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STRONG SOLUTIONS AND PATHWISE CONTROL FOR NON-LINEAR STOCHASTIC SYSTEM WITH POISSON JUMPS IN N-DIMENSIONAL SPACE**

SITU RONG (司徒荣)*

Abstract

The existence of a pathwise unique strong solution for the stochastic differential equation (S. D. E.) with Poisson jumps in n-dimensional space without continuity assumption on drift coefficient, which even can be greater than linear growth, and without Lipschitz condition on diffusion coefficients is obtained. Then the existence of a pathwise stochastic optimal Bang-Bang control for a very much non-linear systemwith Poisson jumps in n-dimensional space is derived. The result is also applied to obtain a maximum likelihood estimate (MLE) of parameter for some continuous, S. D. E. with non-Lipschitz coefficients in n-dimensional space.

§1. Introduction

Recently, some results on the existence of the S. D. E. with non-Lipschitz coefficients in 1-dimensional space have been obtained ([1,2]). But up to now the results in n-dimensional space need more restriction. (For S. D. E. with respect to (w. r. t.) Brownian Motion process (B. M.)[3], and w. r. t. martingale [4,5]. Usually, some monotone condition on the whole coefficients and some continuous, less than linear growth condition on the drift coefficients are required. Moreover, it seems that non-result has been appeared yet for the existence of strong solution to the S. D. E. with Poisson jumps in n-dimensional space without the assumption on the continuity for the drift coefficients, and without condition on the Lipschitzianess of the diffusion coefficients $\sigma(t, x)$ or integrability of $(\partial\sigma/\partial x)^2$ [18,19]. Here we weaken such condition on σ, and the usual monotone condition on the whole coefficients and exclude the continuous assumption on drift, and also replace the less than linear growth condition by a much weakor one on it to get the existence of strong solution for S. D. E. with Poisson jumps in n-dimensional space.

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* Department of Mathematics, Zhongshan University' Guangzhou, Guangdong, China.

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Then the result is used to obtain the existence of a pathwise optimal Bang-Bang stochastic control for a non-linear stochas tic system with Poisson jumps in n-dimensional space, which impliesthe results for the pathwise control on linear system for continuous case and in 1-dimensional space got by [6] and [7], By the way, some useful results on Yamada-Watanabe theorem, pathwise uniqueness, and comparison theorem for such S. D. E. (the last one is in 1-dimentional space) are also obtained. Some interesting examples are also given (See Remark 2 of Theorem 1, Remarks 1^0—5^0 of Theorem 2), At last, the result is also applied to derive a maximum likelihood estimate of parameter for some continuous stochastic system with non-Lipschitz coefficients in n-dimensional space, which is better than [9] and [17] in some sense (In [9] $\sigma=1$, $b_2=0$; in [17] Lipschitz condition is needed: and in both cases it is considered in 1-dimensional space).

§ 2. Strong Solutions and Elliptic Bang-Bang Control

Consider S. D. E. (for $x\in R^n$, $t\in[0, T]$)

$$x_t=x+\int_0^t b(s, x_s)\,ds+\int_0^t \sigma(s, x_s)\,dw_s+\int_0^t\int_Z c(s, x_s, z)\,q(ds, dz), \qquad (2.1)$$

where w_t-n-dimensional standard Brownian Motion process (B. M.),

$$q(ds, dz)=p(ds, dz)-\pi(dz)ds,\ Z=R^n-\{0\}, \pi(dz)=dz/|z|^{n+1}.$$

$p(ds, dz)$-1-dimensional Poisson random point measure with compensator $\pi(dz)ds$, $p(0, Z)=0$, $\sigma(t, x)$-$n\times n$ matrix, defined on $[0, T]\times R^n$, $b(t, x)$, $c(t, x, z)$-n-dimensional vector, defined on $[0, T]\times R^n$ and $[0, T]\times R^n\times Z$, respectively. They are all jointly measurable such that the right hand side of (2.1) makes sense. Let us make some remarks on the assumption first.

Remark 1. Here $\int_0^t$ always means $\int_{(0,t]}$. $p(t, \Gamma)=\int_0^t\int_\Gamma p(ds, dz)$.

Remark 2. Since the compensator of $q(ds, dz)$ is $\pi(dz)ds$, by the proof of Lemma 5.5 in chapter 5 of [9] (pp. 176) for $c(s, x, z)$, which is jointly measurable and for all $x_t(\omega)-\mathscr{F}_t$ measurable such that

$$E\int_0^T\int_Z |c(s, x_s, z)|^2(\pi dz)ds<+\infty,$$

the stochastic integral $\int_0^t\int_Z c(s, x_s, z)\ q(ds, dz)$ is well defined.

Now we introduce some condition for discussing the existence of strong solutions. We say that a Borel measurable function $k(u)>0$, $u>0$, satisfies Condition K. If for any Lebesgue measurable function $x(t)\geqslant 0$, $t\in[0, T]$, let

$$y(t)=\int_0^t k(x(s))ds,$$

$$t_0=\sup(t:\ y(t)=0,\quad t\in[0,T]),$$

if $t_0<T$, then there exists $t\in(t_0, T]$ such that

$$\int_{0+}^{y(t)} ds/k(s)>t-t_0.$$

Example 1. If $k(u)>0$, $u>0$, satisfies

$$\int_{0+} du/k(u)=\infty,$$

then $k(u)$ satisfies condition K. (Cond-K).

For the existence of the pathwise unique strong solution to (2.1) we have

Theorem 1. *Assume that for all $t\in[0,T]$, $x\in R^n$, there exists a constant k_0 such that (denote by $\langle\cdot,\cdot\rangle$ the inner product in R^n, $|\sigma(t,x)|^2=\sum_{i,j=1}^{n}|\sigma_{ij}(t,x)|^2$)*

(i) *there exist constants $d\geqslant 1$ and $m>0$ such that*

$$\langle x, b(t,x)\rangle+|\sigma(t,x)|^2(2n+1)$$
$$\leqslant k_0(1+|x|^2\ln(1+|x|^{2d})\ln(1+\ln(1+\ln(1+|x|^m)))),$$

where $b(t,x)$ and $\sigma(t,x)$ are locally bounded. i. e., for each $r>0$ there exists a constant k_r such that as $|x|\leqslant r$

$$|b(t,x)|+|\sigma(t,x)|\leqslant k_r.$$

(ii) $\int_Z |c(t,x,x)|^i\pi(dz)\leqslant k_0,\ i=1, 2.$

$$\lambda^*\sigma\sigma^*\lambda\geqslant\delta|\lambda|^2,\quad \text{for all } \lambda\in R^n$$

where $\delta>0$ is a constant, $\sigma\sigma^*(t,x)$ is continuous for x uniformly w. r. t. $t\in[0,T]$, and σ is the symmetric positive definite square root of $\sigma\sigma^*$,

(iii) *there exist $d^N(t,\omega)\geqslant 0$ and $k^N(u)>0$, $u>0$; $k^N(u)$ satisfies Cond-K, and*

$$E\int_0^T d^N(t,\omega)dt<\infty,$$

and $k^N(u)$ is increasing, concave such that for $|x|$, $|y|\leqslant N$

$$2\langle x-y, b(t,x)-b(t,y)\rangle+|\sigma(t,x)-\sigma(t,y)|^2+\int_Z|c(t,x,z)-c(t,y,z)|^2\pi(dz)$$
$$\leqslant d^N(t,\omega)k^N(|x-y|^2). \tag{2.2}$$

(iv) $\lim_{x\to y}\int_Z|c(t,x,z)-c(t,y,z)|^2\pi(dz)=0$, *for all $t\in[0,T]$.*

Then S. D. E. (2.1) has a pathwise unique strong solution.

Remark 1. $k(u)=\begin{cases}u\ln(1/u), & \text{as } 0\leqslant u\leqslant a\leqslant e^{-1}<1,\\ a(\ln(1/a))+(d/du)k(a-)\cdot(u-a), & \text{as } u\geqslant a,\end{cases}$

where a is a constant, then $k(u)$ satisfies Cond-K.

Remark 2. Theorem 1 implies Theorem 4.12 in [4] (also [3] and [5]) in some sense. Moreover, drift coefficient in system can be discontinuous and very much non-linear now, e. g., set

$$b_i(t, x)=\begin{cases}-x_i/|x|-x_i|x|^N & \text{as } x\neq 0,\ N \text{ is any positive number,}\\ 0, & \text{as } x=0.\end{cases}$$

Then $\langle x, b(t, x)\rangle\leqslant 0$, and by Schwarz inequality (as $x, y\neq 0$)

$$\langle x-y, b(t, x)-b(t, y)\rangle=-|x|-|y|+\langle x, y\rangle(|x|^{-1}+|y|^{-1})$$
$$-|x|^{N+2}-|y|^{N+2}+\langle x, y\rangle(|x|^N+|y|^N)$$
$$\leqslant(|x|-|y|)(|y|^{N+1}-|x|^{N+1})\leqslant 0.$$

Remark 3. Condition for $b(t, x)$ can not be weakened to

$$|b(t, x)|\leqslant k_0(1+|x|^{1+\delta}),$$

where $\delta>0$ is any constant. Indeed, in this case Mckean [11, p.67] has shown that the explosion of the solution x_t, $t\geqslant 0$, will happen. Now let us give an sxample for $b(t, x)$, which is discontinuous, but condition (iii) is satisfied. This example will be useful for the elliptic Bang-Bang control later.

Example 2. For S. D. E. (2.1) with (A, A^0, A^1—$n\times n$ matrices, b^1—n-dim. vector)

$$b(t, x)=A^0(t)x+A(t)u(t, x)+A^1(t)b^1(t, x) \tag{2.3}$$

assume that conditions (i)—(iv) in Theorem 1 for b^1, σ, c is satisfied, moreover, $A^i(t)$, $i=0, 1$, and $A(t)$ are bounded Borel measurable w. r. t. $t\in[0, T]$ and non-random; $u(t, x)=(u_1(t, x), \cdots, u_n(t, x))$, where ($A^*$-transposition of A)

$$u_i(t, x)=\begin{cases}-a_i\tilde{x}_i/|\tilde{x}|, & \text{as } |\tilde{x}|\neq 0,\ 0\leqslant a_i-\text{constant},\\ 0, & \text{as } |\tilde{x}|=0; \qquad a_1+\cdots+a_n>0;\end{cases} \tag{2.4}$$
$$\tilde{x}=(\tilde{x}_1, \cdots, \tilde{x}_n),\ \tilde{x}_i=a_i(A^*(t)x)_i,\ i=1, \cdots, n.$$

Then (2.1) has a pathwise unique strong solution.

Proof Clearly. (i) in Theorem 1 is satisfied. We only need to prove that (iii) in Theorem 1 holds. After simple evaluation for $x, y\neq 0$

$$\langle x-y, A(t)(u(t, x)-u(t, y))\rangle=-|\tilde{x}|-|\tilde{y}|+\langle\tilde{x}, \tilde{y}\rangle\cdot(|\tilde{x}|^{-1}+|\tilde{y}|^{-1})\leqslant 0.$$

For $x=0$, $y\neq 0$ we will have

$$\langle x-y, A(t)(u(t, x)-u(t, y))\rangle=-|\tilde{y}|\leqslant 0.$$

Applying Theorem 1 we can get the existence of stochastic Bang-Bang control for the bounded controls with elliptic bundary. Suppose we want to minimize

$$J(u, v)=E\int_0^T|x_t^{u,v}|^2dt, \tag{2.5}$$

where $x_t^{u,v}$ is a strong solution of S. D. E. (2.6), which is pathwise unique,

$$x_t^{u,v}=x_0+\int_0^t(A_s^0x_s^{u,v}+A_s^1g(s, |x_s^{u,v}|^2)x_s^{u,v}+B_s^1v_s+B_s^2u^s+B_s^2h(s, x_s^{u,v}))ds+$$
$$\int_0^t(C_s^0+C_s^1Q(s, |x_s^{u,v}|))dw_s+\int_0^t\int_Z D(s, |x_s^{u,v}|^2, z)\,x_s^{u,v}q(ds, dz),$$
$$t\in[0, T]. \tag{2.6}$$

A_t^i, B_t^i, C_t^i, $D(t, x, z)$ are bounded real nonrandom measurable, where the 1st three functions do not depend on x; and $h(t, x)$ is defined by an n-dimensional vector as

$$h(t, x)=(h_1(t, x), h_2(t, x), 0, \cdots, 0); \tag{2.7}$$

$$h_i(t, x)=\begin{cases} a_i^2 x_i/(a_1^2x_1^2+a_2^2x_2^2)^{1/2}, & \text{as } a_1^2x_1^2+a_2^2x_2^2\neq 0, \\ 0, & \text{as } a_1^2x_1^2+a_2^2x_2^2=0, \text{ as } i=1, 2; \end{cases}$$

and a_i, $i=1, 2$, are positive constants. Let $(\mathscr{U}, \mathscr{V})$ be the admissible control set: $(\mathscr{U}, \mathscr{V})=\{u=u(t, x)\in\mathscr{U}, v=v(t, x)\in\mathscr{V}: u(t, x)=u(t, x_1, x_2)$ is jointly measurable on $[0, T]\times R^2$, $v(t, x)$ is jointly measurable on $[0, T]\times R^n$ such that

$$u_1^2(t, x)/a_1^2+u_2^2(t, x)/a_2^2\leqslant 1, \quad u_i=0, \ i=3, 4, \cdots, n;$$

$$v_1^2(t, x)+\cdots+v_n^2(t, x)\leqslant 1.$$

Moreover, $u(t, x)$ and $v(t, x)$ are such that (2.6) has a pathwise unique strong solution $x_t^{u,v}\}$.

Theorem 2. *Assume that*

(i) $A_t^i\leqslant 0$, $B_t^i\geqslant 0$, $i=1, 2$, $C_t^0\cdot C_t^1\leqslant 0$, $C_t^0+C_t^1 Q(t, |x|)\geqslant\delta>0$; $g(t, x)\geqslant 0$, $t\geqslant 0$, $x\geqslant 0$, *is locally bounded and there exist common points* $x_1^0, x_2^0, \cdots, x_m^0$ *such that*

$$|(\partial/\partial x_i)g(t, x)|\leqslant k_N, \quad \textit{as } |x|\leqslant N, \ N=1, 2, \cdots \textit{ for all } x\neq x_i^0, \ i=1, \cdots, m,$$

where k_N *is a constant depending only on* N. *Moreover, there exist constants* $d\geqslant 1$, $m>0$ *such that*

$$g(t, |x|^2)\leqslant k_0(1+\ln(1+|x|^{4d})\ln(1+\ln(1+\ln(1+|x|^m))));$$

(ii) $\int_Z |D(s, |x|^2, z)\, x|^i\pi(dz)\leqslant k_0(1+|x|^i)$, $i=1, 2$; *for all* $s\in[0, T]$, $x\in R^n$;

(iii) *For any* $x, y\in R^1$, $x\geqslant y\Rightarrow(2+D(t, x^+, z))D(t, x^+, z)x^++x\geqslant(2+D(t, y^+, z))D(t, y^+, z)y^++y$;

(iv) $$\int_Z |xD(s, |x|^2, z)-yD(s, |y|^2, z)|^2(dz)\leqslant k^N(|x-y|^2),$$

$$x, y\in R^n.\ |x|, |y|\leqslant N;$$

$$\left|\int_Z (D(t, x^+, z)^2x^+-D(t, y^+, z)^2y^+)\pi(dz)\right|\leqslant k^N(|x-y|),$$

$$x, y\in R^n, \ |x|, |y|\leqslant N;$$

where $k^N(u)$, $u>0$, *is positive, increasing, concave and satisfies* $\int_{0+} du/k^N(u)=\infty$.

(v) $|Q(t, |x|)-Q(t, |y|)|\leqslant\rho^N(|x-y|)$, $x, y\in R^n$, $|x|, |y|\leqslant N$,

$|Q(t, |x|)^2-Q(t, |y|)^2|\leqslant k^N(||x|^2-|y|^2|)$, $x, y\in R^n$, $|x|, |y|\leqslant N$,

where $\rho^N(u)>0$, *as* $u>0$; $\rho^N(0)=0$, *and it is such that*

$$(\rho^N(u))^2\leqslant k^N(u^2), \ \int_{0+} du/\rho^N(u)=+\infty,$$

$k^N(u)$ *is defined in* (iv), *and* $Q(t, x)\leqslant Q(t, y)$, *as* $0\leqslant x\leqslant y$; $|Q(t, x)|\leqslant k_0(1+|x|)$. *Then there exists an admissible control* $u^0\in\mathscr{U}$, $v^0\in\mathscr{V}$ *such that*

$$J(u^0, v^0)=\mathrm{Min}(J(u, v): u\in\mathscr{U}, v\in\mathscr{V}),$$

where $u^0(t, x)=-h(t, x)$ *is defined in* (2.7) *above, and* v^0 *is such that*

$$v^0(t, x)=-x/|x|, \textit{ as } |x|\neq 0; \ v^0(t, x)=0, \textit{ as } |x|=0; \tag{2.8}$$

and x_t^0 *is the pathwise unique strong solution of* (2.6) *with* u^0, v^0; $u^0(t)=u^0(t, x^0(t))$

is called an elliptic Bang-Bang control, $v^0(t)=v^0(t, x^0(t))$ *a circle Bang-Bang control, since they satisfies*

$$\sum_{i=1}^{2} u_i^0(t)^2/a_i^2=1, \ as\,|x^0(t)|^2\neq 0; \ |v^0(t)|=1, \quad as\ x^0(t)\neq 0.$$

Remark. 1^0 If $B_t^2=0$, $A_t^i=0$, $i=0, 1$; $C^1(t)=0$, then we get the usual Bang-Bang control, which implies [6, 7, 12, 13]. Moreover, let

$$g(t, x)=-x, \quad \text{for } t\geqslant 0,\ x\geqslant 0.$$

Then we get a very much non-linear system, which satisfies (i) for g.

2^0 Set

$$\rho^N(u)=u(\ln u^{-2})^{1/2},\ k^N(u)=u\ln(1/u).$$

Then $\rho^N(u)$ and $k^N(u)$ meet the requirement in (v).

3^0 Set

$$Q(t, |x|)=\begin{cases}1/R, & \text{as } |x|<1/R,\\ |x|, & \text{as } |x|\geqslant 1/R,\end{cases} \text{ where } 0<R \text{ is a constant;}$$

and assume that $C_t^0+C_t^1/R\geqslant\delta>0$. Then $Q(t, |x|)$ satisfies condition (v).

4^0 If $\pi(Z)<\infty$, let $D(s, |x|^2, z)=1$, then conditions (ii)—(iv) are all satisfied (For example, as $\pi(dz)=dz/|z|^{n+1}$, let

$Z=R^n-\times_i(-\varepsilon, \varepsilon)_i$, $(-\varepsilon, \varepsilon)_i$ is the interval in the ith coordinate space)

5^0 Let $\pi(z)=dz/|z|^{n+1}$,

$$D(s, |x|^2, z)=f(z)^2|x|^2/(1+f(z)|x|^2) \quad \text{for } s\geqslant 0,\ x\in R^n,\ z\in Z,$$

where ($\varepsilon>0$ is a constant)

$$f(z)=\begin{cases}|z|^{(n+1+\varepsilon)}, & \text{as } |z|\leqslant 1,\\ 1, & \text{as } |z|\geqslant 1.\end{cases}$$

Then it is easily seen that conditions (ii)—(iv) are satisfied.

§ 3, Theorems on Uniqueness and Comparison of Strong Solutions and Yamada-Watanabe Theorem

In order to prove Theorems 1 and 2 we need some auxillary theorems. Actually, they are of interest on their own.

Theorem 3. (Uniqueness). *Assume that* $0\leqslant x(t)$ *is Lebesgue measurable,* $t\in[0, T]$, *which satisfies*

$$0\leqslant x(t)\leqslant\int_0^t k(x(s))ds, \quad \textit{for all } t\in[0, T],$$

where $k(u)>0$, $u>0$, *is increasing and satisfies Cond-K. Then*

$$x(t)=0, \quad \textit{for all } t\in[0, T].$$

Theorem 4. *If condition* (iii) *in Theorem* 1 *holds, then the pathwise uniqueness for S. D. E.* (2.1) *holds.*

Corollary. *Theorems 3 and 4 hold for S. D. E. defined on $t\geqslant 0$, if conditions in them for $t\in[0, T]$ are replaced by conditions for $t\geqslant 0$.*

The proofs of Theorems 3 and 4 are omitted here, or one can prove them as that in [20] and [16] similarly.

Now assume that $x_i(t)$, $i=1, 2$, are two Cadlag processes satisfying

$$x_i(t)=x_i(0)+\int_0^t \beta_i(s, x_i(s), \omega)ds+\int_0^t \sigma(s, x_i(s))dw(s)$$
$$+\int_0^t\int_Z c(s, x_i(s), z)q(ds, dz), \ t\in[0, T], \tag{3.2}$$

where all processes are in 1-dimensional space, w, q are defined in (2.1), but they are also 1-dimensional processes. Then we have

Theorem 5. (Tanaka formula). *Assume that for $N=1, 2, \cdots$*

(i) $|\sigma(t, x)-\sigma(t, y)|\leqslant d_N(t)\cdot\rho^N(|x-y|)$, *as* $|x|$, $|y|\leqslant N$, *where* $\rho^N(u)$, $u>0$, *is local integrable,* $\int_{0+} du/\rho^N(u)^2=\infty$, *and* $\int_0^T d_N(s)^2 ds<\infty$;

(ii) $\sigma(t, x)$ *and* $c(t, x, z)$ *are jointly measurable,* $\beta_i(t, x_i(t), \omega)$ is $\mathscr{F}_t$-adapted, for every cadlag process $x_i(t)$, and there exists a continuous $g(x)$ such that

$$|\beta_i(t, x, \omega)|+|\sigma(t, x)|+\sum_{j=1}^2\int_Z |c_i(t, x, z)|^j\pi(dz)\leqslant|g(x)|.$$

(iii) $x\geqslant y\Rightarrow c(t, x, z)+x\geqslant c(t, y, z)+y$.

Then for all $t\in[0, T]$

$$|x_1(t)-x_2(t)|=|x_1(0)-x_2(0)|+\int_0^t \operatorname{sgn}(x_1(s)-x_2(s))d(x_1(s)-x_2(s)). \tag{3.3}$$

We omit the proof here. Or we can refer the reader to [16].

Remark. Formula (3.3) actually is a generalization of the Tanaka formula from the continuous case [14, 8] to the case with Poisson jumps.

Suppose that there exists a 1-dimensional cadlag process y_t^i satisfying

$$y_t^i=y_0^i+\int_0^t b^i(s, y_s^i, x_s^i)ds+\int_0^t \sigma(s, y_s^i)dw_s$$
$$+\int_0^t\int_Z c(s, y_s^i, z)q(ds, dz), \quad t\in[0, T], \ i=1, 2, \tag{3.3}$$

where y_t^i, w_t and $q(t, \cdot)$ are all in 1-dimensional space, which are random process, B. M. and centralized Poisson random measure, respectively, but x_t^i is a cadlag process in n-dimensional space, $i=1, 2$.

Theorem 6. (Comparison Theorem). *For $i=1, 2$ set $\beta^i(t, y_t^i(\omega), \omega)=b^i(t, y_t^i(\omega), x_t^i(\omega))$. Assume that conditions* (i)—(iii) *in Theorem* 5 *for β^i, σ, c are fulfilled, and*

(iv) $y_0^1\leqslant y_0^2$,

(v) $b^i(t, y, x)$ *is jointly measurable on* $(t, y, x)\in[0, T]\times R^1\times R^n$,

(vi) $b^1(t, y, x)\leqslant b^2(t, y, x)$, *for all* $(t, y, x)\in[0, T]\times R^1\times R^n$.

(vii) $\operatorname{sgn}(y^1-y^2)(b^1(t, y^1, x^1)-b^1(t, y^2, x^2))\leqslant d_N(t)\rho_N(|y^1-y^2|)$, $y^i=f(x^i)$, *for all* $t\in[0, T]$, $|y^1|$, $|y^2|\leqslant N$; *where f is some given function,* $d_N(t)\geqslant 0$, *and* $\rho_N(u)>0$, *as* $u>0$, *and* $\rho_N(u)$ *is increasing, concave and such that*

$$\int_0^t d_N(s)ds<\infty,\ \int_{0+} ds/\rho_N(u)=\infty,\ N=1, 2, \cdots.$$

Then P–a. s.

$$y_t^1\leqslant y_t^2,\quad \text{for all } T\in[0, T].$$

We omit the proof here. Or we can refer the reader to [16].

Corollary. *Theorem* 6 *holds, if replace* (vii) *by*

(vii)′ σ *and* c *satisfy conditions* (*i*)—(*iv*) *in Theorem* 1, *and* $b^i(t, y)=b^i(t, y, x)$, $i=1, 2$, *are jointly, continuous and do not depend on* x, *moreover,*

$$b^1(t, y)<b^2(t, y),\quad \text{for all } (t, y)\in[0, T]\times R^1.$$

Proof In this case there exists a Lipschitz continuous function $b^3(t, y)$

$$b^1<b^3<b^2.$$

Now let us generalize the Yamada–Watanabe theorem (Y–W thm)[8] to the case for S. D. E. with Poisson jumps. We say that x_t is a weak solution of (2.1) iff x_t satisfies (2.1) defined on some probability space with some B. M. and Poisson random point process but with the same original compensator $\pi(dz)dt$. Then we have (The proof of the following theorems is similar to [16])

Theorem 7 (Y–W thm). *If the pathwise uniqueness holds for* (2.1) *and there exists a weak solution for* (2.1), *then* (2.1) *has a pathwise unique strong solution.*

Theorem 8. *If* $(x_t^i, w_t^i, q_t^i(\cdot))$, $i=1, 2$, *are two wark solutions of* (2.1), *then there exists a probability space* $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t, \tilde{P})$ *and a B. M.* $\tilde{w}_t$, *a Poisson random point measure* $\tilde{p}(t, \Gamma, \omega)$ *with compensator* $\pi(\Gamma)\,dt$ *on it such that* $(\tilde{x}_t^1, \tilde{x}_t^2, \tilde{w}_t, \tilde{q}_t(\cdot))$ *is adapted to* $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t, \tilde{p})$, *where* $\tilde{q}(dt, dz)=\tilde{p}(dt, dz)-\pi(dz)dt$, *and the probability law of* $(\tilde{x}_t^1, \tilde{w}_t, \tilde{q}_t(\cdot))$ *and* $(\tilde{x}_t^2, \tilde{w}_t, \tilde{q}_t(\cdot))$ *coincides with that of* $(x_t^1, w_t^1, q_t^1(\cdot))$ *and* $(x_t^2, w_t^2, q_t^2(\cdot))$, *respectively.*

§ 4. Proofs of Theorems 1 and 2

Let us introduce a proposition without proof first.

Proposition 1. *For* $x=(x_1, x_2)$ *set* (a_1, a_2 *are positive constants*)

$$u^0=u^0(t, x)=\begin{cases}(-a_1^2x_1/(a_1^2x_1^2+a_2^2x_2^2)^{1/2},\ -a_2^2x_2/(a_1^2x_1^2+a_2^2x_2^2)^{1/2}), & x\neq 6,\\ (0, 0), & \text{as } x=0.\end{cases}$$

Then

$$\langle x, u^0\rangle\leqslant\langle x, u\rangle$$

for all x, *and* u *satisfying* $(u_1^2/a_1^2)+(u_2^2/a_2^2)\leqslant 1$.

Proof of Theorem 1, Let us extend the definition of $b(t, x)$, $\sigma(t, x)$ and $c(t,$

$x, z)$ to $t\in(T, T_1]$, where $T_1>T$ is a constant, by

$$b(t, x)=b(T, x), \quad \text{as } t\in(T, T_1],$$

etc. And denote $b^N(t, x)$ and $\sigma^N(t, x)$ as [5] such that condition (iii) still holds for b^N, σ^N, c and

$$h^N(t, x)=\begin{cases} h(t, x), & \text{as } |x|\leqslant N,\\ 0, & \text{as } |x|<N+3;\ |h^N|\leqslant|h|, \text{ as } h=b, \sigma. \end{cases}$$

Then by [15] and Theorem 7 there exists a pathwise unique strong solution x_t for the following S. D. E. as $T_1\geqslant t\geqslant 0$, $N=1, 2, \cdots$

$$x_t^N=x_0+\int_0^t b^N(s, x_s^N)ds+\int_0^t \sigma^N(s, x_s^N)dw_s+\int_0^t\int_Z c(s, x_s^N, z)\tilde{q}(ds, dz). \quad (4.1)$$

Now set

$$x_t=x_t^N, \text{ as } t\in[0, \tau_N(x^N)), \quad (4.2)$$

where for arbitrary $x(\cdot)\in D$, $\tau_N(x)=T_1$, as $\sup_{0\leqslant t\leqslant T}|x(t)|\leqslant N$; and

$$\tau_N(x)=\inf(t: |x(t)|>N).$$

It is not difficult to show that it is well defined. Similarly to [1] one has

$$P(\tau_N(x^N)<T_1)\to 0, \quad \text{as } N\to\infty, \quad (4.3)$$

where one needs to apply that (denote $f(x)=1+\ln(1+|x|^{2d})$)

$$\int_Z |f(x+c(t, x, z))-f(x)-\sum_{i=1}^n(\partial f/\partial x_i)(x)c_i(t, x, z)|\pi(dz)\leqslant$$

$$=\int_Z \frac{1}{2}\sum_{i,j}|(\partial^2 f/\partial x_i\partial y_j)(x+\theta c(t, x, z)|\cdot|c_i(t, x, z)|^2\pi(dz)\leqslant k'.$$

By (4.3) it is not difficult to derive that

$$\lim_{N\to\infty}\tau_N(x^N)=T_1, \ P-\text{a. s.}$$

From this by Theorem 4 it is obtained that x_t is the pathwise unique strong solution of (2.1).

Proof of Theorem 2 Firstly, by Example 2 it is ersily seen that condition (iii) in Theorem 1 holds. Hence as the proof of Theorem $1(u^0, v^0)\in(\mathscr{U}, \mathscr{V})$. Secondly, for $(u, v)\in(\mathscr{U}, \mathscr{V})$ follow the approach in the proof of Theorem 2.1 in [8] but apply Theorem 8 here. We have that after applying Ito formula

$$Y_t=|x_0|^2+\int_0^t(2(A_s^0Y_s+A_s^1Y_sg(s, Y_s)+B_s^1x_s^{u,v}\hat{v}_s+B^2x_s^{u,v}h(s, \hat{x}_s^{u,v})+B_s^2\hat{x}_s^{u,v}\hat{u}_s$$

$$+n(C_s^0+C_s^1Q(s, Y_s^{1/2}))^2+\int_Z D(s, Y_s, z)^2Y_s\pi(dz))ds+\int_0^t 2(C_s^0+C_s^1Q(s, Y_s^{1/2}))$$

$$\cdot Y_s^{1/2}d\hat{w}_s^1+\int_0^t\int_Z(D(s, Y_s, z)^2+2D(s, Y_s, z))Y_s\tilde{q}(ds, dz), \ t\in[0, T],$$

$$X_t=|x_0|^2+\int_0^t(2(A_s^0X_s+A_s^1X_sg(s, X_s)-B_s^1X_s^{1/2})+n(C_s^0+C_s^1Q(s, X_s^{1/2}))^2$$

$$+\int_Z D(s, X_s, z)^2X_s\pi(dz))ds+\int_0^t 2(C_s^0+C_s^1Q(s, X_s^{1/2}))X_s^{1/2}d\hat{w}_s^1$$

$$+\int_0^t\int_Z (D(s, X_s, z)^2+2D(s, X_s, z))X_s\hat{q}(ds, dz),\ t\in[0, T],$$

where $\hat{w}_t^1$ is a 1-dimensional B. M. which is the first component of n-dimensional B. M. $\hat{w}_t=(\hat{w}_t^1, \cdots, \hat{w}_t^n)$, $\hat{p}_t(\cdot)$ is 1-dimensional Poisson random point measure with compensator $\pi(dz)dt$, $\hat{p}(dt, dz)=\hat{q}(dt, dz)-\pi(dz)dt$, $Y_t=|\hat{x}_t^{u,v}|^2$, $X_t=|\hat{x}_t^0|^2$, and the probability law of $\hat{x}_t^0$, $\hat{w}_t$, $\hat{p}_t(\cdot)$ and $\hat{x}_t^{u,v}$, $\hat{w}_t$, $\hat{p}_t(\cdot)$ coincides with that of x_t^0, $\bar{w}_t^0$, $p_t(\cdot)$ and $x_t^{u,v}$, $\bar{w}_t^{u,v}$, $p_t(\cdot)$, respectively, where

$$\bar{w}_t^{u,v}=\int_0^t P(x_s^{u,v})dw_s,\ \bar{w}_t^0=\int_0^t P(x_s^0)dw_s,\ \hat{u}_t=u(t, \hat{x}^{u,v}(t));\ \hat{v}_t=v(t, \hat{x}^{u,v}(t))$$

and $P(x)$, $x\in R^n$, is an $n\times n$ matrix, which is orthogonal and Borel measurable (See [8]). Note that

$$-\operatorname{sgn}(X-Y)(F((X^+)^{1/2})-F((Y^+)^{1/2}))\leqslant 0,\ F(x)=x,Q(t, x);$$

$$\operatorname{sgn}(X-Y)(Q(t, (X^+)^{1/2})^2-Q(t, (Y^+)^{1/2})^2)\leqslant k^N(|X^+-Y^+|)\leqslant k^N(|X-Y|);$$

$$\begin{aligned}|Q(t, (X^+)^{1/2})-Q(t, (Y^+)^{1/2})|^2&\leqslant \rho^N(|(X^+)^{1/2}-(Y^+)^{1/2}|)^2\\ &\leqslant k^N(|(X^+)^{1/2}-(Y^+)^{1/2}|^2)\\ &\leqslant k^N(|X^+-Y^+|)\leqslant k^N(|X-Y|).\end{aligned}$$

Hence applying Proposition 1 and Theorem 6 we obtain that P-a. s.

$$X_t\leqslant Y_t,\ \text{for all } t\in[0, T].$$

§5. Application to Parameter Estimation

Theorem 1 can also be applied to search the maximum likelihood estimate (MLE) of some parameter in some general stochastic systems.

Consider S. D. E. in n-dimensional space:

$$x_t=x_0+\int_0^t(\theta b_1(s, x_s)+b_2(s, x_s))ds+\int_0^t\sigma(s, x_s)dw_s,\ t\in[0, T]. \quad (5.1)$$

Assume that

(i) condition (i) in Theorem 1 for b_1 with σ and b_2 with σ, and σ itself is satisfied, respectively; conditions (ii) and (iii) in Theorem 1 hold with $c=0$;

(ii) $P\left(\int_0^T|\sigma^{-1}b_1(s, x_s)|^2ds>0\right)=1$, where x_t is the strong solution of (5.1); and $-\infty<\theta<\infty$ is a real constant parameter.

Theorem 9. *The MLE of θ is (denote $a=\sigma\sigma^*$)*

$$\hat{\theta}_T(x)=\left(\int_0^T\langle a^{-1}b_1(s, x_s), dx_s\rangle-\int_0^T\langle a^{-1}b_1(s, x_s), b_2(s, x_s)\rangle ds\right)\Big/\int_0^T|\sigma^{-1}b_1|^2ds.$$

In order to prove Theorem 9 we need the following lemma, which can be shown as [1].

Lemma 1. *Under the assumption of Theorem 9*

$$E\Phi(y(\cdot))=1,$$

where we denote $b=\theta b_1+b_2$, and

$$\Phi(y(\cdot))=\exp\left(\int_0^T\langle a^{-1}b(s,\ y_s),\ dy_s\rangle-\frac{1}{2}\int_0^T|\sigma^{-1}b(s,\ y_s)|^2ds\right),$$

and y_t is the pathwise unique strong solution of

$$y_t=x_0+\int_0^t\sigma(s,\ y_s)dw_s,\quad t\in[0,\ T].$$

Proof of Theorem 9 By Lemma 1 and Girsanov theorem y_t also satisfies (5.1) with $\bar{w}_t$, where $\bar{w}_t=w_t-\int_0^t\sigma^{-1}b(s,\ y_s)ds$ is a B. M. under probability measure $d\bar{P}=\Phi(y(\cdot))dP$. Since the pathwise uniqueness holds for (5.1), we have for $\Gamma\in\mathscr{B}(\mathscr{C})$ $\mu_{x(\cdot)}(\Gamma)=P(\omega:\ x(\omega,\ \cdot)\in\Gamma)=\bar{P}(\omega:\ y(\omega,\ \cdot)\in\Gamma)=\int_{y(\cdot)\in\Gamma}\Phi(y(\cdot))dP=\int_\Gamma\Phi(x)d\mu_{y(\cdot)}(x)$. Therefore $(d\mu_{x(\cdot)}/d\mu_{y(\cdot)})(x)=\Phi(x)$, and $(d\mu_{x(\cdot)}/d\mu_{y(\cdot)})(x(\cdot))=\Phi(x(\cdot))$, where

$$\Phi(x(\cdot))=\exp\left(\int_0^T\langle a^{-1}(\theta b_1(s,\ x_s)+b_2(s,\ x_s)),\ dx_s\rangle-\frac{1}{2}\int_0^T|\sigma^{-1}(\theta b_1+b_2)|^2ds\right).$$

So the log likelihood function is (cf. [17]).

$$L(\theta)=\int_0^T\langle a^{-1}(\theta_1 b(s,\ x_s)+b_2(s.\ x_s)),\ dx_s\rangle-\frac{1}{2}\int_0^T|\sigma^{-1}(\theta b_1+b_2)|^2ds.$$

Hence the MLE $\hat{\theta}_T(x)$ of θ satisfies the equation

$$\int_0^T\langle a^{-1}b_1(s,x_s),\ dx_s\rangle-\theta\int_0^T|\sigma^{-1}b_1(s,\ x_s)|^2ds-\int_0^T\langle a^{-1}b_1,\ b_2\rangle ds=0.$$

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