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ON DISTRIBUTIVE MODULES AND LOCALLY DISTRIBUTIVE RINGS

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Abstract

Distributive modules over artinian rings are characterized via module diagrams, and it is shown that a left artinian ring is (two-sided) locally distributive in case its left indecomposable injective modules and projective modules are distributive. This latter result is used to show that a locally distributive artinian ring and the endomorphism ring of its minimal cogenerator have identical diagrams.

An artinian ring is locally distributive in case its indecomposable projective modules have distributive submodule lattices. According to [6] such a ring is exact in the sense of Azumaya [3], so our results lend support to Azumaya's conjecture that exact rings have self duality.

A module diagram $\mathscr{M}$ is a finite directed graph with distinguished node 0 such that: (i) there is at most one arrow between any two nodes; (ii) there are no oriented cycles in $\mathscr{M}$ and no arrows emanating from 0; (iii) if $x\neq 0$ then $x\to 0$ in $\mathscr{M}$ if and only if there is no arrow $x\to y\neq 0$ in $\mathscr{M}$. (Cf. [1, 11].) We also let $\mathscr{M}$ denote the set of nodes in the diagram $\mathscr{M}$. A subdiagram $\mathscr{U}\leqslant\mathscr{M}$ satisfies $x\in\mathscr{U}$ and $x\to y$ implies $y\in\mathscr{U}$, and for any subset $X\subseteq\mathscr{M}$, $\mathscr{U}(X)$ denotes the smallest subdiagram of $\mathscr{M}$ containing X.

If $\mathscr{L}$ is any finite distributive lattice and

$$\mathscr{M}=\{x\in\mathscr{L}\mid x \text{ is meet irreducible}\},$$

then $\mathscr{M}$ becomes a module diagram with $x\to y$ in case y is maximal among those elements of $\mathscr{M}$ properly contained in x. (x is meet irreducible in case $x=u\vee v$ implies $x=u$ or $x=v$.) Upon observing that $x\in\mathscr{M}$ and $x\leqslant\bigvee_{i=1}^{m} y_i$ implies $x\leqslant y_i$ for some $i\in\{1,\cdots,m\}$ one easily checks that there is a lattice isomorphism $\delta:\mathscr{L}(\mathscr{M})\to\mathscr{L}$ from the lattice of subdiagrams of $\mathscr{M}$ to $\mathscr{L}$ via $\delta:\mathscr{U}\mapsto\bigvee\mathscr{U}(\mathscr{U}\leqslant\mathscr{M})$. (For a proof, see [7, pp. 82—83].) In this diagram, if $x\to y$ then there is no other path from x to y. We say that a diagram with this property satisfies the lattice condition.

Throughout this paper R is a basic artinian ring with radical J, basic set of

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primitive idempotents $e_1, \cdots, e_n$, and simple modules $S_i \cong Re_i/Je_i$, $i=1, \cdots, n$.

As in [1, 11], a pair $(\mathscr{M}, \delta)$ is a diagram for a finitely generated left R-module M in case Card $(\mathscr{M}\setminus\{0\})=c(M)$, the composition length of M; δ: $\mathscr{L}(\mathscr{M})\to\mathscr{L}(M)$ is an injective lattice homomorphism between the lattices of subdiagrams of $\mathscr{M}$ and submodules of M that satisfies $\delta(\mathrm{Rad}\ \mathscr{U})=\mathrm{Rad}\ \delta(\mathscr{U})$ $(\mathscr{U}\leqslant\mathscr{M})$; and the nodes of $\mathscr{M}$ are labeled by a function $\lambda: \mathscr{M}\setminus\{0\}\to\{1, \cdots, n\}$ such that $\delta(\mathscr{U})/\delta(\mathscr{V})\cong S_{\lambda(x)}$ whenever $\mathscr{V}$ and $\mathscr{V}\cup\{x\}=\mathscr{U}$ are subdiagrams of $\mathscr{M}$.

In order to characterize distributive modules via diagrams we shall use the following lemma, which may prove to have other uses.

If $(\mathscr{M}, \delta)$ is a diagram for ${}_RM$ and $e=e_{i_1}+\cdots+e_{i_k}$ with $1\leqslant i_1<\cdots<i_k\leqslant n$, we let

$$\mathscr{M}_e=\lambda^{-1}(\{i_1, \cdots, i_k\})\cup\{0\}$$

with an arrow $x\to y\neq 0|$ in $\mathscr{M}_e$ in case $x\to y\neq 0$ in $\mathscr{M}$ or $x\to z_1\to\cdots\to z_l\to y\neq 0$ in $\mathscr{M}$ with each $\lambda(z_i)\notin\{i_1, \cdots, i_k\}$, and $x\to 0$ in $\mathscr{M}_e$ otherwise. Then $\mathscr{M}_e$ is a module diagram with labels $\lambda_e=(\lambda|\mathscr{M}_e)$, and if $\mathscr{V}\leqslant\mathscr{M}$ then $\mathscr{V}_e\leqslant\mathscr{M}_e$.

1. **Lemma.** *If $(\mathscr{M}, \delta)$ is a diagram for ${}_RM$ and $e=e_{i_1}+\cdots+e_{i_k}$ with $1\leqslant i_1<\cdots<i_k\leqslant n$ then $(\mathscr{M}_e, \delta_e)$ is a diagram for ${}_{eRe}eM$ where $\delta_e(\mathscr{X})=e\delta(\mathscr{U}(\mathscr{X}))$ for each $\mathscr{X}\leqslant\mathscr{M}_e$.*

Proof Clearly $\delta_e: \mathscr{L}(\mathscr{M}_e)\to\mathscr{L}(eM)$ is a lattice homomorphism. Vpon observing that for each $\mathscr{X}\leqslant\mathscr{M}_e$.

$$\mathscr{U}(\mathscr{X})_e=\mathscr{X} \tag{1}$$

and that Re generates $\delta(\mathscr{U}(\mathscr{X}))$, so that

$$\delta(\mathscr{U}(\mathscr{X}))=Re\,\delta(\mathscr{U}(\mathscr{X}))=R\,\delta_e(\mathscr{X}), \tag{2}$$

one easily checks that δ_e is injective. Also if $\mathscr{V}\leqslant\mathscr{M}$, then applying δ to a composition series for $\mathscr{V}$ yields one for $\delta(\mathscr{V})$,

$$\delta(\mathscr{V})=\delta(\mathscr{V}_0)>\delta(\mathscr{V}_1)>\cdots>\delta(\mathscr{V}_{\mathrm{card}(\mathscr{V}\setminus\{0\})})=0$$

with $\mathscr{V}_{i-1}=\{x_i\}\cup\mathscr{V}_i$, and multiplying the teams by e we see that ${}_{eRe}e\delta(\mathscr{V})$ has composition length

$$c(e\delta(\mathscr{V}))=\mathrm{card}(\mathscr{V}_e\setminus\{0\}). \tag{3}$$

Now observing that $(\mathrm{Rad}\ \mathscr{U}(\mathscr{X}))_e=\mathrm{Rad}\ \mathscr{X}$ we have

$$\begin{aligned} c(e\delta(\mathrm{Rad}(\mathscr{U}(\mathscr{X})))) &\overset{(3)}{=}\mathrm{card}(\mathrm{Rad}(\mathscr{U}(\mathscr{X})_e\setminus\{0\}) \\ &=\mathrm{card}(\mathrm{Rad}(\mathscr{X})\setminus\{0\}) \\ &\overset{(1)}{=}\mathrm{card}(\mathscr{U}(\mathrm{Rad}(\mathscr{X}))_e\setminus\{0\}) \\ &\overset{(3)}{=}c(e\delta(\mathscr{U}(\mathrm{Rad}(\mathscr{X})))) \\ &=c(\delta_e(\mathrm{Rad}(\mathscr{X}))). \end{aligned}$$

So since $\mathscr{U}(\mathrm{Rad}\ \mathscr{X})\subseteq\mathrm{Rad}\ \mathscr{U}(\mathscr{X})$ implies one inclusion, and they have the same composition length, we have

$$\delta_e(\text{Rad } \mathscr{X}) = e\delta(\text{Rad } \mathscr{U}(\mathscr{X})) \tag{4}$$

for any $\mathscr{X} \leqslant \mathscr{M}_e$. Thus δ_e preserves radicals:

$$\delta_e(\text{Rad } \mathscr{X}) \overset{(4)}{=} e\,\delta(\text{Rad } \mathscr{U}(\mathscr{X})) = eJ\delta(\mathscr{U}(\mathscr{X})) \overset{(2)}{=} eJR\delta_e(\mathscr{X}) = eJe\delta_e(\mathscr{X}).$$

Finally, if $\mathscr{X}$, $\mathscr{Y} \leqslant \mathscr{M}_e$ with $\mathscr{X} = \{x\} \cup \mathscr{Y}$ then x is the only element of $\mathscr{U}(x) \setminus \mathscr{U}(\mathscr{Y})$ with label in $\{i_1, \cdots, i_k\}$, so

$$\delta_e(\mathscr{X})/\delta_e(\mathscr{Y}) \cong e(\delta(\mathscr{U}(x))/\delta(\mathscr{U}(x) \cap \mathscr{U}(\mathscr{Y})) \cong S_{\lambda_e(x)}.$$

A module is called distributive in case its lattice of submodules is distributive. In [5, Theorem 1] distributive modules are characterized as those whose quotient modules all have square free socles. According to [10, Lemma 4], a module $_RM$ over the artinian ring R is distributive if and only if, for each $i=1, \cdots, n$, $\{Re_im \mid m \in M\}$ is linearly ordered if and only if the module $_{e_iRe_i}e_iM$ is uniserial for $i=1, \cdots, n$. Moreover, if $_RM$ is distributive then its submodules are all stable under endomorphisms (as in [13, Theorem 2]), and one easily checks that $\mathscr{L}(M)$ is finite.

An R-module X is local in case X/JX is simple, so if $X \leqslant_R M$ then X is meet irreducible in $\mathscr{L}(M)$. Thus we see from an earlier discussion that if $_RM$ is distributive then

$$\mu = \{X \leqslant M \mid X \text{ is local}\} \cup \{0\}$$

with $\delta(\mathscr{U}) = \sum \mathscr{U}$ becoming a diagram, (μ, δ) for M with $X/JX \cong S_{\lambda(X)}$. Now an application of [10, Lemma 4] yields one implication of the first part of the following theorem.

2. **Theorem.** *A module $_RM$ is distributive if and only if there is a diagram $(\mathscr{M}, \delta)$ for M in which any two nodes with the same label are connected by a path.*

Moreover, if these conditions hold then δ is a lattice isomorphism and, up to isomorphism, M has a unique diagram satisfying the lattice condition.

Proof We have just seen that the condition is necessary. For sufficiency, the condition clearly implies that for each $i=1, \cdots, n$, $\mathscr{L}(\mathscr{M}_{e_i})$ is a chain, and so, since δ_{e_i} preserves radicals, $\mathscr{L}(e_iM)$ must also be a chain. Thus M is distributive by [10, Lemma 4].

If $(\mathscr{M}, \delta)$ is a diagram for a distributive module $_RM$ and $\mathscr{U}_1, \cdots, \mathscr{U}_l$ are the maximal subdiagrams of $\mathscr{M}$ then since M/JM is square free[5], $\delta(\mathscr{U}_1), \cdots, \delta(\mathscr{U}_l)$ are the only maximal submodules of M. Thus it follows inductively that $\delta: \mathscr{L}(\mathscr{M}) \to \mathscr{L}(M)$ is actually bijective. Finally, if $\mathscr{M}$ satisfies the lattice condition then $x \mapsto \delta(\mathscr{U}(x))$ is a diagram isomorphism (as defined in [11]) between $\mathscr{M}$ and the diagram $\mathscr{M}$ of local submodules described above.

A finite semigroup $\mathscr{R}$ containing 0 is called an algebra semigroup in case $\mathscr{R} =$

$\{e_1, \cdots, e_n\} \cup \mathscr{J}$ with $e_1, \cdots, e_n$ being orthogonal idempotents and $\mathscr{J}$ a nilpotent ideal such that $\mathscr{R} = \bigcup_{ij} e_i \mathscr{R} e_j$. As in [11], the elements of $\mathscr{R}$ are nodes of diagrams $(\mathscr{R}_l, \delta_l)$ and $(\mathscr{R}_r, \delta_r)$ for the semigroup algebra $K\mathscr{R}$ over a field K. Here $x \to y \neq 0$ in $\mathscr{R}_l$ (in $\mathscr{R}_r$) in case there is an $a \in \mathscr{J} \setminus \mathscr{J}^2$ such that $ax = y$ (resp., $xa = y$) and $x \to 0$ if $\mathscr{J}$ annihilates x; $\delta_l(\mathscr{U}) = K\mathscr{U}$, $\mathscr{U} \leqslant \mathscr{R}_l$; $\lambda_l(x) = i$ if $x = e_i x$; and δ_r and λ_r are defined similarly.

Suppose that the basic left artinian ring R is left locally distributive in the sense that each indecomposable projective Re_i is a distributive module. Then according to Theorem 2 there is a unique (to within isomorphism) diagram $(\mathscr{R}, \delta)$ for ${}_RR$ such that $\mathscr{R}$ satisfies the lattice condition, $\mathscr{R} = \mathscr{P}_1 \cup \cdots \cup \mathscr{P}_n$ with $\mathscr{P}_i \cap \mathscr{P}_j = 0$ if $i \neq j$, and $\delta(\mathscr{P}_i) = Re_i$, $i = 1, \cdots, n$. On the other hand we can associate an algebra semigroup with ${}_RR$ by choosing, for each local left ideal $X = Xe_j \leqslant Je_i$ with $X/JX \cong S_i$, exactly one $x_X = e_i x_X e_j \in X \setminus JX$, and letting

$$\mathscr{J} = \{x_X \mid X \leqslant Je_i,\ X \text{ is local},\ i = 1, \cdots, n\} \cup \{0\}$$

and

$$\mathscr{R}({}_RR) = \{e_1, \cdots, e_n\} \cup \mathscr{J}.$$

Then defining an operation $\cdot$ on $\mathscr{R}({}_RR)$ via

$$x \cdot y = z \in \mathscr{R}({}_RR) \text{ if } Rxy = Rz$$

we obtain the following version of results of Yukimoto in [14].

3. **Theorem.** *If R is left locally distributive then $\mathscr{R} = \mathscr{R}({}_RR)$ is an algebra semigroup such that $(\mathscr{R}_l, \delta_l)$ is a diagram for ${}_RR$, where $\delta_l(\mathscr{U}) = R\mathscr{U}$, $\mathscr{U} \leqslant \mathscr{R}_l$.*

Moreover, if R is also right locally distributive then $(\mathscr{R}_r, \delta_r)$ is a diagram for R_R where $\delta_r(\mathscr{U}) = \mathscr{U}R$, $\mathscr{U} \leqslant \mathscr{R}_r$.

Proof Let $\mathscr{R} = \mathscr{R}({}_RR)$. Using the observation[14] that if $x = xe_j$ and $y = e_j y$ in $\mathscr{R}$, then $Rx \cdot y = Rxy = RxRy$, one easily sees that $(\mathscr{R}, \cdot)$ is an algebra semigroup. If $x, y \in \mathscr{R}$ with $0 \neq Ry$ maximal among the local submodules of $Rx = Re_i x$ then since $Je_i = \sum\{Rae_i \mid a \in \mathscr{J}e_i \setminus \mathscr{J}^2 e_i\}$, Ry is contained in all (hence is equal to some) $Rae_i x = Ra \cdot x$. Conversely, suppose $a \in \mathscr{J}e_i \setminus \mathscr{J}^2 e_i$ and $a \cdot x = y \neq 0$ in $\mathscr{R}$. If $h: Re_i \to Rx$ via $h: re_i \to re_i x$ then $h^{-1}(J^2 x) = J^2 e_i + \operatorname{Ker} h$ cannot cantain Ra (since neither term does), so $Ry = Ra \cdot x$ is maximal among the local left ideals in Rx. Thus we see that $\mathscr{R}_l$ is isomorphic to the diagram $\mathscr{R}$ via $x \mapsto Rx$, and the first statement follows.

Yukimoto's argument (see [14, Proposition 5]) shows that if R is also right locally distributive then $\{e_i x R \mid 0 \neq e_i x \in \mathscr{R}\}$ consists of the distinct local right ideals contained in $e_i R$, and so $(\mathscr{R}_r, \delta_r)$ is a diagram for R_R.

As Yukimoto[14] pointed out, if $\mathscr{R} = \mathscr{R}({}_RR)$ with R left locally distributive then so is $K\mathscr{R}$ (this also follows from Theorem 2), and he showed by example that $K\mathscr{R}$ need not be right locally distributive. We observe next that if $\mathscr{R}$ is any algebra semigroup with $K\mathscr{R}$ left locally distributive then $\mathscr{R}$ satisfies certain necessary conditions for right local distributivity. In particular both the left cancellation condition of [11], $ax_1=ax_2\neq 0$ implies $x_1=x_2$, and its right hand version hold.

4. **Proposition.** *If $\mathscr{R}$ is an algebra semigroup such that $K\mathscr{R}$ is left locally distributive, then $\mathscr{R}$ satisfies both the left and right cancellation conditions and $\mathscr{R}_l$ and $\mathscr{R}_r$ both satisfy the lattice condition.*

Proof Suppose $x_1y=x_2y\neq 0$. Then $\lambda(x_1)=\lambda(x_2)$ in $\mathscr{R}_l$ so by Theorem 2 there is a path from one to the other, say from x_1 to x_2. But then we must have $w\in\mathscr{J}$ with $wx_1=x_2$ and so $w^kx_1y=w^{k-1}x_1y$ for all $k>0$, contrary to nilpotence of $\mathscr{J}$. Thus $\mathscr{R}$ has right cancellation. The lattice condition for $\mathscr{R}_l$ follows, because if $x_1\to x_2\neq 0$ and $x_1\to u_1\to u_2\to\cdots\to u_m\to x_2$ in $\mathscr{R}_l$ then in $\mathscr{R}$ we must have $a\in\mathscr{J}\setminus\mathscr{J}^2$ and $w\in\mathscr{J}^2$ such that $ax_1=x_2=wx_1\neq 0$, contrary to right cancellation. Now if $x_1\neq x_2$ and $ax_1=ax_2=y\neq 0$ with $a\in\mathscr{J}\setminus\mathscr{J}^2$ then $x_1\to y\leftarrow x_2$ appears in $\mathscr{R}_l$ with $\lambda(x_1)=\lambda(x_2)$. But then by local distributivity there would have to be a path from one of x_1 and x_2 to the other, contrary to the lattice condition already established. Thus it follows that $\mathscr{R}$ has left cancellation; and the lattice condition for $\mathscr{R}_r$ holds as it does for $\mathscr{R}_l$.

We note here that Proposition 4, together with the uniqueness part of Theorem 2, shows that if $R=K\mathscr{R}$ is left locally distributive then $\mathscr{R}\cong\mathscr{R}({}_RR)$.

Next we give a characterization of (two-sided) locally distributive rings in terms of their left modules, which we shall subsequently apply to examine the endomorphism rings of their minimal cogenerators.

5. **Theorem.** *A left artinian ring R is locally distributive if and only if its indecomposable left projective modules and injective modules are distributive*

Proof The condition is necessary by [10, Lemma 5].

For sufficiency, let e and f be primitive idempotents in a ring R that satisfies the condition, and let $E=E(Rf/Jf)$ be the injective envelope of Rf/Jf. Then fR and E form a pair in the sense of [9], and we shall prove this implication by using the results of [9] to show that fRe_{eRe} is uniserial. According to [9, Lemma 2.2] $eE(Re/Je)=E(eRe/eJe)$, the left injective envelope of eRe/eJe. Since eRe and eE are uniserial left eRe-modules, eRe is a uniserial ring by [9, Theorem 5.4]. Thus it only remains to show that $fRe/fReJe$ is simple (or zero) over eRe. According to [9, Lemmas 1.1 (a) and 2.3 (b)], for any ideal $I\leqslant{}_RR_R$ and any bisubmodule $Q\leqslant{}_RE_{\mathrm{End}({}_RE)}$ the annihilator conditions

$$Q=r_El_{fR}(Q)=r_E(l_R(Q))$$

and

$$fI=l_{fR}(r_E(fI))=l_{fR}r_E(I)$$

are satisfied. Now let

$$Q=\mathrm{Rer}_E(ReJ)\leqslant{}_RE_{\mathrm{End}({}_RE)}$$

and

$$I=l_R(Q)\leqslant_R R_R.$$

Then since $Q\sqsubseteq r_E(ReJ)$ we have

$$fI=l_{fR}(Q)\sqsupseteq l_{fR}r_E(ReJ)=fPeJ,$$

and since $Ier_E(ReJ)\sqsubseteq IQ=0$ we also have

$$fIe\sqsubseteq l_R(r_E(ReJ))=fReJ,$$

so that

$$fIe=fReJe.$$

Now since E is distributive and $Q\leqslant ReE$, we must have $Q/JQ\cong Re/Je$ by [10, Lemma 4], so since Re is distributive, $Q\cong Re/Ke$ for some ideal $K\leqslant_R R_R$. Thus since $Q=r_E(l_R(Q))=r_E(I)$ we have

$$r_E(I)\cong Re/Ke$$

and so according to [9, Theorem 5.1]

$$fR/fI\cong l_{E(eR/eJ)}(K)\leqslant E(eR/eJ).$$

Finally, applying [9, Lemma 2.2] again we have an embedding of right modules

$$fRe/fReJe=fRe/fIe\hookrightarrow E(eR/eJ)e=E(eRe/eJe)$$

so that $fRe/fReJe$ is simple (or zero) over eRe as promised.

According to the classical theorems of Azumaya [2] and Morita[12], there is a duality between the categories of finitely generated left and right modules over a ring R if and only if R is artinian and the endomorphism ring of the minimal left cogenerator over R is isomorphic to the basic ring of R. Recently it has been proved that serial rings[8, 13] and l-hereditary locally distributive rings[4] have self-duality. Crucial to the proof in [13] is the fact that a basic serial ring and the endomorphism ring of its minimal cogenerator have identical diagrams. Thus we expect that the following result may prove useful in verifying Azumaya's conjecture for locally distributive rings.

6. **Theorem.** *If R is a basic locally distributive ring and S is the endomorphism ring of the minimal left cogenerator over R, then S is locally distributive with the same left and right diagrams as R.*

Proof Let $E=E_1\oplus\cdots\oplus E_n$ with $E_i=E(Re_i/Je_i)$, $i=1, \cdots, n$. Then E is finitely generated, since each E_i is distributive. Thus there is a duality $D=\mathrm{Hom}(-, {}_RE_S)$ between the finitely generated left R-modules and right $S=\mathrm{End}({}_RE)$-modules[2, 12]; so for each finitely generated ${}_RM$ there is a lattice anti-isomorphism $\theta:\mathscr{L}(M)\to\mathscr{L}(D(M))$ via

$$\theta: N\mapsto \mathrm{Ker}(D(i_N)),\ N\leqslant M,$$

where i_N is the inclusion map $N\xrightarrow{i_N}M$. Letting $f_i\in S$ be the idempotent for E_i in the decomposition $E=E_1\oplus\cdots\oplus E_n$, $f_1, \cdots, f_n$ is a basic set of idempotents for S. It follows that the indecomposable right projective S-module $f_iS\cong D(E_i)$ and right injective S-module $D(Re_i)$ are all distributive. Thus by Theorem 5, S is locally distributive. Since all submodules of E_i and e_iR are stable under endomorphisms

$$\varphi: K \mapsto r_{E_i}(K), \quad K \leqslant e_iR$$

defines a lattice anti-isomorphism $\varphi: \mathscr{L}(e_iR) \to \mathscr{L}(E_i)$ by [9, Lemmas 1.1 (a) and 2.3 (b)], so taking $\theta: \mathscr{L}(E_i) \to \mathscr{L}(D(E_i)) = \mathscr{L}(f_iS)$ we have a lattice isomorphism

$$\theta \circ \varphi: \mathscr{L}(e_iR) \to \mathscr{L}(f_iS).$$

Now to see that e_iR and f_iR have isomorphic diagrams we need only to show that if $X \leqslant e_iR$ with $X/XJ \cong e_jR/e_jJ$ then f_jS maps onto $\theta \circ \varphi(X)$. Since φ is a lattice anti-isomorphism $E_i/r_{E_i}(X)$ has simple socle $r_{E_i}(XJ)/r_{E_i}(X)$ which must be isomorphic to Re_j/Je_j, since its annihilator in R is the same as that of X/XJ, according to the proof of [9, Theorem 2.4]. Thus we have an exact sequence

$$0 \to \varphi(X) \xrightarrow{i_{\varphi(X)}} E_i \to E_j$$

which yields an exact sequence

$$D(E_j) \to D(E_i) \xrightarrow{D(i_{\varphi(X)})} D(\varphi(X)) \to 0,$$

so $\theta \circ \varphi(X) = \mathrm{Ker}(D(i_{\varphi(X)}))$ is indeed an epimorph of $D(E_j) \cong f_jS$. Therefore the right (and so by Theorem 3 the left) diagrams of R and S are identical.

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