\left(\frac{d_{\mathscr{B}}^2}{y^2-d_{\mathscr{B}}^2}\right)^i. \tag{4.15}$$

Then, from Lemma 4.4, after properly exchanging the summation signs, the theorem can be obtained.

Theorem 4. 6. *We have*

$$f_{\mathscr{B}}=xz+\sum_{m\geqslant 2}\sum_{t=2}^{\lfloor\frac{m+2}{s}\rfloor}\sum_{\substack{s=\lceil\frac{2m+4}{t}\rceil\\ s\equiv 0(\mathrm{mod}\,2)}}^{m-t+2}\frac{2(t-1)(m-t-1)!}{(m-t-s+2)!(s-2)!}\binom{\frac{s}{2}-1}{m-s+1}z^t x^m. \tag{4.16}$$

Proof Obtained by simply setting $y=1$ in (4.14).

Theorem 4. 7. *We have*

$$h_{\mathscr{B}}=x+\sum_{m\geqslant 2}\frac{x^m}{m}\sum_{\substack{i=1\\ m-1\equiv i(\mathrm{mod}\,2)}}^{\lfloor\frac{m-1}{s}\rfloor}\binom{m}{i}\binom{\frac{m-1-i}{2}-1}{i-1}. \tag{4.17}$$

Proof From (4.10), by setting $y=1$, we obtain

$$h_{\mathscr{B}}=x\left(1+\frac{h_{\mathscr{B}}^3}{1-h_{\mathscr{B}}^2}\right). \tag{4.18}$$

From (4.18), the theorem can be derived directly by using the Lagrangian invertion.

Corollary 4. 8. *We have the following identities: for* $m\geqslant 2$,

$$\sum_{t=2}^{\lfloor\frac{m+2}{s}\rfloor}\sum_{\substack{s=\lceil\frac{2m+4}{t}\rceil\\ s\equiv 0(\mathrm{mod}\,2)}}^{m-t+2}\frac{2(t-1)(m-t-1)!}{(m-t-s+2)!(s-2)!}\binom{\frac{s}{2}-1}{m-s+1}$$

$$=\sum_{\substack{s=\lceil\frac{2m+4}{2}\rceil\\ s\equiv 0(\mathrm{mod}\,2)}}^{m-1}\frac{(m-1)!}{(m-s+1)!(s-1)!}\binom{\frac{s}{2}-2}{m-s}=\sum_{\substack{i=1\\ m-1\equiv i(\mathrm{mod}\,2)}}^{\lfloor\frac{m-1}{s}\rfloor}\frac{1}{m}\binom{m}{i}\binom{\frac{m-1-i}{2}-1}{i-1}. \tag{4.19}$$

Proof From (4.16) with $z=1$, and from (4.9) with $y=1$, the identities hold on the basis of Theorem 4.7.

§5 The Case: $\lambda=\infty$

In this section, we discuss

$$Q(M;\ \mu)=\mu^{\nu(M)}P(M;\ \mu^{-1}),$$

where $\nu(M)$ is the vertex number of a map M, instead of $P(M;\lambda)$, the chromatic polynomial. However, we may easily see that $Q(M;\ \mu)$ is also a polynomial of μ, and its constant term is 1 for any map M. In this point, the case of $\lambda=\infty$ is the same as the case of $\mu=0$. For $Q(M;\mu)$, we may also show that

$$Q(M;\mu)=Q(M-R;\mu)-\mu Q(M\cdot R;\mu) \tag{5.1}$$

for any map M except only for the vertex map; and that

$$Q(M;\mu)=\frac{1}{(1-\mu)\cdots(1-k\mu+\mu)}Q(M_1;\mu)Q(M_2;\mu) \tag{5.2}$$

if the corresponding graph of $M_1\cap M_2$ is the complete graph of order k, $k\geqslant 1$. Of course, in our case, $k\geqslant 3$.

Now, we go back to discuss on $\mathscr{A}$. For convenience, we here still write

$$g_{\mathscr{A}}^{(\mu)}=g_{\mathscr{A}}(x,y_z z;\mu)=\sum_{A\in\mathscr{A}}Q(A;\mu)x^{m(A)}y^{s(A)}z^{t(A)}, \tag{5.3}$$

and similarly,

$$\begin{cases} f_{\mathscr{A}}^{(\mu)}=g_{\mathscr{A}}(x,1,z;\mu),\\ d_{\mathscr{A}}^{(\mu)}=g_{\mathscr{A}}(x,y,1;\mu),\\ h_{\mathscr{A}}^{(\mu)}=g_{\mathscr{A}}(x,1,1;\mu).\end{cases} \tag{5.4}$$

Thus, our purpose here is to determine $g_{\mathscr{A}}^{(0)}$, $f_{\mathscr{A}}^{(0)}$, $d_{\mathscr{A}}^{(0)}$, and $h_{\mathscr{A}}^{(0)}$.

Theorem 5. 1. *$g_{\mathscr{A}}^{(0)}$ satisfies the following equation:*

$$g_{\mathscr{A}}^{(0)}=xy^2z+xz\left(\frac{d_{\mathscr{A}}^{(0)}}{y-d_{\mathscr{A}}^{(0)}}\right)g_{\mathscr{A}}^{(0)}. \tag{5.5}$$

Proof It is similar to the discussion in § 2 with $\mu=0$. In this case, we have

$$g_{\mathscr{A}}^{(0)}=g_{\mathscr{A}_0}^{(0)}+g_{\mathscr{A}_{(0)}}^{(-)} \tag{5.6}$$

and $g_{\mathscr{A}_{(0)}}^{(\cdot)}$ disappears from (5.1). It is easily seen that

$$g_{\mathscr{A}_0}^{(0)}=(1-\mu)xy^2z=xy^2z. \tag{5.7}$$

By Lemma 2.2 and (5.2) for $k=1$, we have

$$g_{\mathscr{A}_{(0)}}^{(-)}=xz\left(\frac{d_{\mathscr{A}}^{(0)}}{y-d_{\mathscr{A}}^{(0)}f}\right)g^{(0)}. \tag{5.8}$$

Therefore, from (5.6)-(5.8), the theorem follows.

Theorem 5. 2. *We have*

$$d_{\mathscr{A}}^{(0)}=\sum_{m\geqslant 1}x^m\sum_{s=\lceil\frac{m+3}{2}\rceil}^{m+1}\frac{(m-1)!(s-3)!y^s}{(s-1)!(2s-m-3)!(m-s-1)!(m-s)!} \tag{5.9}$$

Proof From (5.5), by setting $z=1$, we may see

$$d_{\mathscr{A}}^{(0)}=x\left(\frac{(d_{\mathscr{A}}^{(0)})^2}{y-d_{\mathscr{A}}^{(0)}}+y^2\right). \tag{5.10}$$

From (5.10), by using the Lagrangian inversion, the theorem can be obtained.

Lemma 5. 3. *For $n\geqslant 1$, we have*

$$\left(\frac{d_{\mathscr{A}}^{(0)}}{y-d_{\mathscr{A}}^{(0)}}\right)^n=\sum_{j\geqslant n}x^j\sum_{i=\lceil\frac{j+d}{2}\rceil}^{j}\frac{n(j-1)!i!y^i}{i!(j-i)!(2i-n-j)!(j-i-n)!}. \tag{5.11}$$

Proof According to (5.10), by Lagrangian inversion, the lemma can be found.

Theorem 5. 4. *For $g_{\mathscr{A}}^{(0)}$, we have*

$$g_{\mathscr{A}}^{(0)}=xy^2z+\sum_{m\geq 3}\sum_{t=2}^{\lfloor\frac{m+1}{2}\rfloor}\sum_{s=\lceil\frac{m+t}{2}\rceil}^{m-t+2}\frac{(t-1)(m-t-1)!x^my^sz^t}{(m-t-s+2)!(2s-m-3)!(m-s+1)!} \quad (5.12)$$

Proof From (5.5), $g_{\mathscr{A}}^{(0)}$ can be expressed as

$$g_{\mathscr{A}}^{(0)}=xy^2z\sum_{n\geq 0}(zx)^n\left(\frac{d_{\mathscr{A}}^{(0)}}{y-d_{\mathscr{A}}^{(0)}}\right)^n. \quad (5.13)$$

Substituting (5.11) into (5.12), after properly treatments of then idicies, we may obtain the theorem.

Theorem5. 5. *For $f_{\mathscr{A}}^{(0)}$, we have*

$$f_{\mathscr{A}}^{(0)}=xz+\sum_{m\geq 3}\sum_{t=2}^{\lfloor\frac{m+1}{2}\rfloor}\sum_{s=\lceil\frac{m+t}{2}\rceil}^{m-t+2}\frac{(t-1)(m-t-1)!x^mz^t}{(m-t-s+2)!(2s-m-3)!(m-s+1)!}. \quad (5.14)$$

Proof Obtained directly by setting $y=1$ in (5.12).

Theorem 5. 6. *For $h_{\mathscr{A}}^{(0)}$, we have*

$$h_{\mathscr{A}}^{(0)}=\sum_{m\geq 1}\sum_{s=\lceil\frac{m+t}{2}\rceil}^{m+1}\frac{(m-1)!(s-3)!x^m}{(s-1)!(2s-m-3)!(m-s-1)!(m-s)!} \quad (5.15)$$

Proof A direct resultof Theorem 5.2.

Corollary 5. 7. *We have the following identity:*

$$\sum_{t=2}^{\lfloor\frac{m+1}{2}\rfloor}\sum_{s=\lceil\frac{m+t}{2}\rceil}^{m-t+2}\frac{(t-1)(m-t-1)!}{(m-t-s+2)!(2s-m-3)!(m-s+1)!}$$
$$=\sum_{s=\lceil\frac{m+t}{2}\rceil}^{m+1}\frac{(m-1)!(s-2)!}{(s-1)!(2s-m-3)!(m-s-1)!(m-s)!}. \quad (5.16)$$

Proof Obtained from Theorems 5.5—5.6.

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