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THE EXISTENCE OF ALMOST PERIODIC SOLUTIONS AND PERIODIC SOLUTIONS

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Abstract

In this paper, it is obtained that a periodic system has an almost periodic solution if it has a solution $x=\varphi(t)$ uniformly stable with respect to Ω_φ, and has a periodic solution if $x=\varphi(t)$ is weakly uniformly asymptotically stable with respect to Ω_φ. Meanwhile, it is also obtained that a uniformly almost periodic system has an almost periodic solution if it has a solution $x=\varphi(t)$ uniformly asymptotically stable with respect to A_φ^f

§ 1. Introduction

L. G. Deysach and G. R. Sell [1], proved that a periodic system has an almost periodic solution if it has a uniformly stable bounded solution. C. R. Sell[2] proved that a periodic system has a periodic solution if it has a uniformly asymptotically stable bounded solution. But these conditions are difficult to satisfy. For example, the system

$$\frac{dx}{dt}=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}x$$

has a periodic solution

$$x(t)=\begin{pmatrix}\cos t\\ \sin t\\ 0\\ 0\end{pmatrix},$$

but it is not uniformly stable. The main aims of this paper is to weaken the conditions of [1] and [2].

§ 2. Almost Periodic Solutions of Periodic Systems

Consider the n dimensional system

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$$\frac{dx}{dt}=f(x, t), \tag{1}$$

where f: $R^n \times R \to R^n$. is a continuous vector function.

In following proposition, we suppose that system (1) and its hull system satisfy the conditions of the uniqueness of solution.

Definition 2.1. *The solution $x=\varphi(t)$ of system (1) is uniformly stable with respect to E (or USR E) if $x=\varphi(t)$ is bounded in R and for any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that if $x(t, x_0, t_0)$ is a solution of (1) and $x\in E\cap N(\varphi(t_0), \delta(x))$, then*

$$\|\varphi(t)-x(t, x_0, t_0)\|<\varepsilon \quad (t\geqslant t_0),$$

where

$$E\subset R^n,\ N(\varphi(t_0), \delta(\varepsilon))=\{x\mid \|x-\varphi(t_0)\|<\varepsilon\}.$$

Lemma 2.1. *If the solution $x=\varphi(t)$ of system (1) is USR E_1, and $E_2\subset E_1$, then it is USR E_2.*

Proof From Defiintion 2.1 we can get the proof.

If f: $R^n\times R\to R^n$, g: $R^n\times R\to R^n$, φ: $R\to R^n$, we take some notations.

(i) $f(x, t+t_k) \xrightarrow{\text{loc}} g(x, t)$ denotes $\{f(x, t+t_k)\}$ uniformly converges to $g(x, t)$ in any compact subset of $R^n\times R$. $f(x, t+t_k)\xrightarrow{\text{unif}} g(x, t)$ denotes $\{f(x, t+t_k)\}$ uniformly converges to $g(x, t)$ in $V\times R$, where V is any compact subset of R^n.

(ii) $H(f)=\{g\mid$ there is a sequence $\{t_k\}$, $f(x, t+t_k)\xrightarrow{\text{loc}} g(x, t)\}$,

$\Omega(f)=\{g\mid$ there is a sequence $\{t_k\}$, with $t_k\to+\infty$, $f(x, t+t_k) \xrightarrow{\text{loc}} g(x, t)\}$.

(iii) $H_\varphi=\{x\mid$ there is a sequence $\{t_k\}$, $\varphi(t_k)\to x\}$.

$\Omega_\varphi=\{x\mid$ there is a sequence $\{t_k\}$ with $t_k\to+\infty$, $\varphi(t_k)\to x\}$.

Lemma 2.2. *If the solution $x=\varphi(t)$ of (1) is USR E and $(\bar\varphi, \bar f)\in H(\varphi, f)$, then the solutian $x=\bar\varphi(t)$ of*

$$\frac{dx}{dt}=\bar f(x, t) \tag{2}$$

is USR E and $\delta(\varepsilon)$ in Definition 2.1 are inherited by the hull system (2).

Proof Refer to the proof of [4. Theorem 5.3].

Lemma 2.3. *If there is a sequence $\{t_k\}$ with $t_k\geqslant 0$ and $\varphi(t+t_k)\to\bar\varphi(t)$, then $\Omega_{\bar\varphi}\subset\Omega_\varphi$.*

Proof Any $x_0\in\Omega_{\bar\varphi}$, there is a sequence $\{\bar t_k\}$ with $\bar t_k\to+\infty$ such that $\bar\varphi(\bar t_k)\to X_0$. Hence for $\varepsilon_m=1/2m>0$ there exist k_m such that

$$\|\bar\varphi(\bar t_{k_m})-x_0\|<1/2m. \tag{3}$$

Fixed $\bar t_{k_m}$, because $\varphi(\bar t_{k_m}+t_k)\to\bar\varphi(\bar t_{k_m})$ as $k\to\infty$, there is an r_m such that

$$\|\varphi(\bar t_{k_m}+t_{r_m})-\bar\varphi(\bar t_{k_m})\|<1/2m. \tag{4}$$

From (3) and (4) we obtain

$$\|\varphi(\bar t_{k_m}+t_{r_m})-x_0\|<1/m.$$

Take $t'_m = \bar{t}_{k_m} + t_{r_m}$, therefore

$$\|\varphi(t'_m) - x_0\| < 1/m \tag{5}$$

or $\lim_{m\to\infty} \varphi(t'_m) = x_0$. From $t_{r_m} \geqslant 0$ and $\bar{t}_{k_m} \to +\infty$ we know $t'_m \to +\infty$. Hecne $x_0 \in \Omega_\varphi$, so $\Omega_{\bar{\varphi}} \subset \Omega_\varphi$.

Corollary 2.1. *Assume that the solution $x=\varphi(t)$ of (1) is USR Ω_φ and there is a sequence $\{t_k\}$ with $t_k \geqslant 0$ such that $\varphi(t+t_k) \xrightarrow{\text{loc}} \bar{\varphi}(t)$, $f(x, t+t_k) \xrightarrow{\text{loc}}$ loc $\bar{f}(x, t)$. Then the solution $x=\bar{\varphi}(t)$ of (2) is USR $\Omega_{\bar{\varphi}}$ and the estimates $\delta(\varepsilon)$ in Definition 2.1 are inherited by system (2).*

Proof From Lemma 2.3, we know $\Omega_{\bar{\varphi}} \subset \Omega_\varphi$. From Lemma 2.1, the solution $x=\varphi(t)$ of (1) is USR $\Omega_{\bar{\varphi}}$. From Lemma 2.2, we get the solution $x=\bar{\varphi}(t)$ of (2) is USR $\Omega_{\bar{\varphi}}$ and $\delta(\varepsilon)$ are inherited.

Theorem 2.1. *If system (1) is a periodic system and the solution $x=\varphi(t, x_0)$ of (1) is USR Ω_φ, then $x=\varphi(t, x_0)$ is an asymptotic almost periodic function.*

Proof Assume that $f(x, t+\omega) = f(x, t)$, whree $\omega > 0$. For any sequence $\{t_k\}$ with $t_k \to +\infty$ we can suppose that $t_k = m_k\omega + \tau_k$, where m_k is natural number and $0 \leqslant \tau_k < \omega$. Then

$$\varphi(t+t_k, x_0) = \varphi(t+\tau_k+m_k\omega, x_0) = \varphi(t+\tau_k, \varphi(m_k\omega, x_0)).$$

Because $\{\tau_k\}$, $\{\varphi(m_k\omega, x_0)\}$ are bounded sequences, we can suppose $\tau_k \to \tau_0$, $\varphi(m_k\omega, x_0) \to y_0$ (or else, we take their subsequences). From

$$f(x, t+t_k) = f(x, t+m_k\omega+\tau_k) = f(x, t+\tau_k) \xrightarrow{\text{loc}} f(x, t+\tau_0),$$

we obtain

$$\varphi(t+t_k, x_0) \xrightarrow{\text{loc}} \varphi(t+\tau_0, y_0).$$

FromCorollary 2.1, the solution $x=\varphi(t+m_k\omega, x_0)$ of (1) is USR Ω_φ and the estimates $\delta(\varepsilon)$ in Definition 2.1 are inherited. From $\varphi(m_k\omega, x_0) \to y_0$, we know that there is a K_1 such that

$$\|\varphi(m_k\omega, x_0) - y_0\| < \delta(\varepsilon) \quad \text{when } k \geqslant K_1.$$

Hence for $k \geqslant K_1$,

$$\|\varphi(t, x_0) - x(t, y_0, m_k\omega)\| < \varepsilon \quad (t \geqslant m_k\omega),$$

that is

$$\|\varphi(t+m_k\omega, x_0) - x(t+m_k\omega, y_0, m_k\omega)\| < \varepsilon \quad (t \geqslant 0).$$

Because $x(t+m_k\omega, y_0, m_k\omega) = \varphi(t, y_0)$, we get

$$\|\varphi(t+m_k\omega, x_0) - \varphi(t, y_0)\| < \varepsilon \quad (t \geqslant 0).$$

So, for $k \geqslant K_1$, we have

$$\|\varphi(t+\tau_0+m_k\omega, x_0) - \varphi(t+\tau_0, y_0)\| < \varepsilon \quad (t \geqslant 0). \tag{6}$$

Assume that $H = \sup_{t\in R} \|\varphi(t, x_0)\|$ and $M = \sup_{\|x\|\leqslant H} \|f(x, t)\|$. From $\tau_k \to \tau_0$, we know that there is a K_2 such that $|\tau_k - \tau_0| < \varepsilon/M$ when $k \geqslant K_2$, Hence, for $k \geqslant K_2$,

$$\|\varphi(t+\tau_0+m_k\omega, x_0)-\varphi(t+t_k, x_0)\| \leqslant \sup_{t\in R}\|f[\varphi(t, x_0), t]\|\cdot|\tau_k-\tau_0| \leqslant \varepsilon \quad (t \geqslant 0). \tag{7}$$

Take $K=\max(K_1, K_2)$. Combining (6), (7) we obtain, for $k \geqslant K$,

$$\|\varphi(t+t_k, x_0)-\varphi(t+\tau_0, y_0)\| < 2\varepsilon \quad (t \geqslant 0),$$

That is, the sequence $\{\varphi(t+t_k, x_0)\}$ converges to $\varphi(t+\tau_0, y_0)$ uniformly in R^+. So $\varphi(t, x_0)$ is an asymptotic almost periodic function.

Theorem 2.2. *Under the condition of Theorem 2.1, there exists an almost periodic solution $x=\bar{\varphi}(t)$ of (1) and it is USR $\Omega_{\bar{\varphi}}$ and $\bar{\varphi}\in\Omega(\varphi)$.*

Proof From Theorem 2.1, system (1) has an asympto tic almost periodic solution $x=\varphi(t)$. We take sequence $\{m\omega\}$ with $m\to+\infty$, where m is a natural number. From the definition of asymptotic almost periodic there is a subsequence $\{m_k\omega\}$ of $\{m\omega\}$ such that $\varphi(t+m_k\omega)\to\bar{\varphi}(t)$. Therefore $\bar{\varphi}\in\Omega(\varphi)$ and $\bar{\varphi}(t)$ is an almost periodic function. But $f(x, t+m_k\omega)=f(x, t)$, so $x=\bar{\varphi}(t)$ is the solution of system (1). From Corollary 2.1, we see that the solution $x=\bar{\varphi}(t)$ of (1) is USR $\Omega_{\bar{\varphi}}$. This completes the proof of Theorem 2.2.

Lemma 2.4. *Assume that $x=\varphi(t)$ is an almost periodic solution of autonomous system*

$$\frac{dx}{dt}=f(x). \tag{8}$$

Then $x=\varphi(t)$ is USR H_φ.

Proof Refer to [5, Chapter 5, Corollary 2 of Theorem 36].

Theorem 2.3. *The following statements are equivalent:*

(i) *System (6) has an almost periodic solution,*

(ii) *There is a solution $x=\varphi(t)$ of (6) with USR H_φ.*

(iii) *There is a solution $x=\varphi(t)$ of (6) with USR Ω_φ.*

Proof (i)⇒(ii) from Lemma 2.4.

(ii)⇒(iii) from Lemma 2.1.

(iii)⇒(i) from Theorem 2.2.

§ 3. Periodic Solutions of Periodic Systems

Definition 3.1. *The solution $x=\varphi(t)$ of (1) is weakly uniformly asymptotically stable with respect to E (or WUASR E) if the solution $x=\varphi(t)$ of (1) is USR E and there is a $\delta_0>0$ such that for any $x_0\in E\cap N(\varphi(t_0), \delta_0)$,*

$$\lim_{t\to+\infty}\|\varphi(t)-x(t, x_0, t_0)\|=0,$$

where $x(t, x_0, t_0)$ is a solution of (1).

Lemma 3.1. *If the solution $x=\varphi(t)$ of (1) is WUASR E_1 and $E_2\subset E_1$, then it is WUASR E_2.*

Proof From Lemma 2.1 and Definition 3.1, we can easily get the proof of Lemma 3.1.

Lemma 3.2. *Assume that system (1) is a periodic system and the solution $x=\varphi(t)$ of (1) is WUASR E. Then the solution $x=\bar{\varphi}(t)$ of (2) is WUASR E and the estimates $\delta(\varepsilon)$, δ_0 in Definition 2.1 and Definition 3.1 are inherited if (i) $H_\varphi\subset E$ and $(\bar{\varphi}, \bar{f})\in H(\varphi, f)$ or (i′) $\Omega_\varphi\subset E$ and $(\bar{\varphi}, \bar{f})\in\Omega(\varphi, f)$.*

Proof We prove the conclusion of (i) only. The conclusion of (i′) is the same as that of (i).

From Lemma 2.1 the solution $x=\bar{\varphi}(t)$ of (2) is USR E and $\delta(\varepsilon)$ in Definition 2.1 is inherited. We assume that $\varphi(t+t_k)\to\bar{\varphi}(t)$, $f(x, t+t_k)\to\bar{f}(x, t)$, because $(\bar{\varphi}, \bar{f})\in H(\varphi, f)$. For any $x_0\in E\cap N(\bar{\varphi}(t_0), \delta_0)$, where δ_0 is the same as in Definition 3.1, suppose that

$$\|x_0-\bar{\varphi}(t_0)\|=\eta<\delta_0. \tag{9}$$

Since $\varphi(t_0+t_k)\to\bar{\varphi}(t_0)$, there is a K_1 such that

$$\|\varphi(t_0+t_k)-\bar{\varphi}(t_0)\|<\frac{1}{2}(\delta_0-\eta) \quad (k\geqslant K_1). \tag{10}$$

Suppose that $t_k=m_k\omega+\tau_k$ where ω is the period of $f(x, t)$, m_k is an integer and $0\leqslant\tau_k<\omega$. We assume that $\tau_k\to\tau_0$ (or else we take its subsequence). Then

$$f(x, t+t_k)=f(x, t+\tau_k)\xrightarrow{\text{loc}}\bar{f}(x, t)=f(x, t+\tau_0).$$

Because $\varphi(t)$ is a bounded function, we can suppose that $\|\varphi(t)\|\leqslant H$ and $M=\sup\limits_{\|x\|\leqslant H}\|f(x, t)\|$. Hence

$$\|\varphi(t+t_k)-\varphi(t+\tau_0+m_k\omega)\|\leqslant M|\tau_k-\tau_0|\to 0, \tag{11}$$

that is, there exists a K_2 such that

$$\|\varphi(t+t_k)-\varphi(t+\tau_0+m_k\omega)\|\leqslant\frac{1}{2}(\delta_0-\eta) \quad (k\geqslant K_2). \tag{12}$$

Take $K_0=\max(K_1, K_2)$. From (9), (10), (12) we get

$$x_0\in E\cap N(\varphi(t_0+\tau_0+m_k\omega), \delta_0) \quad (k\geqslant K_0).$$

Hence, from the conditions of the lemma, we have

$$\lim_{t\to+\infty}\|\varphi(t)-x(t, x_0, t_0+\tau_0+m_k\omega)\|=0 \quad (k\geqslant K_0),$$

or, for $k\geqslant K_0$

$$\lim_{t\to+\infty}\|\varphi(t+\tau_0+m_k\omega)-x(t+\tau_0+m_k\omega, x_0, t_0+\tau_0+m_k\omega)\|=0.$$

Because

$$x(t+\tau_0+m_k\omega, x_0, t_0+\tau_0+m_k\omega)=x(t+\tau_0, x_0, t_0+\tau_0),$$

we obtain

$$\lim_{t\to+\infty}\|\varphi(t+\tau_0+m_k\omega)-x(t+\tau_0, x_0, t_0+\tau_0)\|=0 \quad (k\geqslant K_0). \tag{13}$$

Since the solution $x=\varphi(t)$ of (1) is USR E, for any $\varepsilon'>0$ there is a $\delta(\varepsilon')>0$ such that if

$$\bar{\varphi}(t_0)\in E\cap N(\varphi(t_0+\tau_0+m_k\omega),\ \delta(\varepsilon')),$$

we have

$$\|\varphi(t)-x(t,\ \bar{\varphi}(t_0),\ t_0+\tau_0+m_k\omega)\|<\varepsilon'\quad (t\geqslant t_0+\tau_0+m_k\omega). \tag{14}$$

From (11) and $\varphi(t_0+t_k)\to\bar{\varphi}(t_0)$ we get $\varphi(t_0+\tau_0+m_k\omega)\to\bar{\varphi}(t_0)$, that is, there is a K^0 such that

$$\|\varphi(t_0+\tau_0+m_k\omega)-\bar{\varphi}(t_0)\|<\delta(\varepsilon')\quad (k\geqslant K^0).$$

But $\bar{\varphi}(t_0)\in H_\varphi\subset E$, so we get

$$\bar{\varphi}(t_0)\in E\cap N(\varphi(t_0+\tau_0+m_k\omega),\ \delta(\varepsilon'))\quad (k\geqslant K^0).$$

Then, when $k\geqslant K^0$, (14) is true, that is,

$$\|\varphi(t+\tau_0+m_k\omega)-x(t+\tau_0+m_k\omega,\ \bar{\varphi}(t_0),\ t_0+\tau_0+m_k\omega)\|<\varepsilon'\quad (t\geqslant t_0).$$

Since

$$x(t+\tau_0+m_k\omega,\ \bar{\varphi}(t_0),\ t_0+\tau_0+m_k\omega)=x(t+\tau_0,\ \bar{\varphi}(t_0),\ t_0+\tau_0),$$

we obtain

$$\|\varphi(t+\tau_0+m_k\omega)-x(t+\tau_0,\ \bar{\varphi}(t_0),\ t_0+\tau_0)\|<\varepsilon'\quad (t\geqslant t_0). \tag{15}$$

Take $K=\max(K_0,\ K^0)$. Combining (13), (15) we get

$$\lim_{t\to+\infty}\|x(t+\tau_0,\ x_0,\ t_0+\tau_0)-x(t+\tau_0,\ \bar{\varphi}(t_0),\ t_0+\tau_0)\|\leqslant\varepsilon'.$$

But ε' is any small positive number, so

$$\lim_{t\to+\infty}\|x(t+\tau_0,\ x_0,\ t_0+\tau_0)-x(t+\tau_0,\ \bar{\varphi}(t_0),\ t_0+\tau_0)\|=0. \tag{16}$$

Suppose that $y(t,\ x_0,\ t_0)$ is the solution of (2) with $y(t_0,\ x_0,\ t_0)=x_0$. Then

$$y(t,\ x_0,\ t_0)=x(t+\tau_0,\ x_0,\ t_0+\tau_0),$$

$$\bar{\varphi}(t)=y(t,\ \bar{\varphi}(t_0),\ t_0)=x(t+\tau_0,\ \bar{\varphi}(t_0),\ t_0+\tau_0)\ \text{since}\ \bar{f}(x,\ t)=f(x,\ t+\tau_0).$$

From (16) we obtain

$$\lim_{t\to+\infty}\|y(t,\ x_0,\ t_0)-\bar{\varphi}(t)\|=0.$$

This completes the proof of Lemma 3.2.

Theorem 3.1. *Assume that system* (1) *is a periodic system and the solution* $x=\varphi(t)$ *of* (1) *is WUASR* Ω_φ. *Then system* (1) *has a periodic solution* $x=\bar{\varphi}(t)$ *with* $\bar{\varphi}\in\Omega_\varphi$ *and the solution* $x=\bar{\varphi}(t)$ *of* (1) *is WUASR* $\Omega_{\bar{\varphi}}$.

Proof From Theorem 2.1, the solution $x=\varphi(t)$ of (1) is an asymdtotic almost periodic function. We take sequence $\{m\omega\}$, where m is a natural number and ω is the period of $f(x,\ t)$. Then there is a subsequence $\{m_k\omega\}$ of $\{m\omega\}$ such that $\varphi(t+m_k\omega)\to\bar{\varphi}(t)$ and $\bar{\varphi}(t)$ is an almost periodic function, so $\bar{\varphi}\in\Omega_\varphi$. But $f(x,\ t+m_k\omega)=f(x,\ t)$, so $x=\bar{\varphi}(t)$ is the solution of (1). From Lemma 3.2 the solution $x=\bar{\varphi}(t)$ of (1) is WUASR Ω_φ and $\delta(\varepsilon)$, δ_0 are inherited. From Lemma 3.1, the solution $x=\bar{\varphi}(t)$ of (1) is WUSAR $\Omega_{\bar{\varphi}}$.

Now we prove that $x=\bar{\varphi}(t)$ is a periodic function.

For any $x_0\in\Omega_{\bar{\varphi}}$, there exists a sequence $\{\bar{t}_k\}$ with $\bar{t}_k\to+\infty$ such that $\bar{\varphi}(\bar{t}_k)\to x_0$. Furthermore we can suppose

$$\bar{\varphi}(t+\bar{t}_k) \xrightarrow{\text{unif.}} y(t, x_0, 0),$$

$$f(x, t+\bar{t}_k) \xrightarrow{\text{unif.}} \bar{\bar{f}}(x, t).$$

(or else we take their subsequence). Assume that $\bar{t}_k = r_k\omega + \tau_k$, where r_k is natural number, $0 \leqslant \tau_k < \omega$. We can even suppose that $\tau_k \to \tau_0$, $r_k < r_{k+1}$, then $\bar{\varphi}(r_k\omega+\tau_0) \to x_0$. So there exists a K such that

$$\|\bar{\varphi}(r_k\omega+\tau_0) - x_0\| < \delta_0 \quad (k \geqslant K),$$

that is

$$x_0 \in \Omega_{\bar{\varphi}} \cap N(\bar{\varphi}(r_k\omega+\tau_0), \delta_0) \quad (k \geqslant K).$$

Hence

$$\lim_{t\to+\infty} \|x(t, x_0, \tau_0+r_k\omega) - \bar{\varphi}(t)\| = 0 \quad (k \geqslant K),$$

that is

$$\lim_{t\to+\infty} \|x(t+\tau_0+r_k\omega, x_0, \tau_0+r_k\omega) - \bar{\varphi}(t+\tau_0+r_k\omega)\| = 0 \quad (k \geqslant K).$$

Since

$$x(t+\tau_0+r_k\omega, x_0, \tau_0+r_k\omega) = x(t+\tau_0, x_0, \tau_0) = y(t, x_0, 0),$$

we obtain

$$\lim_{t\to+\infty} \|y(t, x_0, 0) - \bar{\varphi}(t+\tau_0+r_k\omega)\| = 0 \quad (k \geqslant K). \tag{17}$$

we especially have

$$\lim_{t\to+\infty} \|y(t, x_0, 0) - \bar{\varphi}(t+\tau_0+r_{K+1}\omega)\| = 0,$$

$$\lim_{t\to+\infty} \|y(t, x_0, 0) - \bar{\varphi}(t+\tau_0+r_{K+2}\omega)\| = 0.$$

So

$$\lim_{t\to+\infty} \|\bar{\varphi}(t+\tau_0+r_{K+1}\omega) - \bar{\varphi}(t+\tau_0+r_{K+2})\| = 0.$$

But $\bar{\varphi}(t+\tau_0+r_{K+1}\omega)$, $\bar{\varphi}(t+\tau_0+r_{K+2}\omega)$ are almost periodic functions, and then $\bar{\varphi}(t+\tau_0+r_{K+1}\omega) \equiv \bar{\varphi}(t+\tau_0+r_{K+2}\omega)$, that is, $\bar{\varphi}(t) \equiv \bar{\varphi}(t+(r_{K+2}-r_{K+1})\omega)$. Take $\omega_0 = (r_{K+2} - r_{K+1})\omega$. Hence $\bar{\varphi}(t+\omega_0) \equiv \bar{\varphi}(t)$, that is, $\bar{\varphi}(t)$ is a periodic function. This completes the proof of Theorem 3.1.

§ 4. Almost Periodic Solutions of Almost Periodic Systems

Definition 4.1. *The solution $x=\varphi(t)$ of* (1) *isa uniformly asymptotically stable withrespect to E* (or UASR E) *if it is USR E and there exists $\delta_0>0$ such that for any $\varepsilon'>0$ there is a $T(\varepsilon')>0$, such that when $x_0 \in E \cap N(\varphi(t_0), \delta_0)$ we have*

$$\|\varphi(t) - x(t, x_0, t_0)\| < \varepsilon' \quad (t \geqslant t_0 + T(\varepsilon')).$$

Lemma 4.1. *Assume that the solution $x=\varphi(t)$ of* (1) *is UASR E_1 and $E_2 \subset E_1$. Then it is UASR E_2 and $\delta(\varepsilon)$, δ_0 and $T(\varepsilon')$ are all inherited.*

Proof From Lemma 2.1 and Definition 4.1, we come to the conclusion of

Lemma 4.1.

A notation:

$A^f_\varphi=\{x\,|\,$there exist $\bar{t}\in R$, $x_0\in\Omega_\varphi$ and $g\in H(f)$ with $x=x_g(\bar{t}, x_0, 0)\}$, where $x_g(t, x_0, 0)$ is the solution of system

$$\frac{dx}{dt}=g(x, t) \quad g\in H(f) \tag{18}$$

with

$$x_g(0, x_0, 0)=x_0,$$

It is ease to prove that if $x=\varphi(t)$ is the solution of autonomous system (6) then $A^f_\varphi=\Omega_\varphi$.

Lemma 4.2. *If there exists sequence $\{t_k\}$ with $t_k\geqslant 0$ such that $\varphi(t+t_k)\xrightarrow{\text{loc}}\bar{\varphi}(t)$, $f(x, t+t_k)\xrightarrow{\text{loc}}\bar{f}(x, t)$, then $A^{\bar{f}}_{\bar{\varphi}}\subset A^f_\varphi$.*

Proof Refer to Lemma 2.3 and its proof.

Lemma 4.3. *Assume that the solution $x=\varphi(t)$ of (1) is UASR E and $(\bar{\varphi}, \bar{f})\in H(\varphi, f)$. Then the solution $x=\bar{\varphi}(t)$ of (2) is UASR E and $\delta(\varepsilon)$, δ_0, $T(\varepsilon')$ are inherited.*

Proof Refer to the proof of [6, Theorem 6].

Theorem 4.1. *If system (1) is a uniformly almost periodic system and the solution $x=\varphi(t)$ of (1) is UASR A^f_φ, then $x=\varphi(t)$ is an asymptotic almost periodic function.*

Proof For any sequence $\{t_k\}$ with $t_k\to+\infty$, it is sufficient to prove that there exists a subsequence $\{t'_k\}$ of $\{t_k\}$ such that $\{\varphi(t+t'_k)\}$ converges uniformly on R.

Because $\{\varphi(t+t_k)\}$ is a bounded sequence and $f(x, t)$ is uniformly almost periodic function, we can suppose that $\varphi(t+t_k)\xrightarrow{\text{loc}}\bar{\varphi}(t)$, $f(x, t+t_k)\xrightarrow{\text{loc}}\bar{f}(x, t)$, (or else we can take their subsequences). Assume that $x(t, x_0, t_0)$ is a solution of (1) with $x(t_0, x_0, t_0)=x_0$ and $x_k(t, x_0, t_0)$ is a solution of

$$\frac{dx}{dt}=f(x, t+t_k) \tag{19}$$

with $x_k(t_0, x_0, t_0)=x_0$. Because the solution $x=\varphi(t)$ of (1) is UASR A^f_φ, so for any $\varepsilon>0$ we can take $\delta'=\min(\delta(\varepsilon), \delta_0, \varepsilon)$ such that when $x_0\in A^f_\varphi\cap N(\varphi(t_0), \delta')$, we have

$$\|\varphi(t)-x(t, x_0, t_0)\|<\begin{cases}\varepsilon & (t\geqslant t_0),\\ \delta'/2 & (t\geqslant t_0+T(\delta'/2)).\end{cases} \tag{20}$$

Since $\varphi(t_k)\to\bar{\varphi}(0)$, there exists a K_1 such that $\|\varphi(t_k)-\bar{\varphi}(0)\|<\delta'(k\geqslant K_1)$. So from (20) we see that when $k\geqslant K_1$

$$\|\varphi(t)-x(t, \bar{\varphi}(0), t_k)\|<\varepsilon \quad (t\geqslant t_k),$$

that is, when $k\geqslant K_1$

$$\|\varphi(t+t_k)-x(t+t_k,\ \bar{\varphi}(0),\ t_k)\|<\varepsilon \quad (t\geqslant 0). \tag{21}$$

But $x(t+t_k,\ \bar{\varphi}(0),\ t_k)=x_k(t,\ \bar{\varphi}(0),\ 0)$, then when $k\geqslant K_1$,

$$\|\varphi(t+t_k)-x_k(t,\ \bar{\varphi}(0),\ 0)\|<\varepsilon \quad (t\geqslant 0). \tag{22}$$

Suppose that $y(t,\ x_0,\ t_0)$ is the solution of (2) with $y=(t_0,\ x_0,\ t_0)=x_0$.

(i) We prove that the sequence $\{x_k(t,\ x_0,\ t_0)\}$ converges uniformly to $y(t,\ x_0,\ t_0)$ on $t_0\in R$, $\|x_0\|\leqslant H$ and $t\in[t_0,\ t_0+T(\delta'/2)]$.

Proof If not, there are $t_0^m\in R$, $\|x_0^m\|\leqslant H$, $t^m\in[t_0^m,\ t_0^m+T(\delta'/2)]$ and $\eta_0>0$ such that

$$\|x_m(t^m,\ x_0^m,\ t_0^m)-y(t^m,\ x_0^m,\ t_0^m)\|\geqslant\eta_0. \tag{23}$$

Take $\tau^m=t^m-t_0^m\in[0.\ T(\delta'/2)]$, so from (23) we get

$$\|x_m(t_0^m+\tau^m,\ x_0^m,\ t_0^m)-y(t_0^m+\tau^m,\ x_0^m,\ t_0^m)\|\geqslant\eta_0. \tag{24}$$

We can suppose that $\tau^m\to\bar{\tau}$, $x_0^m\to\bar{x}_0$ and $\bar{f}(x,\ t+t_0^m)\xrightarrow{\text{unif}}\bar{\bar{f}}(x,\ t)$, (or else we can take their subsequences). From $f(x,\ t+t_m)\xrightarrow{\text{unif.}}\bar{f}(x,\ t)$ and $\bar{f}(x,\ t+t_0^m)\xrightarrow{\text{unif}}\bar{\bar{f}}(x,\ t)$, we get

$$f(x,\ t+t_m+t_0^m)\xrightarrow{\text{unif.}}\bar{\bar{f}}(x,\ t) \tag{25}$$

Because $x_m(t+t_0^m,\ x_0^m,\ t_0^m)$ is the solution of

$$\frac{dx}{dt}=f(x,\ t+t_m+t_0^m),$$

$y(t+t_0^m,\ x_0^m,\ t_0^m)$ is the solution of

$$\frac{dx}{dt}=\bar{f}(x,\ t+t_0^m),$$

and from (25), we know that the sequences $\{x_m(t+t_0^m,x_0^m,\ t_0^m)\}$, $\{y(t+t_0^m,\ x_0^m,\ t_0^m)\}$ converge uniformly to $\bar{x}(t,\ \bar{x}_0,\ 0)$ on $t\in[0,\ T(\delta'/2)]$, where $\bar{x}(t,\ \bar{x}_0,\ 0)$ is the solution of

$$\frac{dx}{dt}=\bar{\bar{f}}(x,\ t).$$

that is, there is a K_2 such that when $m\geqslant K_2$,

$$\|x_m(t+t_0^m,\ x_0^m,\ t_0^m)-y(t+t_0^m,\ x_0^m,t_0^m)\|<\eta_0/2 \quad (0\leqslant t\leqslant T(\delta'/2)).$$

In particular we take $t=\tau^m$. Then when $m\geqslant K_2$,

$$\|x_m(t_0^m+\tau^m,\ x_0^m,\ t_0^m)-y(t_0^m+\tau^m,\ x_0^m,\ t_0^m)\|<\eta_0/2.$$

This contradicts (24), so (i) is true.

From (i) we see that there is a K_3. such that when $k\geqslant K_3$,

$$\|x_k(t,\ x_0,\ t_0)-y(t,\ x_0,\ t_0)\|<\delta'/2 \quad (t_0\in R,\ \|x_0\|\leqslant H,\ t\in[t_0,\ t_0+T(\delta'/2)]). \tag{26}$$

(ii) We prove that for any m,

$$\|x_k(m\cdot T(\delta'/2),\ \bar{\varphi}(0),\ 0)-\bar{\varphi}(m\cdot T(\delta'/2))\|<\delta' \quad (k\geqslant K_3). \tag{27}$$

Proof We prove (27) with mathematical induction. Obviously if $m=0$, (27) is true. Suppose (27) is true for $m=i$. For $m=i+1$, we write $T=T(\delta'/2)$. Then

$$\|x_k((i+1)T,\ \bar{\varphi}(0),\ 0)-\bar{\varphi}((i+1)T)\|$$

$$= \|x_k((i+1)T,\ x_k(iT,\ \bar\varphi(0),\ 0),\ iT) - \bar\varphi((i+1)T)\|$$
$$\leqslant \|x_k((i+1)T,\ x_k(iT,\ \bar\varphi(0),\ 0),\ iT) - y((i+1)T,\ x_k(iT,\ \bar\varphi(0),\ 0),\ iT)\|$$
$$+ \|y((i+1)T,\ x_k(iT,\ \bar\varphi(0),\ 0),\ iT) - \bar\varphi((i+1)T)\|. \tag{28}$$

From (26), we obtain

$$\|x_k((i+1)T,\ x_k(iT,\ \bar\varphi(0),\ 0),\ iT) - y((i+1)T,\ x_k(iT,\ \bar\varphi(0),0),\ iT)\| < \delta'/2 \quad (k\geqslant K_3). \tag{29}$$

From Lemma 4.3, we know that the solution $x=\bar\varphi(t)$ of (2) is UASR A^t_φ and $\delta(\varepsilon)$, δ_0, $T(\delta'/2)$ are inherited.

From the supposition of induction,

$$\|x_k(iT,\ \bar\varphi(0),\ 0) - \bar\varphi(iT)\| < \delta'$$

and $x_k(iT,\ \bar\varphi(0),\ 0)\in A^t_\varphi$, we get $x_k(iT,\ \bar\varphi(0),\ 0)\in A^t_\varphi \cap N(\bar\varphi(iT),\ \delta')$. Hence

$$\|y(t,\ x_k(iT,\ \bar\varphi(0).\ 0).\ iT) - \bar\varphi(t)\| < \begin{cases}\varepsilon, & t\geqslant iT\\ \delta'/2, & t\geqslant (i+1)T.\end{cases}$$

In particular we take $t=(i+1)T$. Then

$$\|y((i+1)T,\ x_k(iT,\ \bar\varphi(0),\ 0),\ iT) - \bar\varphi((i+1)T)\| < \delta'/2. \tag{30}$$

From (28), (29) and (30), we get

$$\|x_k((i+1)T,\ \bar\varphi(0),\ 0) - \bar\varphi((i+1)T)\| < \delta'.$$

Therefore (27) is true for $m=i+1$, that is (ii) is true.

From (ii),

$$x_k(mT,\ \bar\varphi(0),\ 0)\in A^t_\varphi\cap N(\bar\varphi(mT),\ \delta') \quad (k\geqslant K_3).$$

Because the solution $x=\bar\varphi(t)$ of (2) is UASR A^t_φ and $\delta(\varepsilon)$, δ_0, $T(\delta'/2)$ are inherited we have for $k\geqslant K_3$,

$$\|y(t,\ x_k(mT,\ \bar\varphi(0),\ 0),\ mT) - \bar\varphi(t)\| < \varepsilon \quad (t\geqslant mT). \tag{31}$$

For any $t\in[0,\ +\infty)$, we suppose $t\in[mT,\ (m+1)T)$. From (26), (31) we get, for $k\geqslant K_3$,

$$\|x_k(t,\ \bar\varphi(0),\ 0) - \bar\varphi(t)\| = \|x_k(t,\ x_k(mT,\ \bar\varphi(0),\ 0),\ mT) - \bar\varphi(t)\|$$
$$\leqslant \|x_k(t,\ x_k(mT,\ \bar\varphi(0),\ 0),\ mT) - y(t,\ x_k(mT,\ \bar\varphi(0),\ 0),\ mT)\|$$
$$+ \|y(t,\ x_k(mT,\ \bar\varphi(0),\ 0),\ mT) - \bar\varphi(t)\| < \delta' + \varepsilon \leqslant 2\varepsilon. \tag{32}$$

Take $K=\max(K_1,\ K_3)$. Then from (22), (32), we have

$$\|\varphi(t+t_k) - \bar\varphi(t)\| < 3\varepsilon \quad (t\geqslant 0)$$

when $k\geqslant K$, that is, sequence $\{\varphi(t+t_k)\}$ converges uniformly to $\bar\varphi(t)$ on R^+. This completes the proo of Theorem 4.1.

Lemma 4.4. *If $f(x,\ t)$ isa uniformly almost periodic function, then $f\in\Omega(f)$.*

Proof From the definition of almost periodic function we can easily come to the conclusion.

Theorem 4.2. *Under the supposition of Theorem 4.1, there is an almost periodic solution $x=\bar\varphi(t)$ of (1) with $\bar\varphi\in\Omega(\varphi)$ and it is UASR $A^t_{\bar\varphi}$.*

Proof From Theorem 4.1, $x=\varphi(t)$ is an asymptotic almost poriodic function.

From Lemma 4.4, there exists a sequence $\{t_k\}$ with $t_k \to +\infty$ such that

$$f(x, t+t_k) \xrightarrow{\text{unif}} f(x, t).$$

But sequence $\{\varphi(t+t_k)\}$ is a uniformly bounded and equi-continuous sequence, so there exists a subsequence $\{\varphi(t+t_{k_m})\}$ of $\{\varphi(t+t_k)\}$ such that

$$\varphi(t+t_{k_m}) \xrightarrow{\text{loc}} \bar{\varphi}(t),$$

that is, $\bar{\varphi} \in \Omega(\varphi)$. Since $\varphi(t)$ is an asymptotic almost periodic function, $\bar{\varphi}(t)$ is an almost periodic function. It is obviously that $\bar{\varphi}(t)$ is the solution of (1). From Lemma 4.3, the solution $x=\bar{\varphi}(t)$ of (1) is USAR A^f_{φ}. From Lemma 4.2, $A^f_{\bar{\varphi}} \subset A^f_{\varphi}$. From Lemma 4.1, we conclude that the solution $x=\bar{\varphi}(t)$ of (1) is UASR $A^f_{\bar{\varphi}}$. This is the proof of Theorem 4.2.

References

[1] Deysach, L. C. & Sell, C. R., On the existence of almost periodic motions, *The Michgan Mathematical Journal*, **12**: 1—4 (1965), 87—95.

[2] Sell, G. R., Periodic solutiou and asymptotic stability, *J. Differential Equation*, **2** (1966), 143—157.

[3] Cronin, J., Periodic solution in n-dimensions and Volterra equations, *J. Differential Equation*, **19** (1975), 21—35.

[4] Artstein, Z., The limiting equations of nonautonomous ordinary differential equations, *J. Differential Equations*, **25** (1977). 184—202.

[5] Nemitskii, V. V. & Stepanov, V. V., Qualitative Theorem of Differential Equations, 1949 (Russian).

[6] Artstein, Z., Uniformly asymptotic stabilty via the limiting equation, *J. Differential Equation*, **27** (1978), 172—189.