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A NEW CHARACTERIZATION OF NONDETERMINISTICALLY RECOGNIZABLE FAMILIES OF LANGUAGES**

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Abstract

In this paper, the author establishes the concepts of relative self-compatibility and relative finite derivability of languages and obtains the relation between self-compatibility and relative self-compatibility. From this it is proved that a family of languages is nondeterministically recognizable if and only if it is relatively self-compatible and relatively finitely derivable to some set of families of langugaes.

Motivated by problem solving, Havel established the (deterministic) finite branching automaton and the (deterministically) recognizable family of languages recognized by the automaton in [3] and [5]. He gave an automaton-independent characterization of recognizable families of languages at the same time. The concepts of the nondeterministic finite branching automaton and the nondeterministically recognizable family of languages were defined in [1]. In the present paper, we give an automaton-independent characterization of nondeterministically recognizable families of languages.

§1. Preliminaries

An alphabet Σ is a finite nonempty set of objects called letters. We denote by Σ^* the free monoid generated by Σ under concatenation. The identity element in Σ^* is the empty string Λ. We call any element and any subset of Σ^* a word and a language over Σ respectively. For any word x over Σ, we denote the number of letters in x by $|x|$, called the length of x. For any u, $v\in\Sigma^*$, u is called a prefix of v, denoted by $u\leqslant v$, iff $v=uw$ for some $w\in\Sigma^*$.

In the context we shall use the following notations:

$$\mathscr{L}(\Sigma)=2^{\Sigma^*}-\{\phi\};$$

for any $L\subseteq\Sigma^*$, $w\in\Sigma^*$

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$$\mathrm{Pref}(L)=\{u\mid u\in\Sigma^*,\ (\exists v\in L)\ u\leqslant v\};$$
$$\mathrm{Fst}(L)=\mathrm{Pref}\ (L)\ \cap\Sigma;$$
$$\mathrm{Fst}_\Lambda(L)=(\mathrm{Pref}(L)\cap\Sigma)\cup(L\cap\{\Lambda\});$$
$$\partial_w(L)=\{u\mid u\in\Sigma^*,\ wu\in L\}.$$

Any subset X of $\mathscr{L}(L)$ is called a family of languages over Σ (in short, a family).

Definition 1. 1[3]. *A (deterministic) finite branching automaton (fb-automaton) is a quintuple*

$$\mathscr{B}=\langle Q,\ \Sigma,\ \delta,\ q_0,\ B\rangle,$$

where $\langle Q,\ \Sigma,\ \delta,\ q_0\rangle$ is an ordinary finite automaton without final states and B is a subset of $Q\times 2^{\Sigma_\Lambda}$, where $\Sigma_\Lambda=\Sigma\cup\{\Lambda\}$.

Definition 1.2[3]. *A language $L\in\mathscr{L}(\Sigma)$ is accepted by an fb-automaton $\langle Q,\ \Sigma,\ \delta,\ q_0,\ B\rangle$ iff for each $w\in$ Pref(L),*

$$(\delta(q_0,\ w),\ \mathrm{Fst}_\Lambda(\partial_w(L)))\in B.$$

We denote by $T(\mathscr{B})$ the family of all languages accepted by $\mathscr{B}$. Obviously, $T(\mathscr{B})\subseteq\mathscr{L}(\Sigma)$.

Definition 1.3[3]. *A family $X\subseteq\mathscr{L}(\Sigma)$ is recognizable iff $X=T(\mathscr{B})$ for some fb-automaton $\mathscr{B}$.*

Generally, we denote $\mathrm{Rec}_{fb}\Sigma=\{X\mid X$ is a recognizable family of languages over $\Sigma\}$.

Definition 1. 4[3]. *For every $u\in\Sigma^*$, we define a binary replacement operator R_u as follows: for each $L_1,\ L_2\subseteq\Sigma^*$,*

$$R_u(L_1,\ L_2)=(L_1-u\Sigma^*)\cup uL_2.$$

Definition 1. 5[3]. *A family $X\subseteq\mathscr{L}(\Sigma)$ has the replacement property iff for each $L_1,\ L_2\in X$ and each $u\in$ Pref$(L_1)\cap$ Pref(L_2),*

$$R_u(L_1,\ \partial_u(L_2))\in X.$$

Definition 1. 6.[3] *Let $X\subseteq\mathscr{L}(\Sigma)$ and $w\in\Sigma^*$. We define*

$$\partial_w(X)=\{\partial_w(L)\mid L\in X\}-\{\phi\},$$
$$\mathscr{D}(X)=\{\partial_w(X)\mid W\in\Sigma^*\}.$$

We call the family $\partial_w(X)$ the derivative of X with respect to w.

Definition 1. 7.[3] *Let $X\subseteq\mathscr{L}(\Sigma)$ and $L\in\mathscr{L}(\Sigma)$. We say that L is compatible with X iff for each $w\in\Sigma^*$ there is a language $L_w\in X$ such that*

$$\mathrm{Fst}_\Lambda(\partial_w(L))=\mathrm{Fst}_\Lambda(\partial_w(L_w)).$$

We denote by $C(X)$ the family of all languages compatible with X. X is called self-compatible iff $X=C(X)$.

Lemma 1. 1.[4] *For every $X\subseteq\mathscr{L}(\Sigma)$,*

$$C(X)=\{L\mid L\in\mathscr{L}(\Sigma),\ (\exists L_u\in X)\ \mathrm{Fst}_\Lambda(\partial_u(L))=\mathrm{Fst}_\Lambda(\partial_u(L_u))\ \text{for any}\ u\in\mathrm{Pref}(L)\}.$$

Theorem 1. 1.[3] *A family $X\subseteq\mathscr{L}(\Sigma)$ is recognizable iff X is self-compatible and $\mathscr{D}(X)$ is finite.*

Definition 1. 8[1]. *A nondeterministic finite branching automaton (nfb-automaton) consists of an ordinary nondeterministic finite automaton $\langle Q, \Sigma, \delta, I\rangle$ without final states and a branching relation*

$$B\subseteq Q\times 2^{\Sigma_A}.$$

Definition 1. 9[1]. *Let $\mathscr{B}=\langle Q, \Sigma, \delta, I, B\rangle$ be an nfb-automaton. We call a partial function $f: \Sigma^*\to Q$ a decision rule for $\mathscr{B}$ iff it satisfies the following two conditions for any $w\in\Sigma^*$ and $a\in\Sigma$,*

(1) *$f(\Lambda)\in I$ if $I\neq\phi$ ($f(\Lambda)$ is undefined otherwise);*

(2) *$f(wa)\in f(w)a$ if $f(w)$ is defined and $f(w)a\neq\phi$ ($f(wa)$ is undefined otherwise).*

Definition 1. 10[1]. *A language $L\in\mathscr{L}(\Sigma)$ is accepted by an nfb-automaton iff there exists a decision rule $f: \Sigma^*\to Q$ for $\mathscr{B}$ such that* $\mathrm{Pref}(L)\subseteq \mathrm{Dom}(f)$ *and* $(f(w), \mathrm{Fst}_A(\partial_w(L)))\in B$.

According to [1], we shall use the notations

$$|\mathscr{B}|=\{L\,|\,L\in\mathscr{L}(\Sigma) \text{ and accepted by } \mathscr{B}\}$$

and

$$\mathrm{Rec}_{nfb}\,\Sigma=\{|\mathscr{B}|\,|\,\mathscr{B}=\langle Q, \Sigma, \delta, I, B\rangle \text{ is an } nfb\text{-automaton}\}$$

and call the elements of $\mathrm{Rec}_{nfb}\,\Sigma$ the nondeterministically-recognizable families of languages over Σ.

Lemma 1. 2.[1] $\mathrm{Rec}_{fb}\,\Sigma\subsetneq\mathrm{Rec}_{nfb}\,\Sigma$.

§ 2. The Relative Self-Compatibility of Families

Definition 2. 1. *Let R be a set of families of languages. We say that R is locked under derivatives iff for each $Y\in R$ and each $a\in\Sigma$, there exists a finite subset $R[Y, a]$ of R such that*

$$\partial_a(Y)=\bigcup_{X\in R[Y,\,a]} X.$$

In general, $R[Y, a]$ corresponding to Y and a is not unique. We select one of them and denote it by $d^a(Y)$. Therefore, for any $w\in\Sigma^*$, $Y\in R$ we define $d^W(Y)$ as follows

1) $d^A(Y)=\{Y\}$,

2) if $w=w'a$, then $d^w(Y)=\bigcup_{Z\in d^{w'}(Y)} d^a(Z)$.

Clearly, if R is locked under derivatives, then for any $w\in\Sigma^*$, $Y\in R$, $d^w(Y)$ is a finite subset of R.

Definition 2. 2. *Let X be a family, R a set of families of languages which is locked under derivatives. X and R are called depeudent iff there exists a finite subset R_0*

of R such that

$$X=\bigcup_{Y\in R_0} Y.$$

For any $w\in\Sigma^*$, we define the derivative of X with respect to w relative to R as

$$D_R^w(X)=\bigcup_{Y\in R_0} d^w(Y).$$

It is is easy to see that if X and R are dependent then for any $w\in\Sigma^*$, $D_R^w(X)$ is a finite set of families.

Definition 2. 3. *Let X and R be dependent. We say X is finitely derivable relative to R iff $\{D_R^w(X)\mid w\in\Sigma^*\}$ is finite.*

Lemma 2. 1. *Let X and R be dependent. If X is finitely derivable relative to R, then X is finitely derivable.*

Proof Since X and R are dependent, there exists a finite subset R_0 of R such that $X=\bigcup_{Y\in R_0} Y$, and for any $w\in\Sigma^*$, $\partial_w(X)=\bigcup_{Y\in R_0}\partial_w(Y)$.

Now we first prove that for any $w\in\Sigma^*$, $Y\in R$, $\partial_w(Y)=\bigcup_{Z\in d^w(Y)} Z$ by induction on $|w|$. When $|w|=0$, it is trival. Assume the result holds for $|w|=n$. Consider $w=w_1a$, $|w|=n$, $a\in\Sigma$. We have

$$\begin{aligned}\partial_w(Y)&=\partial_{w_1a}(Y)=\partial_a(\partial_{w_1}(Y))=\partial_a\Big(\bigcup_{Z\in d^{w_1}(Y)} Z\Big)=\bigcup_{Z\in d^{w_1}(Y)}\partial_a(Z)\\&=\bigcup_{Z\in d^{w_1}(Y)}\Big(\bigcup_{X\in d^a(Z)} X\Big)=\Big(\bigcup_{X\in\bigcup_{Z\in d^{w_1}(Y)} d^a(Z))} X=\bigcup_{X\in d^w(Y)} X.\end{aligned}$$

Consequently, by induction hypothesis for any $w\in\Sigma^*$, $Y\in R$,

$$\partial_w(Y)=\bigcup_{Z\in d^w(Y)} Z.$$

By the dependence of X and R, we know that for any $w\in\Sigma^*$,

$$\begin{aligned}\partial_w(X)&=\bigcup_{Y\in R_0}\partial_w(Y)=\bigcup_{Y\in R_0}\Big(\bigcup_{Z\in d^w(Y)} Z\Big)\\&=\bigcup_{Z\in\bigcup_{Y\in R_0} d^w(Y)} Z=\bigcup_{Z\in D_R^w(X)} Z.\end{aligned}$$

So

$$\{\partial_w(X)\mid w\in\Sigma^*\}=\{\bigcup_{Z\in D_R^w(X)} Z\mid w\in\Sigma^*\}.$$

Evidently, $\{d_R^w(X)\mid w\in\Sigma^*\}$ is finite implies that $\{\partial_w(X)\mid w\in\Sigma^*\}$ is finite. Thus X is finitely derivable.

Definition 2. 4. *Let X be a family, R a set of families. We say L is compatible with X through R iff*

1) *For any $w\in$ Pref(L), there exist $X_w\in D_R^w(X)$ and $L_w\in X_w$ such that*

$$\mathrm{Fst}_A(\partial_w(L))=\mathrm{Fst}_A(L_w),$$

2) *All the X_w's determined in 1) satisfy the condition*

$$w=w_1a\in \mathrm{Pref}(L)\Rightarrow X_w\in D_R^a(X_{w_1}).$$

We denote by $C_R(X)$ the family of all languages compatible with X through R.

X is called relatively self-compatible to R iff $X = C_R(X)$.

Lemma 2. 2. *Let $R = \mathscr{D}(X) = \{\partial_w(X) \mid w \in \Sigma^*\}$. Then $C_R(X) = C(X)$.*

Proof Let $L \in C(X)$. By the definition of $C(X)$, for any $w \in \mathrm{Pref}(L)$ there exists $L_w \in X$ such that $\mathrm{Fst}_\Lambda(\partial_w(L)) = \mathrm{Fst}(\partial_w(L_w))$. $\partial_w(L_w) \in \partial_w(X) (\in R)$ is evident. So for any $w \in \mathrm{Pref}(L)$ there exist $X_w = \partial_w(X) \in R$ and a language $\partial_w(L_w) \in \partial_w(X)$ such that $\mathrm{Fst}_\Lambda(\partial_w(L)) = \mathrm{Fst}_\Lambda(\partial_w(L_w))$. Hence $L \in C_R(X)$. This implies $C(X) \subseteq C_R(X)$. The proof of the other inclusion is the same as above by Lemma 1.1. Thus we obtain $C_R(X) = C(X)$.

Lemma 2.3. *If X and R are dependent, then $X \subseteq C_R(X) \subseteq C(X)$.*

Proof Consider the first inclusion. Let $L \in X$. By Definitions 2.1, 2.2 and 2.3, there exists a finite $R_0 \subseteq R$ such that $X = \bigcup_{X' \in R_0} X'$. Then L must belong to some X' of R_0. Denote the special X' and L by X_Λ and L_Λ respectively. Hence $\mathrm{Fst}_\Lambda(\partial_\Lambda(L)) = \mathrm{Fst}_\Lambda(\partial_\Lambda(L_\Lambda))$. For any $a \in \mathrm{Pref}(L) \cap \Sigma$, we have $D_R^a(X_\Lambda) \subseteq D_R^a(X)$ and there exists $X' \in D_R^a(X_\Lambda)$ such that $\partial_a(L) \in X'$. Denote the X' by X_a, then there exists $L_a = \partial_a(L) \in X_a (\in D_R^a(X))$ such that $\mathrm{Fst}_\Lambda(\partial_a(L_a)) = \mathrm{Fst}_\Lambda(\partial_a(L))$. Step by step, for any $(\Lambda \neq)$ $w \in \mathrm{Pref}(L)$ we can find $X_w \in D_R^w(X)$ and $L_w = \partial_w(L) \in X_w$ such that $\mathrm{Fst}_\Lambda(\partial_w(L)) = \mathrm{Fst}_\Lambda(L_w)$. $X_w \in D_R^a(X_{w1})$ is clear, where $w = w_1 a$, $a \in \Sigma$. Therefore $L \in C_R(X)$, i. e. $X \subseteq C_R(X)$.

Consider the second inclusion. Let $L \in C_R(X)$. By Definition 2.4 for any $w \in \mathrm{Pref}(L)$ there exist $X_w \in D_R^w(X)$ and $L_w \in X_w$ such that $\mathrm{Fst}_\Lambda(\partial_w(L)) = \mathrm{Fst}_\Lambda(L_w)$. From the definition of $D_R^w(X)$, we know that there exists $L'_w \in X$ such that $L_w = \partial_w(L'_w)$. Thereof for any $w \in \mathrm{Pref}(L)$ there exists L'_w such that $\mathrm{Fst}_\Lambda(\partial_w(L)) = \mathrm{Fst}_\Lambda(\partial_w(L'_w))$. By Lemma 1.1, $L \in C(X)$, i. e. $C_R(X) \subseteq C(X)$.

Theorem 2. 1. *Let X be a self-compatible family, R a set of families and X and R be dependent. Then X is relatively self-compatible to R.*

Proof Since X and R are dependent, $X \subseteq C_R(X) \subseteq C(X)$ (Lemma 2.3). From the self-compatibility of X (namely $C(X)$), we have $X = C_R(X) = C(X)$. Thus X is relatively self-compatible to R.

We shall know that the above theorem is only necessary, in other words, there really exist X and R such that $X \subsetneq C_R(X) \subsetneq C(X)$. But we still have the following theorem.

Theorem 2.2. *Let X be relatively self-compatible to R and have the replacement property. Then X is self-compatible.*

The proof depends on the following lemma, which we state and prove first.

Lemma 2. 4. *Let X be relatively self-compatible and have the replacement property. If $L \in C(X)$, then for any non-negative integer k there exists a language $L_k \in X$ satisfying the two conditions.*

1) *for any* $w\in \mathrm{Pref}(L)\cap\left(\bigcup_{i=0}^{k}\Sigma^i\right)$,

$$\mathrm{Fst}_A(\partial_w(L))=\mathrm{Fst}_A(\partial_w(L_k));$$

2) $\mathrm{Pref}\ (L)\cap\left(\bigcup_{i=0}^{k+1}\Sigma^i\right)=\mathrm{Pref}(L_k)\cap\left(\bigcup_{i=0}^{k+1}\Sigma^i\right)$.

Proof By $L\in C(X)$, for any $u\in \mathrm{Pref}\ (L)$ there exists $L_u\in X$ such that

$$\mathrm{Fst}_A(\partial_u(L_u))=\mathrm{Fst}_A(\partial_u(L)).$$

Now we proceed to prove the lemma by induction on k. The conclusion is immediate for the case $k=0$. Suppose the lemma is true for $k=l$. Let $k=l+1$. Construct a language L_{l+1} as

$$L_{l+1}=(L_l-\bigcup_{u\in\Sigma^{l+1}\cap\mathrm{Pref}(L_l)}u\partial_u(L_l))\cup(\bigcup_{u\in\Sigma^{l+1}\cap\mathrm{Pref}(L)}u\partial_u(L)).$$

By induction hypothesis

$$\mathrm{Pref}\ (L)\cap\left(\bigcup_{i=0}^{l+1}\Sigma^i\right)=\mathrm{Pref}\ (L_l)\cap\left(\bigcup_{i=0}^{l+1}\Sigma^i\right).$$

Therefore

$$\Sigma^{l+1}\cap\mathrm{Pref}\ (L)=\Sigma^{l+1}\cap\mathrm{Pref}\ (L_l). \qquad (*)$$

From the replacement property of X and the definition of L_{l+1}, $L_{l+1}\in X$.

Now we prove L_{l+1} satisfies condition 1). We divide it into two cases.

Case (1), $|w|=l+1$. Let $a\in \mathrm{Fst}_A(\partial_w(L)$. Clearly, $wa\in$ Pref $(w\partial_w(L))$. Hence $wa\in$ Pref (L_{l+1}) (by the definition of L_{l+1}), $a\in \mathrm{Fst}_A(\partial_w(L_{l+1}))$. So $\mathrm{Fst}_A(\partial_w(L))\subseteq \mathrm{Fst}_A(\partial_w(L_{l+1}))$. Conversely, let $a\in \mathrm{Fet}_A(\partial_w(L_{l+1}))$, then $wa\in \mathrm{Pref}(L_{l+1})$. Hence $wa\in \mathrm{Pref}\ (\bigcup_{u\in\Sigma^{l+1}\cap\mathrm{Pref}(L)}u\partial_u(L))$. This is because the length of any word in

$$\mathrm{Pref}\ (L_l-\bigcup_{u\in\Sigma^{l+1}\cap\mathrm{Pref}(L)}u\partial_u(L_l))$$

is less than $l+1$, but $|wa|=l+1$. Thus $a\in \mathrm{Fst}_A(\partial_w(L))$, i. e., $\mathrm{Fst}_A(\partial_w(L_{l+1}))\subseteq \mathrm{Fst}_A(\partial_w(L))$. This formula together with $\mathrm{Fst}_A(\partial_w(L))\subseteq \mathrm{Fst}_A\ (\partial_w(L_{l+1}))$ implies $\mathrm{Fst}_A(\partial_w(L))=\mathrm{Fst}_A(\partial_w(L_{l+1}))$.

Case (2). $|w|<l+1$. Let $a\in \mathrm{Fst}_A(\partial_w(L))$. Then $wa\in \mathrm{Pref}(L)$. If $wa\in \mathrm{Pref}\ (\bigcup_{u\in\Sigma^{l+1}\cap\mathrm{Pref}(L)}u\partial_u(L))$, then $wa\in \mathrm{Pref}(L_{l+1})$ (by the definition of L_{l+1}). Therefore $a\in \mathrm{Fst}_A(\partial_w(L_{l+1}))$. If $wa\notin \mathrm{Pref}(\bigcup_{u\in\Sigma^{l+1}\cap\mathrm{Pfef}(L)}u\partial_u(L))$, then for any $u\in\Sigma^{l+1}\cap$ Pref (L), $wa\not\leqslant u$. From the formula $(*)$, for any $u\in\Sigma^{l+1}\cap$ Pref (L_l), $wa\not\leqslant u$. Hence $wa\notin \bigcup_{u\in\Sigma^{l+1}\cap\mathrm{Pref}(L)}u\partial_u(L_l)$. By induction hypothesis

$$wa\in \mathrm{Pref}\ (L)\cap\left(\bigcup_{i=0}^{l+1}\Sigma^i\right)=\mathrm{Pref}(L_l)\cap\left(\bigcup_{i=0}^{l+1}\Sigma^i\right).$$

Then $wa\in \mathrm{Pref}(L_l)$. Therefore $wa\in \mathrm{Pref}\ (L_l-\bigcup_{u\in\Sigma^{l+1}\cap\mathrm{Pref}(L_l)}u\partial_u\ (L))$. Thus $wa\in$ Pref (L_{l+1}). This implies $a\in \mathrm{Fst}_A(\partial_w(L_{l+1}))$, i. e. $\mathrm{Fst}_A(\partial_w(L)\subseteq \mathrm{Fst}(\partial_w(L_{l+1}))$. Conversely, let $a\in \mathrm{Fst}_A(\partial_w(L_{l+1}))$, then we have $wa\in \mathrm{Pref}(L_{l+1}))$. If $wa\in \mathrm{Pref}\ (\bigcup_{u\in\Sigma^{l+1}\cap\mathrm{Pref}(L)}u\,\partial_u(L))$, then $wa\in \mathrm{Pref}(L)$. Therefore $a\in \mathrm{Fst}_A(\partial_w(L))$. If $wa\notin \mathrm{Pref}(\bigcup_{u\in\Sigma^{l+1}\cap \mathrm{Pref}(L)} u\partial_u(L))$, then $wa\in \mathrm{Pref}(L_l-\bigcup_{u\in\Sigma^{l+1}\cap \mathrm{Pref}(L_l)} u\partial_u(L))$. Therefore $wa\in \mathrm{Pref}(L_l)$. By induction hypothesis $wa\in \mathrm{Pref}(L_l)\cap\left(\bigcup_{i=0}^{l+1}\Sigma^i\right)=\mathrm{Pref}(L)\cap\left(\bigcup_{i=0}^{l+1}\Sigma^i\right)$. Consequently, $wa\in \mathrm{Perf}(L)$ and $a\in \mathrm{Fst}_A(\partial_w(L))$. Thus $\mathrm{Fst}_A(\partial_w(L_{l+1}))=\mathrm{Fst}_A(\partial_w(L))$. This completes the proof of condition 1).

Finally, we prove L_{l+1} satisfies condition 2). Let $w\in \mathrm{Pref}(L)\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right)$. When $|w|=0$, it is immediate that $w\in \mathrm{Pref}(L_{l+1})\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right)$. When $|w|\geqslant 1$, we have $w=w_1a$, $a\in\Sigma$ and $w_1\in \mathrm{Pref}(L)\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right)$. From condition 1), we know $a\in \mathrm{Fst}_A(\partial_{w1}(L))=\mathrm{Fst}_A(\partial_{w1}(L_{l+1}))$. Hence $w_1a=w\in \mathrm{Pref}(L_{l+1})\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right)$. Therefore

$$\mathrm{Pref}(L)\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right)\sqsubseteq \mathrm{Pref}(L_{l+1})\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right).$$

Now we prove the other inclusion. Let $w\in \mathrm{Pref}(L_{l+2})\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right)$. When $|w|\leqslant l+1$, we have $w\in \mathrm{Pref}(L_{l+1})$. If $w\in \mathrm{Perf}(\bigcup_{u\in\Sigma^{l+1}\cap \mathrm{Pref}(L)} u\partial_u(L))$, then $w\in \mathrm{Perf}(L)\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right)$. If $w\notin \mathrm{Pref}(\bigcup_{u\in\Sigma^{l+1}\cap \mathrm{Pref}(L)} u\partial_u(L))$, then $w\in \mathrm{Perf}(L_l-\bigcup_{u\in\Sigma^{l+1}\cap \mathrm{Pref}(L_l)} u\partial_u(L_l))$. Hence $w\in \mathrm{Pref}(L_l)$. Therefore $w\in \mathrm{Pref}(L_l)\cap\left(\bigcup_{i=0}^{l+1}\Sigma^i\right)$ by $|w|\leqslant l+1$. By induction hypothesis $w\in \mathrm{Pref}(L)\cap\left(\bigcup_{i=0}^{l+1}\Sigma^i\right)$. So $w\in \mathrm{Pref}(L)\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right)$. When $|w|=l+2$, we have $w=w_1a$, $a\in\Sigma$ and $|w_1|=l+1$. Similarly, we can show that $w_1\in \mathrm{Pref}(L_{l+1})$ and $a\in \mathrm{Fst}_A(\partial_{w_1}(L_{l+1}))$. By the definition of L_{l+1}, $w_1\in \mathrm{Pref}(L)$. From condition 1), we see that $a\in \mathrm{Fst}_A(\partial_{w1}(L_{l+1}))=\mathrm{Fst}_A(\partial_w(L))$. Hence $w_1a=w\in \mathrm{Pref}(L)$. That proves

$$\mathrm{Pref}(L)\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right)\sqsupseteq \mathrm{Pref}(L_{l+1})\cap\left(\bigcup_{i=r}^{l+2}\Sigma^i\right).$$

Thus

$$\mathrm{Pref}(L)\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right)=\mathrm{Pref}(L_{l+1})\cap\left(\bigcup_{i=0}^{l+2}\Sigma^i\right).$$

Proof of Theorem 2.2. Let $L\in C(X)$. From Lemma 2.4, we know that for any non-negative integer k, there exists a language $L_k\in X$ satisfying condition 1) and 2). Thus we obtain a sequence of languages in X,

$$L_0,\ L_1,\ L_2,\ \cdots,\ L_k,\ \cdots. \qquad (*)$$

By $L_i\in X$, $L_i\in C_R(X)$, $i=0, 1, 2, \cdots$, each L_i of this sequence satisfies the condition: for any $w\in \mathrm{Pref}(L_i)$ there exist $X_w\in D_R^w(X)$ and $L_w\in X_w$ such that $\mathrm{Fst}_A(\partial_w(L_i))=\mathrm{Fst}_A(L_w)$ and that $w=w_1a\in \mathrm{Pref}(L_i)$ implies $X_w\in D_R^a(X_{w_1})$. This is equivalent to saying each L_i can determine a partial function f_i: $\Sigma^*\dashrightarrow R$ defined by

$$f_i(w)=\begin{cases}X_w & \text{if } w\in \mathrm{Pref}(L_i),\\ \text{undefined} & \text{if } w\notin \mathrm{Pref}(L_i)\end{cases}$$

which satisfies the condition: if f_i is defined on $w=w_1a$ then $f_i(w)\in D_R^w(X)\cap D_R^a(f_i(w_1))$, and for any $w\in \mathrm{Pref}\ (L_i)$ there exists $L_w\in f_i(w)$ such that $\mathrm{Fst}_A(\partial_w(L_i))=\mathrm{Fst}_A(L_w)$. In view of this, we obtain a sequence of partial functions corresponding to the above sequence of languages,

$$f_0,\ f_1,\ f_2,\ \cdots,\ f_k,\ \cdots. \tag{1}$$

By Lemma 2.4 and the definition of the function sequence, the sequence satisfies the following condition: for any $i_1,\ i_2,\ 0\leqslant i_1\leqslant i_2$,

$$\mathrm{Dom}\ (f_{i1})\cap\left(\bigcup_{i=0}^{i_1+1}\Sigma^i\right)=\mathrm{Dom}\ (f_{i2})\cap\left(\bigcup_{i=0}^{i_1+1}\Sigma^i\right).$$

For any f_i in sequence (1), it is clear that $f_i(\Lambda)\in D_R^\Lambda(X)$. Since $D_R^\Lambda(X)$ is finite, there is at least one infinite subsequence of (1) with the same value on Λ. Define the following subsequence

$$f_{00},\ f_{01},\ f_{02},\ \cdots,\ f_{0k},\ \cdots, \tag{2}$$

as one of such infinite subsequences.

The same as above, for any f_{0i} and $a\in\Sigma$, $f_{0i}(a)\in D_R^a(X)$. Since $D_R^a(X)$ is finite for any $a\in\Sigma$, we can also obtain a subsequence of (2) as

$$f_{01},\ f_{11},\ f_{12},\ \cdots,\ f_{1k},\ \cdots, \tag{3}$$

all the elements of which have the same restriction on Σ.

Similarly, we can obtain a subsequence of (3) as

$$f_{20},\ f_{21},\ f_{22},\ \cdots,\ f_{2k},\ \cdots, \tag{4}$$

whose elements have the same restriction on Σ^2.

Step by step, we can obtain a series of sequences

$$(f_i),\ (f_{0i}),\ (f_{1i}),\ (f_{2i}),\ \cdots,\ (f_{ki}),\ \cdots,$$

where each element (f_{ki}) is a subsequence of last one and the whole elements of (f_{ki}) have the same restriction on $\bigcup_{i=0}^{k}\Sigma^i$.

Let us now define a partial function f: $\Sigma^*\to R$ by

$$f(w)=f_{nn}(w)$$

for $w\in\Sigma^*$ and $n=|w|$.

Clearly this definition is well defined. About f, we have the two following concluions:

1) That $f(\Lambda)\in D_R^\Lambda(X)$, and that $f(w_1a)$ is defined implies $f(w_1a)\in D_R^{w_1a}(X)$ and $f(w_1a)\in D_R^a(f(w_1))$, where $a\in\Sigma$.

2) For any $w\in \mathrm{Pref}(L)$ there exists $L_w\in f(w)$ such that

$$\mathrm{Fst}_A(L_w)=\mathrm{Fst}_A(\partial_w(L)).$$

Proof of 1) Obviously, $f(\Lambda)\in D_R^\Lambda(X)$ and that $f(w_1a)$ is defined implies

$f(w_1a)\in D_R^{w_1a}(X)$. By $w_1a\in$ Pref(L), $|w_1|=n-1$ and $a\in\Sigma$, we know that $f(w_1)=f_{(n-1)(n-1)}(w_1)$, $f(w)=f_{nn}(w)$. Then $f(w)\in D_R^a(f_{nn}(w_1))$. From the definition of (f_{ni}), we see that $f_{nn}(w_1)=f_{(n-1)(n-1)}(w_1)$ (by $|w_1|=n-1$). Thus $f(w)=f_{nn}(w)\in D_R^a(f_{nn}(w_1))=D_R^a(f_{(n-1)(n-1)}(w_1))=D_R^a(f(w_1))$.

Proof of 2) Let $w\in$ Pref(L) and $|w|=n$, the corresponding language of f_{nn} be L_m (an element of sequence (*)). Clearly, $m\geqslant n$. By Lemma 2.4, $w\in$ Pref(L_m) (since $|w|=n\leqslant m$). Hence there exists $L_w\in f_{nn}(w)=f(w)$ such that $\mathrm{Fst}_A(\partial_w(L_m))=\mathrm{Fst}_A(L_w)$. So $\mathrm{Fst}_A(\partial_w(L_m))=\mathrm{Fst}_A(\partial_w(L))$ by Lemma 2.4. Thus $\mathrm{Fst}_A(L_w)=\mathrm{Fst}_A(\partial_w(L))$.

From the above conclusions and the definition of f, we see that for any $w\in$ Pref(L), there exist $X_w=f(w)\in D_R^w(X)$ and $L_w\in X_w$ such that $\mathrm{Fst}_A(\partial_w(L))=\mathrm{Fst}_A(L_w)$, and all the X_w's satisfy the condition that if $w=w_1a\in$ Pref(L), then $X_w\in D_R^a(X_w)$. This is equivalent to saying $L\in C_R(X)$. Hence $L\in X$. Therefore $C(X)\subseteq X$. Thus $C(X)=X$.

§ 3. The New Characterization of Nondeterministically Recognizable Families

From Lemma 1, 2, we know that recognizability and nondeterministic recognizability of families are not equivalent. In this section, we give a characterization of nondeterministically recognizable families.

Definition 3. 1. *Let $\mathscr{B}_1$, $\mathscr{B}_2$ be two nfb-automata. $\mathscr{B}_1$, $\mathscr{B}_2$ are called equivalent iff $|\mathscr{B}_1|=|\mathscr{B}_2|$.*

Definition 3. 2. *Let $\mathscr{B}=\langle Q, \Sigma, \delta, I, B\rangle$ be an nfb-automaton, $Q=\{q_1, q_2, q_3, \cdots, q_k\}$. For any $q_i\in Q$, denote $\mathscr{B}_{q_i}=\langle Q, \Sigma, \delta, \{q_i\}, B\rangle$. We say that $\mathscr{B}$ is irreducible iff for any q_i, q_j, $q_j\neq q_j$ implies $|\mathscr{B}_{q_i}|\neq|\mathscr{B}_{q_j}|$.*

From the proof of Theorem 5 in [2], we obtain the following lemma easily.

Lemma 3. 1. *Let $\mathscr{B}$ be a reducible nfb-automaton. Then there exists an nfb-automaton $\mathscr{B}'$ equivalent to $\mathscr{B}$, but it has less states than $\mathscr{B}$.*

Theorem 3. 1. *For any nfb-automaton, there exists an irreducible nfb-automaton equivalent to it.*

Proof Suppose $\mathscr{B}$ is an *nfb*-automaton. If $\mathscr{B}$ is irrducible, then the result is immediate. If $\mathscr{B}$ is reducible, then there exists an *nfb*-automaton $\mathscr{B}'$ which is equivalent to $\mathscr{B}$, but it has less states than $\mathscr{B}$. If $\mathscr{B}'$ is irreducible, then the result is immediate too. If not, then there exists another *nfb*-automaton $\mathscr{B}''$ which is equivalent to $\mathscr{B}'$ (of course to $\mathscr{B}$), but it has less states than $\mathscr{B}'$. Since all the *nfb*-automata with only one state are irreducible, using the above method successively we can find an irreducible *nfb*-automaton equivalent to $\mathscr{B}$.

Corollary 3. 1. *For any nondeterministically recognizable family, there is an irreducible nfb-automaton to recognizes it.*

Now we come to the main result of this section.

Theorem 3. 2 (Characterization Theorem). *A family X over Σ is nondeterministically recognizable if and only if there exists a set of families R such that*

1) X and R are dependent;

2) X is relatively self-compatible and relatively finitely derivable to R.

Proof Consider the "if" part. Since X is relatively finitely derivable, i. e., $\{D_R^w(X) \mid w\in \Sigma^*\}$ is finite, $\bigcup_{w\in\Sigma^*} D_R^w(X)$ is a finite set of families. Construct an *nfb*-automaton $\mathscr{B}=\langle Q, \Sigma, \delta, I, B\rangle$ as

$$Q=\bigcup_{w\in\Sigma^*} D_R^w(X),\ I=D_R^\Lambda(X),$$

$$\delta(X_i, a)=D_R^a(X_i),\ \text{for } X_i\in Q,\ a\in\Sigma,$$

$$B=\{(X_i, \mathrm{Fst}_\Lambda(L)) \mid X_i\in Q,\ L\in X_i\}.$$

Now we prove that the above automaton recognizes X. Let $L\in X$. We define a partial function $f: \Sigma^*\to Q$ by

$$f_L(w)=\begin{cases} X_w & \text{if } w\in \mathrm{Pref}(L); \\ \text{undefined} & \text{if } w\in \Sigma^*-\mathrm{Pref}(L) \end{cases}$$

for $w\in\Sigma^*$, where X_w is the one determined by w as in Definition 2.4. By this definition, it is easy to see that $f_L(\Lambda)\in I$ (i. e., $D_R^\Lambda(X)$) and for any $wa\in \mathrm{Pref}(L)$, $f_L(wa)\in\delta(f(w), a)$, $a\in\Sigma$. Hence f_L is a decision rule of $\mathscr{B}$, and for any $w\in\mathrm{Pref}(L)$ there exists an L_w in $f(w)$ such that $\mathrm{Fst}_\Lambda(\partial_w(L))=\mathrm{Fst}_\Lambda(L_w)$ (Definition 2.4). Therefore $(f(w), \mathrm{Fst}_\Lambda(\partial_w(L)))\in B$. That is to say $L\in|\mathscr{B}|$. Thus $X\subseteq|\mathscr{B}|$. Conversely, let $L\in|\mathscr{B}|$, then $\mathscr{B}$ has a decision rule f_L: $\Sigma^*\to Q$ such that $\mathrm{Pref}(L)\subseteq\mathrm{Dom}(f_L)$, and for any $w\in\mathrm{Pref}(L)$, $(f_L(w), \mathrm{Fst}_\Lambda(\partial_w(L)))\in B$. From the definition of $\mathscr{B}$ and Definitions 2.1 and 2.2, we know $f_L(w)\in D_R^w(X)$. Let $X_w=f_L(w)$. Then for any $a\in\Sigma$, if $w_1a\in\mathrm{Pref}(L)$ then $X_{w_1a}\in D_R^a(X_{w_1})$. By $(f_L(w), \mathrm{Fst}_\Lambda(\partial_w(L)))\in B$ and the definition of B, there exists $L_w\in X_w$(i. e., $f_L(w)$) such that $\mathrm{Fst}_\Lambda(L_w)=\mathrm{Fst}_\Lambda(\partial_w(L))$. Hence $L\in C_R(X)$, i. e., $L\in X$. Therefore $|\mathscr{B}|\subseteq X$. Thus $X=|\mathscr{B}|$.

Consider the "only if" part. Let X be a nondeterministically recognizable family. From Corollary 3.1, we see there exists an irreducible *nfb*-automaton $\mathscr{B}$ such that $|\mathscr{B}|=X$. Let $\mathscr{B}=\langle Q, \Sigma, \delta, I, B\rangle$, $Q=\{q_1, q_2, \cdots, q_k\}$, and $\mathscr{B}_{q1}, \mathscr{B}_{q2}, \cdots, \mathscr{B}_{qk}$ be the k automata defined in Definition 3.2. Denote the k families recognized by above k automata by $X_1, X_2, \cdots, X_k$ respectively, and let $R=\{X_1, X_2, \cdots, X_k\}$. Now we prove that

1) X and R are dependent;

2) X is relatively self-compatible to R;

3) X is relatively finitely derivable to R.

1) For any $X_{i_0}\in R$, it is clear that $\partial_a(X_{i_0})=\bigcup_{q_i\in\delta(q_{i_0},a)} X_i$. Since $\delta(q_{i_0}, a)$ is finite, we know $\{X_i \mid q_i\in\delta(q_{i_0}, a)\}$ is a finite subset of R. Thus X and R are dependent since $X=\bigcup_{q_{i_0}\in I} X_i$. We define

$$d_R^a(X_{i0})=\{X_i \mid q_i\in\delta(q_{i_0}, a)\}.$$

Obviously, it is well defined by the irreducibility of $\mathscr{B}$.

2) It is very intuitive that X is relatively finitely derivable since X is finite.

3) Let $L\in C_R(X)$. Then for any $w\in \mathrm{Pref}(L)$, there exist $X_w\in D_R^w(X)$ and $L_w\in X_w$ such that $\mathrm{Fst}_A(\partial_w(L))=\mathrm{Fst}_A(L_w)$, and $w=w_1a\in \mathrm{Pref}(L)$, $a\in\Sigma$ implies $X_{w_1a}\in D_R^a(X_{w_1})$. From this we can define a partial function f: $\Sigma^*\to Q$ by

$$f(w)=\begin{cases} q_w & \text{if } w\in \mathrm{Pref}(L),\\ \text{undefined} & \text{if } w\in \Sigma^*-\mathrm{Pref}(L),\end{cases}$$

where q_w is the initial state of $\mathscr{B}_{q_i}$ such that $|\mathscr{B}_{q_i}|=X_w$. This definition is also well defined by the irreducibility of $\mathscr{B}$. Clearly, for any $w\in\mathrm{Pref}(L)$, $X_w\neq\phi$. Hence $X_{w_1}\neq\varphi$ and $X_w\in D_R^a(X_{w_1})$, if $w=w_1a$, $a\in\Sigma$. Thus $q_w\in\delta(y_{w_1}, a)$, i. e.

$$f(w)\in f(w_1)\cdot a=\delta(f(w_1), a).$$

Since $f(\Lambda)\in I$, f is a decision rule of $\mathscr{B}$ and for any $w\in \mathrm{Pref}(L)$, we have $(f(w),\mathrm{Fst}_A(\partial_w(L)))\in B$. Hence $L\in|\mathscr{B}|$. Therefore $L\in X$. Thus $C_R(X)\subseteq X$. That is to say X is relatively self-compatible to R.

From Theorem 3.2, Theorem 2.2 and Lemma 2.1, we have

Corollary 3. 2. *If X is a nondeterministically recognizable family and has the replacement property, then X is recognizable.*

The following result was given in [4].

Lemma 3. 3. *The class of recognizable families is closed under union, but the class of nondeterministically recognizable families is not.*

Thus we have the following result.

Corollary 3.3. *Let X_1, X_2 be two recognizable families. If $X_1\cup X_2$ has the replacement property, then $X_1\cup X_2$ is recognizable.*

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References

[1] Havel, I. M., Nondeterministically recognizable sets of languages, Math. Found. of Computer Science, 1975.

[2] Havel, I. M., On the branching structure of languages, Math. Found. of Computer Science 45, Springer-Verlag, Berlin, 1976.

[3] Havel, I. M., Finite branching automata: automata theory motivated by problem solving, Math. Found. of Co mputer Science (a. blikle, Ed.), Lecture Notes in Computer Science 28, Springer-Verlag, Berlin.

1975.

[4] Wang Shuiting & Guo Yuqi, Integration of families of languages and the replacement property and semirecognizability of families of languages, *Journal of Lanzhou University, Natural Science Edition*, **18**: 3 (1982). (in Chinese).

[5] Havel, I. M., Finite branching automata, *Kybernetika*, **10** (1974).

[6] Eilenberg, S., Automata, Languages and Machines, Volume A, Academic Press, New York and London, 1974.