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ON ORDER RINGS OF SEMI-PRIMARY RINGS

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Abstract

The main results of this paper are stated as follows. Let R be an orderring in the semi-primary ring Q. Suppose that R satisfies the maximal condition for nil right ideals of R, Then we have (i) if an ideal I of R has a finite length as right R-module, then I also has a finite length as left R-module; (ii) denote by $A(R)$ the Artinian radical of R, and N the nil radical of R, then $A(R)+N/N=A(R/N)$, if R satisfies the commutative condition on the zero product of prime ideals of R.

§0. Induction

It is of interest to consider the structure of the Noetherian rings which have Artinian or Noetherian quotient rings [1-8]. In this paper we study on basis of [1, 9—14] the rings with semi-primary quotient rings, which satisfy the maximal condition for nil one-sided ideals. Then we can show that there always exist Artinian radicals in these rings. Forthermore we will give some description of Artinian radicals.

All concepts which are not specially explained in this paper are cited from [1].

§1. On the Length of One-Sided Ideal

We always denote by R the associative ring with the identity and denote by Q the semi-primary quotient ring of R. A ring Q is called semi-primary if the Jacobson radical N' of Q is nilpotent and Q/N' is Artinian. Let I be an ideal of R, let $C(I)$ be the set of all the elements c of R such that $c+I$ is regular element of R/I. In special case $C(0)$ is the set of all the regular elements of R. We say that R satisfies the maximal condition on nil right ideals if every accending chain on nil right ideals $L_1 \subseteq L_2 \subseteq \cdots$ terminates after a finite number of steps. It is clear by [15] that there exists an integer m such that $N^m=(0)$ for the nil radical N of R.

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In this section we study the Artinian radical of an order in the semi-primary ring Q. For the sake of convenience we restate here the Lemma 2.9 in [1]: Let R be an order in the semi-primary ring Q, let N be the nil radical of R, let N' be the Jacobson radical of Q, then $N'=NQ=QN$, R/N is semi-prime Goldie ring, Q/N' is quotient ring of R/N and $C(0)=C(N)$.

After this prelimiary we can state the following

Lemma 1.1. *Let R be an order in the semi-primary ring, let I be an ideal of R such that I contains nil radical N of R. Then IQ is an ideal of Q. If $\mathfrak{p}$ denotes a prime ideal of R, then $\mathfrak{p}Q$ is a prime ideal of Q.*

Proof We prove the first assertion. To do this we need only to show that $c^{-1}IQ=IQ$ for any $c\in C(0)$. Since $C(0)=C(N)$ by Lemma 9.2 in [1], Q/N' is a quotient ring of R/N and Q/N' is u semi-simple Artinian ring. Write $\bar{Q}=Q/N'$, then $\bar{I}\bar{Q}=IQ+N'$ is an ideal of $\bar{Q}$ by Theorem 1.31 in [1]. Thus $\bar{c}^{-1}\bar{I}\bar{Q}=c^{-1}IQ+N' \subseteq \bar{I}\bar{Q}$ for any element $c\in C(0)$. Since $N'\subset IQ$, $c^{-1}IQ\subset IQ$.

Now we prove the last assertion. It is clear that $N\subset\mathfrak{p}$; hence $\mathfrak{p}Q$ is an ideal of Q and $N'\subseteq\mathfrak{p}Q$. Since $\bar{Q}$ is a semi-simple Artinian ring, $\bar{Q}=[\bar{a}_1]\oplus\cdots\oplus[\bar{a}_l]$, where $[\bar{a}_i]$ is a minimal ideal of $\bar{Q}$ generated by $a_i\in R$ for $i=1, 2, \cdots, l$. We denote by (a_i) the principal ideal of R. From $[\bar{a}_i][\bar{a}_j]=[0]$ it follows that $(a_i)(a_j)\subset\mathfrak{p}$. Hence $(a_i)\subset\mathfrak{p}$ or $(a_j)\subset\mathfrak{p}$. Thus $[a_i]\subset\mathfrak{p}Q$ or $[a_j]\subset\mathfrak{p}Q$. This completes the Proof.

Lemma 1.2 *Let R be an order in the semi-primary ring Q, let $\mathfrak{p}$ be a prime ideal of R; then either $\mathfrak{p}$ is minimal prime ideal or $\mathfrak{p}Q=Q$. Suppose that P is a prime ideal of Q; then $P\cap R=\mathfrak{p}$ is prime.*

Proof Since the Jacobson radical N' of Q is nilpotent, $N'=P_1\cap\cdots\cap P_n$, where P_i is a maximal prime ideal of Q. Hence every prime ideal $P\in\{P_1, \cdots, P_n\}$. Since $N=N'\cap R=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n$ with $\mathfrak{p}_i=P_i\cap R$, it is easy to see that every prime ideal $\mathfrak{p}$ of R must contain some $\mathfrak{p}_i$. Clearly $\mathfrak{p}_iQ\subseteq\mathfrak{p}Q$. By Lemma 1 $\mathfrak{p}Q$ is a prime ideal. Hence either $\mathfrak{p}Q=Q$ or $\mathfrak{p}Q=P$ is a proper ideal of Q. From $\bar{Q}=[\bar{a}_1]\oplus\cdots\oplus[\bar{a}_l]$ it follows that $\bar{P}_i=[\bar{a}_1]\oplus\cdots\oplus[\bar{a}_{i-1}]\oplus[\bar{a}_{i+1}]\oplus\cdots\oplus[\bar{a}_l]$ for all $a_i\in R$. Hence $(a_j)\subset\mathfrak{p}_i\subset P$ for $j\neq i$. Clearly $P=P_i$. therefore $\mathfrak{p}\subseteq\mathfrak{p}Q\cap R=P_i\cap R=\mathfrak{p}_i$. This completes the proof.

Corollary 1. *In the preceding expression $N=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n$, where $\mathfrak{p}_i=P_i\cap R$, all $\mathfrak{p}_i$ are different from each other.*

Lemma 1. 3. *Let R be an order in the semi-primary ring Q, let P be a prime ideal of Q and let $\mathfrak{p}=P\cap R$; then $R/\mathfrak{p}$ is a prime Goldie ring and also an order of Q/P. Suppose that $R/\mathfrak{p}$ has minimal one-sided ideal; then $R/\mathfrak{p}$ is a simple Artinian ring.*

Proof It is clear that φ: $r+\mathfrak{p}\to r+P$ is a ring is omorphism from $R/\mathfrak{p}$ into Q/P. Let $c\in C(\mathfrak{p})$, $cq\in P$, $q=rd^{-1}$, $d\in C(\mathfrak{p})$; then $cr=cqd\in\mathfrak{p}$, $r\in\mathfrak{p}$. This proves that $q\in P$. Hence $C(\mathfrak{p})\subset C(P)$. On the other hand, since Q/P is simple Artinian,

$C+P$ is a unit of Q/P for any $c\in C(\mathfrak{p})$. Finally, since any element of Q/P is $q+P = rc^{-1}+P=(r+P)(c^{-1}+P)+P$, where $c\in C(\mathfrak{p})\subset C(P)$, $R/\mathfrak{p}$ is an order of Q/P.

Now we prove the other assertion. Since Q/P is simple Artinian, $R/\mathfrak{p}$ is a prime Goldie ring. By assumption $R/\mathfrak{q}$ has nonzero sole; hence $R/\mathfrak{p}$ is simple Artinian by Theorem 1.4 in [1].

Definition 1. 1. *We say that R satisfies the from left (right) to right (left) finite length condition on nil ideals if and only if every nil ideal of R with a finite lengh as a left (right) ideal has always a finite length as a right (left) ideal.*

Theorem 1.1. *Let R be an order in the semi-primary ring Q, let R satisfy the from left to right finite length condition on nil ideals of R. Suppose that I is an ideal of R such that I has a finite length as a left ideal; then I also has a finite length as a right ideal.*

Proof By assumption I has a descending chain of ideals of R

$$I=I_1\supset I_2\supset\cdots\supset I_n=0$$

such that I_j/I_{j+1} is a minimal ideal of R/I_{j+1}. suffi ces to show that I_j/I_{j+1} has a finite length as a right R/I_{j+1}-module for $j=1, \cdots, n-1$. We prove it by induction. If $j=n-1$, then $I_{j+1}=I_n=(0)$, I_n has a finite length. Assume that I_{j+1} has a finite length as a right ideal; we want to show that I_j has a finite length as a right ideal. To do this we denote $\bar{R}=R/I_{j+1}$, $\bar{I}_j=I_j/I_{j+1}$ and divide the problem in two cases to discuss.

(i) $\bar{I}_j^2=(\bar{0})$. We merely show that $N+I_j=N+I_{j+1}$. In fact, since $I_{j+1}\subset I_j$, it follows that

$$N+I_j=N+I_{j+1}+y_1R+y_2R+\cdots \qquad (1)$$

By Lemma 1.1$(I_j+N)Q$ is an ideal of Q. Hence $(I_j+N)Q=[a_1]+\cdots+[a_m]+N'$, where $[\bar{a}_i]=[a_i]+N'$ is a minimal ideal of Q/N' and $[\bar{a}_i]^2=[\bar{a}_i]$. From $(I_j+N)(I_j+N)\subset I_{j+1}+N$ it follows that

$$(I_j+N)(I_j+N)Q+N'=[a_1]+\cdots+[a_m]+N'=(I_j+N)Q=(I_{j+1}+N)Q. \qquad (2)$$

We take the element y which appeared in expression (1). Then from $y_i\in(I_{j+1}+N)Q$ it follows that $y_i=(x+n)q=(x'+n')d_i^{-1}$, where x, $x'\in I_{j+1}$, n, $n'\in N$ and $d_i\in C(0)$. Since I_{j+1} has a finite length as a left ideal, we have a descending chain on left ideals $I_{j+1}\supseteq I_{j+1}d\supseteq\cdots$ terminating after a finite number of steps, i. e. there exists an integer t such that $I_{j+1}d^t=I_{j+1}d^{t+1}$. Hence $I_{j+1}d=I_{j+1}$. From this it follows that $y_i\in(I_{j+1}d^{-1}+Nd^{-1})\cap R\subseteq I_{j+1}+N$. This proves that $I_{j+1}+N=I_j+N$ and $I_j=I_{j+1}+(N\cap I_j)$. By the hypothesis of the theorem the ideal$N\cap I_j$ has a finite length as a right ideal. On the other hand I_{j+1} has a finite length as a right ideal by the assumption of induction, I_j has a finite length as a right ideal.

(ii) $\bar{I}^2\neq(\bar{0})$. First we prove that $\bar{I}_j^2=\bar{I}_j$. In fact, let $x\in\bar{I}_j$, $x\bar{I}_j=(\bar{0})$; then $(x)_{\bar{R}}\bar{I}_j=(\bar{0})$, where $(x)_{\bar{R}}$ denotes a principal ideal of $\bar{R}$. Since $\bar{I}_j$ is a minimal ideal,

$(x)_{\bar R}=(\bar 0)$. This means that if $x\neq 0$, then $\bar I_j\times\bar I_j=\bar I_j$; hence $\bar I_j$ is a simple ring. It is easy to see that $\bar I_j$ does not contain non-zero nilpotent one-sided ideal. On the other hand $\bar I_j$ has a finite length as a left $\bar R$-module; hence there exists a minimal left ideal $L\subseteq\bar I_j$ of $\bar R$. Choosing $\bar 0\neq x\in L$ we have $\bar I_j x\neq(\bar 0)$. Therefore $\bar I_j x$ is a left ideal of $\bar R$ and $L=\bar I_j x$. Hence $\bar I_j=\sum_i\oplus L_i$, where L_i is aminimal left ideal of $\bar I_j$. Clearly L_i is also a minimal left ideal of $\bar R$, $\bar I_j L_i=L_i$. Thus we have

$$\bar I_i=\sum_{i=1}^{l}\oplus\bar R x_i=\sum_{i=1}^{l}\oplus\bar I_j x_i=\sum_{k=1}^{l}\oplus y_k\bar I_j=\sum_{k=1}^{l}\oplus y_k\bar R, \tag{3}$$

where $\bar I_j x_i=\bar R x_i$, $y_k\bar I_j=y_k\bar R$ are minimal left and right ideals of $\bar R$ respectively. This indicates that $\bar I_j$ has a finite length as a right ideal.

Corollary 1.1. *Let R be an oredr in the semi-primary ring Q. let N be the nil radical of R and N contains no minimal left ideal of R. Suppose that an ideal I of R has a finite length l as a left ideal. Then I also has length l as a right ideal.*

Proof It is enough to show that the case (i) in the proof of Theorem 1, 1 cannot happen. In fact, if the case (i) would happen, then we would have proved in the proof of Theorem 1.1 that $I_j=I_{j+1}+(N\cap I_j)$. But $I_j\cap N$ has a finite length as a left ideal; hence N could contain minimal left ideal of R. This contradicts the assumption of N. Therefore $\bar I_j^2\neq(\bar 0)$. In that case we have formula (3). This proves that $\bar I_j$ has length 1 as both left and right ideals.

Corollary 1. 2. *Let R be an order in the semi-primary ring Q. let N be the nil radical of R. Then every ideal of R with a finite length as a left ideal has also a finite length as a right ideal if and only if R satisfies the from left to right finite length condition on nil ideals.*

Proof The sufficiency of the condition has been proved by Theorem 1.1 and it is obvious that the condition is alsc necessary.

Corollary 1. 3. *Let R be an order in the semi-primary ring Q. such that R satisfies both the from left to right and the right to left finite length conditions on nil ideals. Then every ideal I of R has a finite length as a left ideal if and only if I has a finite length as a right ideal.*

Theorem 1. 2. *Let R be an order in the semi-primary ring Q such that R satisfies maximal condition on nil right ideals. Then every ideal I of R with finite length as a left ideal has a finite length as a right ideal.*

Proof By the hypothesis we have a descending chain on ideals of R, $I=I_1\supset I_2\supset\cdots\supset I_{n-1}\supset I_n=(0)$ such that I_j/I_{j+1} is a minimal ideal of R/I_{j+1}. It is enough to show that I_j/I_{j+1} has a finite length as an R/I_{j+1}-module. By the hypothesis of Theorem 1.1 we may assume that $I\subset N$. Set $\mathfrak{p}=\{r\in R\,|\,I_j r\subset I_{j+1}\}$, it is clear that $\mathfrak{p}$ is a prime ideal of R. Let $\bar R=R/I_{j+1}$, $S=\bar R/\bar{\mathfrak{p}}$ and $Z(\bar I_j)=\{\bar x\in\bar I_j\,|\,\bar x\bar c=0\}$ for some $\bar c\in C(\bar{\mathfrak{p}})$. By Lemma 1.3 S is a prime Goldie ring, since $\bar I_j$ is a right S-module.

Hence for any element $\bar{x}\in Z(\bar{I}_j)$ and $\bar{a}\in S$ there exists an essential right ideal L such that $\bar{a}L\subseteq \mathrm{r}(\bar{x})$, where $\mathrm{r}(\bar{x})$ is the right annihilator of $\bar{x}$ in S. Since L contains an element $\bar{d}\in C(\bar{\mathfrak{p}})$ such that $\bar{x}\bar{a}\bar{d}=\bar{0}$, $\bar{x}\bar{a}\in Z(\bar{I}_j)$. Now we want to show that $Z(\bar{I}_j)=(0)$. If it is not true, then $Z(\bar{I}_j)=\bar{I}_j$, since $\bar{I}_j$ is a minimal ideal. On the other hand $\bar{I}_j$ has a finite length as a left R-module; hence $Z(\bar{I}_j)=\bar{I}_j=\sum_{i=1}^{k}\bar{R}\bar{x}_i$. Therefore for any element $\bar{x}_1$ there exists $\bar{c}\in C(\bar{\mathfrak{p}})$ such that $\bar{x}_1\bar{c}_1=\bar{0}$ and $Z(\bar{I}_j)\bar{c}_1=\bar{R}\bar{x}_2\bar{c}_1+\cdots+\bar{R}\bar{x}_k\bar{c}_1$. Since $\bar{x}^2\bar{c}_1\in Z(\bar{I}_j)$, there exists $\bar{c}_2\in C(\bar{\mathfrak{p}})$ such that $\bar{x}_2\bar{c}_1\bar{c}_2=\bar{0}$. We continue this process and obtain $\bar{c}=\bar{c}_1\bar{c}_2\cdots\bar{c}_k\in C(\bar{\mathfrak{p}})$ such that $Z(\bar{I}_j)\bar{c}=\bar{0}$ i. e. $\bar{I}_i\bar{c}=\bar{0}$. Hence $\bar{c}\in\bar{\mathfrak{p}}$, this is impo-ssible. This indicates that $Z(\bar{I}_j)=(\bar{0})$.

Let $\bar{d}\in C(\bar{\mathfrak{p}})$. Then we have $\bar{I}_j\supseteq\bar{I}_j\bar{d}\supseteq\bar{I}_j\bar{d}^2\supseteq\cdots$ Since $\bar{I}_j$ has a finite length as a left $\bar{R}$-module, there exists an integer t such that $\bar{I}_j\bar{d}^t=\bar{I}_j\bar{d}^{t+1}$. Thus we have $\bar{I}_j=\bar{I}_j\bar{d}$. Let Q' be the quotient ring of S. Then by Goldie theorem Q' is a simple Artinian ring. Hence $\bar{I}_j$ can be regarded as a right Q'-module. It is clear that there exists a simple right Q'-module M such that $\bar{I}_j$ is a direct sum of copies of M. Since R satisfies the maximal condition on nil right ideals of R and $I\subset N$, $\bar{I}_j$ is finitely generated as a right ideals of R. Hence M is finitely generated as a right S-module. Since Q' is simple Artian ring, Q' is finitely generated as a right S-module. Now we want to show that $R/\mathfrak{p}$ is Noetherian rignt $\bar{R}$-module. In fact, $\bar{I}_j=\bar{R}\bar{x}_1+\cdots+\bar{R}\bar{x}_n$. Let $M'=\{\bar{x}_1, \cdots, \bar{x}_n\}$. Then $\bar{\mathfrak{p}}=\mathrm{r}(M')$ is right annihilator of M' in $\bar{R}$. We define f: $\bar{R}\to\bar{x}_1\bar{R}\oplus\cdots\oplus\bar{x}_n\bar{R}$ by $f(\bar{r})=(\bar{x}_1\bar{r}, \cdots, \bar{x}_n\bar{r})$ for all $\bar{r}\in\bar{R}$. Then f is a right $\bar{R}$-module homomorphism and ker $(f)=\mathrm{r}(\bar{x}_1)\cap\cdots\cap\mathrm{r}(\bar{x}_n)=\mathrm{r}(M')$. Hence $\bar{R}/\mathrm{r}(M')=\bar{R}/\bar{\mathfrak{p}}$ can be embedded in $\bar{x}_1\bar{R}\oplus\bar{x}_2\bar{R}\oplus\cdots\oplus\bar{x}_n\bar{R}$. But for each i we have $\bar{x}_i\bar{R}\subset N/I_{j+1}=\bar{N}$. By the hypothesis of the theorem each $\bar{x}_i\bar{R}$ is Noetherian as a right $\bar{R}$-module; hence $\bar{R}/\bar{\mathfrak{p}}$ is Noetherian as a right $\bar{R}$-module. By Lemma 1.29 in [1] $Q'=S=\bar{R}/\bar{\mathfrak{p}}$. This proves that $\bar{I}_j$ has a finite length as a right $\bar{R}$-module. Together with the above results we see that R satisfies the from left to right length condition on nil ideals; therefore the condition of Theorem 1.1 is satisfied. Using Theorem 1.1 we completes the prooof.

Corollary 1. 4. *Let R be an order in the semi-primary ring Q. Suppose that R satisfies the maximal conditions on both nil left and right ideals of R. Then every ideal I of R has a finite length as a left ideal if and only if I has a finite length as a right ideal.*

Proposition 1. 1. *Let R be an order in tue semi-primary ring Q, let I be an ideal of R such that I has a finite length as a left R-module. Let $I=I_1\supset I_2\supset\cdots\supset I_n=(0)$ be a chain of ideals such that I_j/I_{j+1} is a minima ideal of R/I_{j+1}. Suppose that R satisfies the maximal condition on nil right ideals. Then $R/\mathfrak{p}_j$ is a simple Artinian ring, where $p_j=\{r\in R\,|\,I_jr\subset I_{j+1}\}$ for $j=1, \cdots, n-1$.*

Proof Denote $\bar{R}=R/I_{j+1}$, $\bar{I}_j=I_j/I_{j+1}$. We divide the problem in two cases to discuss. (i) $\bar{I}^2=(\bar{0})$. By the proof of Theorem 1.2 $S=\bar{R}/\bar{\mathfrak{p}}_j$ is a simple Artinian ring and $R/\mathfrak{p}_j\cong\bar{R}/\bar{\mathfrak{p}}_j$. This proves the theorem. (ii) $\bar{I}^2\neq(\bar{0})$. By the proof of Theorem 1.1 (ii) $\bar{I}_j$ is a simple ring without non-zero nilpotent one-sided ideals. Let B be a minimal right ideal of $\bar{I}_j$. Then it is well-known that $B=\bar{e}\bar{I}_j$, $\bar{e}^2=\bar{e}\in\bar{I}_j$ and $\bar{K}=\bar{e}\bar{I}_j\bar{e}$ is a division ring. By (3) in the proof of Theorem 1.1 $B=\bar{K}\bar{I}_j=\sum_{i=1}^{t}\bar{K}\bar{\alpha}_i$, where $\bar{\alpha}_i=\bar{e}\bar{\sigma}_i x_i$, and $\bar{\sigma}_i$, $x_i\in\bar{I}_j$. Hence B is a vector space over $\bar{K}$. Since B is finite dimensional, $S=\mathrm{End}_{\bar{K}}A$. This means that S is a simple Artinian ring.

Corollary 1.5. *Let B be an order in the semi-primary ring Q such that R satisfies the from left to right finite length condition on nil ideals. Let I be an ideal of R such that I has a finite length as a left R-module. Denote a chain of ideals by* $I=I_1\supset I_2\supset\cdots\supset I_n=(0)$ *such that* I_j/I_{j+1} *is a minimal ideal of* R/I_{j+1} *for* $j=1,\cdots,n-1$. *Then* $R/\mathfrak{p}_j$ *is a simple Artinian ring, where* $\mathfrak{p}_j=\{r\in R\,|\,I_jr\subset I_{j+1}\}$ *for* $j=1,\cdots,n-1$.

Proposition 1.2. *Let R be a ring with identity, let I be an ideal of R such that I has afi nite length as a left ideal. Denote by X an ideal of R such that I/IX is finitely generated as a right R-module. Then there exists an ideal Y of R such that* $YI\subset IX$ *and R/Y has a finite length as a left R-module.*

Proof Let $\bar{R}=R/IX$ and $\bar{I}=I/IX$. By assumption we have $\bar{I}=\sum_{i=1}^{n}\bar{x}_i\bar{R}$. Denote $M=\{\bar{x}_1,\cdots,\bar{x}_n\}$; then $l(M)=l(\bar{I})$. Define f: $\bar{R}\to\bar{R}\bar{x}_1\oplus\cdots\oplus\bar{R}\bar{x}_n$; then ker $f=l(M)$. Hence $\bar{R}/l(M)$ can be imbedded in $\bar{R}\bar{x}_1\oplus\cdots\oplus\bar{R}\bar{x}_n$ as a left $\bar{R}$-module. Since $\bar{I}$ has a finite length as a left $\bar{R}$-module and $\bar{R}\bar{x}_i\subset\bar{I}$, $\bar{R}/l(M)$ also has a finite length as a left$\bar{R}$-module. Clearly $l(M)\bar{I}=(\bar{0})$. Therefore there exists an ideal Y of R such that $\bar{Y}=l(M)$ and R/Y has a finite length as a left R-module and $YI\subseteq IX$

Corollary 1.6. *Let R be an order in the semi-primary ring Q, let I be an ideal of R such that I has a finite length has a left iedal. Denote by X an ideal of R. Suppose tha R satisfies the from left (right) to right (left) finite length condition on nil ideals. Then there exists an ideal Y of R such that* $YI\subseteq IX$ *and R/Y has a finite lentgh as a left R-module.*

Proof By Theorems 1.1 and 1.2 $I=\sum_{i=1}^{n}x_iR$. Using Proposition 1.2 we obtain the proof.

It is now proper to recall the concept of Artinian radical. We define the Artinian radical $A(R)$ of R to be the sum of all the Artinian right ideals of R. When R is left and right Noe-therian, it is well-known that $A(R)$ is also the sum of all the Artinian left ideals of R, But in genearl this is not the case for a ring which is only right Noetherian.

Theorem 1.3. *Let R be an order in the semi-primary ring Q. If one of the* *following two conditions is astisfied;*

(i) *R satisfies the from both left to right and right to left finite length conditions on nil ideals, and*

(ii) *R satisfies the maximal conditions on nil left and right ideals,*
then $A(R)$ is also the sum of all the Artinian left ideals of R.

Proof By Lemma 9.8 in [1] every Artinian right (left) ideal has a finite length. Applying Theorems 1.1 and 1.2 we get imme-diately the proof.

§ 2. Some Properties of Artinian Radical

Definition 2. 1. *We say that R satisfies the commutative condition on zero product of prime ideals if and only if the zero product $\mathfrak{p}_1\cdots\mathfrak{p}_n=(0)$ of prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ always implies the product $\mathfrak{p}_{i1}\cdots\mathfrak{p}_{in}=(0)$ for any permutation $\{i_1, \cdots, i_n\}$ of $\{1, 2, \cdots, n\}$.*

Theorem 2. 1. *Let R be an order in the semi-primary ring Q such that R satisfies the maximal condition on nil right ideals. Suppose that Q satisfies the commutative condition on zero product of prime ideals. Then $R/\mathfrak{p}$ has a finite length as a right R-module for every prime $\mathfrak{p}$ if and only if the right Artinian radical $A(R)\not\subset\mathfrak{h}$.*

Proof Necessity: Let T the be set of all ideals I' of R such that R/I' has a finite length as a right R-module. Since by assumption $\mathfrak{p}\in T$ and $N=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n$, $N^m=(0)$, we have $\mathfrak{p}_1\cdots\mathfrak{p}_n=(0)$, where $\mathfrak{p}_i$ are prime ideals. It is clear that $\mathfrak{p}\in\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. We first prove that if all $\mathfrak{p}_1, \cdots, \mathfrak{p}_n\in T$ then $R=A(R)$. In fact, let $I=\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}\neq(0)$; then by Lemma 1.3 $R/\mathfrak{p}_k$ is a simple Artinian ring, since $R/\mathfrak{p}_k$ has a finite length as a right R-module. Hence $I=\sum_{i\in\Gamma}\oplus x_iR$ where x_iR is a minimal right ideal of R. It is easy to see that $IQ=\sum_{i\in\Gamma}\oplus x_iQ$. In fact, if $y\in x_iQ\cap\sum_{j\neq i}x_jQ$ then $y=x_iq=zq'$, $z\in\sum_{j\neq i}x_jR$. There exists an element $d\in C(0)$ such that $yd=x_ir=zr'$, $r, r'\in R$. From this it follows that $yd=0$, $y=0$. Put $\bar{Q}=Q/N'$; then $\overline{IQ}=\sum_{j\in\Gamma}\bar{x}_j\bar{Q}=\sum_{j\in\Gamma'}\oplus\bar{x}_j\bar{Q}$, where $\Gamma'\subset\Gamma$. But $\bar{Q}$ is a semi-simple Artinian ring, Γ' is finite. It is easy to see that $\sum_{j\in\Gamma}\oplus x_jQ=\sum_{j\in\Gamma'}\oplus x_jQ+(N'\cap\sum_{j\in\Gamma}x_jQ)$. On the other hand

$$N'\cap\sum_{j\in\Gamma}x_jQ=(N\cap\sum_{j\in\Gamma}x_jR)Q=\sum_{k=1}^{n_0}n_kR)Q\subseteq\sum_{k=1}^{n_0'}x_kQ,$$

where n_0' is an integer; this implies that Γ is a finite set. Therefore

$$\sum_{j\in\Gamma}\oplus x_jQ=\sum_{j=1}^{m'}\oplus x_jQ.$$

This shows that I has a finite length as a right ideal.

We now consider right $R/\mathfrak{p}_{k-1}$-module $\mathfrak{p}_1\cdots\mathfrak{p}_{k-2}/\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}$. Since $R/\mathfrak{p}_{k-1}$ has a finite length as a right R-module, by Lemma 1.3 $\mathfrak{p}_1\cdots\mathfrak{p}_{k-2}=\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}+\sum_{i\in J}y_iR$, where $\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}+y_iR$ is a minimal right ideal of $R/\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}$. Hence $\mathfrak{p}_1\cdots\mathfrak{p}_{k-2}Q=\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q+\sum_{i\in J}y_iQ$. We want to prove that $(\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q+y_iQ)\cap(\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q+\sum_{j\neq i}y_jQ)=\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q$. In fact, let $y=zq+y_iq_i=z'q'+rq^*\in(\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q+y_iQ)\cap(\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q+\sum_{j\neq i}y_jQ)$, $z, z'\in\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}$, $r\in\sum_{j\neq i}y_jR$. We denote $q=sd^{-1}$, $q_i=s_id_i^{-1}$, $q'=s'd'^{-1}$, $q^*=s^*d^*d^{*-1}$; then $y=\tilde{z}d^{-1}+\tilde{y}_id_i^{-1}=\tilde{z}'d'^{-1}+\tilde{r}d^{*-1}$.Since $d^{-1}d_i=c_i=c_ic^{-1}$, $yd_ic=\tilde{z}d^{-1}d_ic+\tilde{y}_ic\in\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}+y_iR$, $yd^*c'\in\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}+\sum_{j\neq i}y_jR$, d_ic, $d^*c'\in C(0)$. There exists an element $c^*\in C(0)$ such that $yc^*\in(\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}+y_iR)\cap(\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}+\sum_{j\neq i}y_jR)=\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}$. Hence $y\in\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q$ as required. On the other hand, there are at most a finite number of y_iQ mentioed above contained in N'. Since $Q=Q/N'$ is a semi-simlpe Artinian ring and since $N'+\mathfrak{p}_1\cdots\mathfrak{p}_{k-2}Q=N'+\mathfrak{p}_1\cdots\mathfrak{p}_{k-2}Q+\sum_{i\in J}y_iQ$,it is easy to see that $J=\{1, \cdots, m\}$ is a finit set this means that $\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q+\sum_{i=1}^{m}y_iQ+N'=p_1\cdots p_{k-2}Q+N'$. Hence $\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q=\sum_{i\in J}y_iQ=(\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q+\sum_{i=1}^{m}y_iQ)+N'\cap(\sum_{i\in J}y_iQ+\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q=(\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q+\sum_{i=1}^{m}y_iQ)+(N\cap(\sum_{i\in J}y_iR+\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}))Q=\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}Q+\sum_{i=1}^{m'}y_iQ$, where m' is an integer. Thus we have

$$\mathfrak{p}_1\cdots\mathfrak{p}_{k-2}=\mathfrak{p}_1\cdots\mathfrak{p}_{k-1}+\sum_{i=1}^{m'}y_iR.$$

Analogically we have

$$\mathfrak{p}_1\cdots\mathfrak{p}_{k-3}=\mathfrak{p}_1\cdots\mathfrak{p}_{k-2}+\sum_{i=1}^{m'}z_iR,$$

where $z_iR+\mathfrak{p}_1\cdots\mathfrak{p}_{k-2}$ is a minimal right ideal of $R/\mathfrak{p}_1\cdots\mathfrak{p}_{k-2}$. We continue in this way to get $R=A(R)$.

Now we consider the case $\mathfrak{p}_1\cdots\mathfrak{p}_k=(0)$. We want to show that $\mathfrak{p}_{i1}\cdots\mathfrak{p}_{ik}=(0)$ for any permutation $\{i_1, \cdots, i_k\}$. In fact we have $\mathfrak{p}_1Q\cdots\mathfrak{p}_kQ=(0)$. But $\mathfrak{p}_iQ$ is a minimal prime ideal of Q; hence it follows that $\mathfrak{p}_{i_1}Q\mathfrak{p}_{i_2}Q\cdots\mathfrak{p}_{i_k}Q=(0)$ by the hypothesis of the theorem. Then $\mathfrak{p}_{i_1}\cdots\mathfrak{p}_{i_k}=(0)$. In this case we can assume that $X=\mathfrak{p}_1\cdots\mathfrak{p}_t$ $\mathfrak{p}_1, \cdots, \mathfrak{p}_t\notin T$, $Y=\mathfrak{p}_{t+1}\cdots\mathfrak{p}_k$, $\mathfrak{p}_{t+1}, \cdots, \mathfrak{p}_k\in T$ and $XY=0$. As above we can show that $X=\mathfrak{p}_1\cdots\mathfrak{p}_t$ has a finite length as a right R-module. Hence $A(R)\not\subset\mathfrak{p}$.

Sufficiency: We first construct the $A(R)$. Let E_0 be right socle of R, let E_1 contains E_0 such that $\bar{E}_1=E_1/E_0$ is right socle of $\bar{R}=R/E_0$. We continue in this way to get a chain $E_0\subset E_1\subset\cdots\subset E_n\subset\cdots$, where $\bar{E}_{n+1}=E_{n+1}/E_n$ is right socle of $\bar{R}=R/E_n$. Thus $A(R)=\bigcup_{i=1,\cdots}E_i$. Suppose that $A(R)\not\subset\mathfrak{p}$; then there exists $a\in A(R)a\notin\mathfrak{p}$, hence we can find E_n such that $a\in E_n$ and $E_n\notin\mathfrak{p}$. Clearly E_n is a right R-module and has a finite length.For the sake of convenience we write $E=E_n$.It is clear that E has a chain of ideals of R: $E=I_1\supset I_2\supset\cdots\supset I_{n-1}\supset I_m=(0)$ such that I_j/I_{j+1} is a minimal ideal of R/I_{j+1}. Since $E\not\subset\mathfrak{p}$, we may assume that $I_{j+1}\subset\mathfrak{p}$, $I_j\not\subset\mathfrak{p}$. Hence $I_j=I_{j+1}+(a_j)$, where (a_j) is a principal ideal of R, $(a_j)\not\subset\mathfrak{p}$. It is easy to see that $\mathfrak{p}_j=\{r\in R\mid I_jr\subseteq I_{j+1}\}=\{r\in R\mid (a_j)r\subseteq I_{j+1}\}$ is a prime ideal. By Proposition 1.1 $R/\mathfrak{p}_j$ is simple Artinian. We want to show that $\mathfrak{p}_j=\mathfrak{p}$. Now we prove this assertion. We can show $(a_j)\cap\mathfrak{p}\subset I_{j+1}$, if it is not true, then we have $I_{j+1}+((a_j)\cap\mathfrak{p})=I_{j+1}+(a_j)=I_j\subset\mathfrak{p}$, this is impossible. Thus $(a_j)\cap\mathfrak{p}\subset I_{j+1}$, $(a_j)\mathfrak{p}\subset I_{j+1}$. Therefore we obtain $\mathfrak{p}\subset\mathfrak{p}_j$. By Corollary 1.1 $\mathfrak{p}=\mathfrak{p}_j$, This completes the proof.

Theorem 2.2. *Let R be an order in the semi-primary ring Q such that R satisfies the maximal condition on nil right ideals. Suppose that Q satisfies the commutative condition on zeroproduct of prime ideals. Then $A(R)+N/N$ is the right Artinian radical of R/N, where $A(R)$ is the right Artinian radical of R and N is the nil radical of R.*

Proof Let A' be the right Artinian radical of R/N, then clearly $A(R)+N/N\subset A'$. We prove the converse. Denote $\bar{R}=R/N$ and by $\bar{L}$ an ideal of $\bar{R}$ such that $\bar{L}$ has a finite length as a right ideal· Thus we have a chain of ideals $\bar{L}=\bar{L}_1\supset\bar{L}_2\supset\cdots\supset\bar{L}_n=(\bar{0})$ such that $\bar{L}_j/\bar{L}_{j+1}$ is a minimal ideal of $\bar{R}/\bar{L}_{j+1}$. Denote $\bar{\mathfrak{p}}_j=\{\bar{r}\in\bar{R}\mid\bar{L}_j\bar{r}\subset\bar{L}_{j+1}\}$. It is clear that $\bar{\mathfrak{p}}_j$ is a prime ideal of $\bar{R}$. By Lemma 9.2 in [1] $\bar{R}$ is an order of $\bar{Q}=Q/N'$ and $\bar{Q}$ is a semi-simple Artinian ring. By Proposition 1.1 we see that $\bar{R}/\bar{\mathfrak{p}}_j$ is a simple Artinian ring, and also $R/\mathfrak{p}_j$ is a simple Artinian ring. Now let $\bar{L}_{n-1}=(\bar{l}_1)$, clearly $(\bar{l}_1)$ is a minimal ideal of $\bar{R}$ and $(\bar{l}_1)^2=(\bar{l}_1)$. Therefore $(\bar{l}_1)\oplus\bar{\mathfrak{p}}_1\bar{R}$, since $\bar{\mathfrak{p}}_1$ is maximal. From this it follows that $\bar{L}_{n-2}=(\bar{L}_1)\oplus\bar{\mathfrak{p}}_1\cap\bar{L}_{n-2}$. It is obviovs that $\bar{\mathfrak{p}}_1\cap\bar{L}_{n-2}$ is a minimal ideal of $\bar{R}$, we denote it by $(\bar{l}_2)$. Thus $\bar{L}_{n-2}=(\bar{l}_1)\oplus(\bar{l}_2)$ with $(\bar{l}_2)^2=(\bar{l}_2)$. Since $(\bar{l}_1)\subset\bar{\mathfrak{p}}_2$, $(\bar{l}_1)\oplus\bar{\mathfrak{p}}_1\cap\bar{\mathfrak{p}}_2=\bar{\mathfrak{p}}_2$, $(\bar{l}_1)\oplus(\bar{l}_2)\ \oplus\bar{\mathfrak{p}}_1\cap\bar{\mathfrak{p}}_2=\bar{R}$. Now we use induction. Assume thao $\bar{L}_{n-j}=(\bar{l}_1)\oplus(\bar{l}_2)\oplus\cdots\oplus(\bar{l}_j)$, where $(\bar{l}_i)^2=(\bar{l}_i)$ is minimal ideal of R. Then $\bar{R}=(\bar{l}_1)\oplus\cdots\oplus(\bar{l}_j)\ \oplus\bar{\mathfrak{p}}_1\cap\cdots\cap\bar{\mathfrak{p}}_j$, $(\bar{l}_i)\oplus\bar{\mathfrak{p}}_i=\bar{R}$ for $i=1,\cdots,j$. Since $(\bar{l}_i)\subset\bar{\mathfrak{p}}_{j+1}$, we have $(\bar{l}_1)\oplus\cdots\oplus(\bar{l}_j)\oplus\bar{\mathfrak{p}}_1\cap\cdots\cap\bar{\mathfrak{p}}_j\cap\bar{\mathfrak{p}}_{j+1}=\bar{\mathfrak{p}}_{j+1}$, $i=1,\cdots,j$. and $\bar{L}_{n-j-1}=(\bar{l}_1)\oplus\cdots\oplus(\bar{l}_j)\oplus\bar{\mathfrak{p}}_1\cap\cdots\cap\bar{\mathfrak{p}}_j\cap\bar{L}_{n-j-1}$. Since $\bar{\mathfrak{p}}_1\cap\cdots\cap\bar{\mathfrak{p}}_j\cap\bar{L}_{n-j-1}$ is a minimal ideal of $\bar{R}$, denoting it by $(\bar{l}_{j+1})$, we have $(\bar{l}_{j+1})\oplus\bar{\mathfrak{p}}_{j+1}=\bar{R}$ wiih $(\bar{l}_{j+1})^2=(\bar{l}_{j+1})$. Hence $\bar{R}=(\bar{l}_1)\oplus\cdots\oplus(\bar{l}_{j+1}(\oplus\bar{\mathfrak{p}}_1\cap\cdots\bar{\mathfrak{p}}\cap\bar{\mathfrak{p}}_{j+1}$, $\bar{L}_{n-j+1}=(\bar{l}_1)\oplus\cdots\oplus(\bar{l}_{j+1})$. This completes the proof of induction. Finally we have

$$\begin{gathered}\bar{L}=\bar{L}_1=(\bar{l}_1)\oplus\cdots\oplus(\bar{l}_{n-1}),\\ \bar{R}=(\bar{l}_1)\oplus\cdots\oplus(\bar{l}_{n-1})\oplus\bar{\mathfrak{p}}_1\cap\cdots\cap\bar{\mathfrak{p}}_{n-1},\\ (\bar{l}_i)\oplus\bar{\mathfrak{p}}_i=\bar{R},\ R/\mathfrak{p}_i \text{ is simple Artinian ring.}\end{gathered}\tag{2.1}$$

By Theorem 2.1 $A(R)\not\subset\mathfrak{p}_i$ for $i=1,\cdots,n-1$. We take any direct summand $(\bar{l}_i)$ of $\bar{L}$ in form (2.1); then by (2.1) $(\bar{l}_i)\oplus\bar{\mathfrak{p}}_i=\bar{R}$, and $R/\mathfrak{p}_i$ is a simple Artinian ring. Therefore by Theorem 2.1 $A\not\subset\mathfrak{p}_i$. Recall the proof in the sufficient part of Theorem 2.1 we had an ideal chain $A(R)=I_1\supset I_2\subset\cdots\supset I_{m-1}\supset I_m=(0)$ such that I_j/I_{j+1} is a minimal ideal of R/I_{j+1}, $I_j=I_{j+1}+(a_j)$. Without loss of generality we may assume that $I_{j+1}\subset\mathfrak{p}_i$, $I_j\not\subset\mathfrak{p}_i$; then $(a_j)\cap\mathfrak{p}_i\subseteq I_{j+1}$, because otherwise we would have $I_j\subset\mathfrak{p}_i$. Hence we have $(a_j)\mathfrak{p}_i\subseteq I_{j+1}$. By Lemma 1.1 $\mathfrak{p}_iQ$ is a prime ideal of Q and $p_iQ\neq Q$. Let $\bar{\mathfrak{p}}_iQ=P$ and let $[a_j]$ be a principal ideal of Q; then clearly $\overline{[a_j]}\not\subset P$. Hence there exists a minimal ideal $\overline{[a'_j]}$ of $\bar{Q}$ such that $\overline{[a'_j]}\subset\overline{[a_j]}$ and $\overline{[a'_j]}\oplus\bar{P}=\bar{Q}$, $a'_j\in(a_j)$, where (a_j) is a principalideal of R. It is clear that $I_j=I_{j+1}+(a'_j)$, $(a'_j)\mathfrak{p}_i\subseteq I_{j+1}$. For the sake of convenience we let $\overline{[a_j]}=\overline{[a'_j]}$. On the other hand, since $\overline{[l_i]}\not\subset\bar{P}$, $\overline{[a_j]}\cap\overline{[l_i]}=\overline{[a_j]}$, because otherwise we would get $\overline{[a_j]}\subset\bar{P}$ or $\overline{[l_i]}\subset\bar{P}$, which is impossible. Thus $\overline{[a_j]}\subseteq\overline{[l_i]}$. From this it follows that $\bar{a}_j=\sum_j\bar{q}_j\bar{l}_i\bar{q}'_j=\bar{d}^{-1}\bar{l}^*_i\bar{c}^{-1}$, where $\bar{d}$, $\bar{c}\in(N)$, $\bar{l}^*\in(\bar{l}_i)$. Since $(\bar{l}_i)$ is a minimal ideal of $\bar{R}$, $(\bar{l}^*_i)=(\bar{l}_i)$. It follows therefore that $\bar{d}\bar{a}_j\bar{c}=I^*_i$, $\overline{[a_j]}=[\bar{l}^*]=\overline{[l_i]}$ and $[a_j]+N'=[l_i]+N'$.

Since $N'=P\cap(\bigcap_{P\neq\mathfrak{p}_k}P_k)$, $N=\mathfrak{p}_i\cap(\bigcap_{\mathfrak{p}_k\neq\mathfrak{p}_i}\mathfrak{p}_k)$, where $\mathfrak{p}_i=P\cap R$, $\mathfrak{p}_k=P_k\cap R$. But $[a_j]\subset P_k$, $(a_j)\subset\mathfrak{p}_k(l_i)\subset\mathfrak{p}_k$ for $\kappa\neq i$. Therefore from $(a_j)+\mathfrak{p}_j=R$ and $(l_i)+\mathfrak{p}_i=R$ it follows that

$$(a_j)+N=(a_j)+\mathfrak{p}_i\cap(\bigcap_{\mathfrak{p}_i\neq\mathfrak{p}_k}\mathfrak{p}_k)=(l_i)+\mathfrak{p}_i\cap(\bigcap_{\mathfrak{p}_i\neq\mathfrak{p}_k}\mathfrak{p}_k)=)l_i)+N. \tag{2.2}$$

Since $(\bar{l}_i)$ is a direct summand of $\bar{L}$ in (2.1), we have $\bar{L}\subseteq A(R)+N/N$ by (2.2). This completes the proof.

References

[1] Chatters, A. W. and Hajarnavis, C. R., Ring with chain conditions, Pitman Advanced Publishing Program, Boston. London. Melbourne, 1980.
[2] Ginn, S. M. and Moss, P. B., *Bull. London Math. Soc.*, **9** (1977), 177—181.
[3] Gordon, R., *J. of Algebra*, **35** (1975), 304—307.
[4] Krause, G. Lenagan, T. H., and Stafford, J. T., *J .of Algebra*, **55** (1978), 154—154.
[5] Lengan, T H., Artinian quotient rings of Macaulay rings, *Lecture notes in Mathematics*, 545(1973), Springer Verlag.
[6] Robson, J. C., *Prob. London Math. Soc*, **17**; 3 (1967), 600—616.
[7] Stafford, J. T., *J. of Algebra*, **47** (1977), 244—267.
[8] Stafford, J. T., *J. of Algebra*, **52** (1978), 218—235.
[9] Lenagan, T. H., *Proc. Amer, Math. Soc.*, **51** (1975), 499—500.
[10] Small, L. W., *J. of Algebra*, **4** (1986), 13—41.
[11] Small. L. W., *J. of Algebra*, **9** (1968), 266—273.
[12] Chatters, A. W., and Robson, J. C. ,(to appear in Comm. in Algebra).
[13] Ginn, S. M. and Moss, P. B., *Bull. Amer. Math. Soc*,. 81 (1975), 709—710.
[14] Ginn, S. M., and Moss, P. P., *Bull. London Math. Soc.*, **9** (1977), 177—181.