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# SIGN TYPES AND KAZHDAN-LUSZTIG CELLS

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**Abstract**

This paper studies the relations between sign types and left cells in an affine Weyl group. It is proved that any left cell is a union of finitely many connected sets in the sense of [5]. Also, a geometric explanation of the finiteness of cells in an affine Weyl group is given.

## § 1. Introduction

Let $\Phi$ be an irreducible reduced root system in a real vector space $E$ with a positive definite inner product $\langle\ ,\ \rangle$ such that $|\alpha|=\langle\alpha,\alpha\rangle=1$ for any short root $\alpha$ in $\Phi$. Let $\Delta=\{\alpha_1,\cdots,\alpha_l\}$ be a simple root system in $\Phi$ and $\Phi^+$ be the corresponding set of positive roots. For any $\alpha\in\Phi^+$, $k\in\mathbf{Z}$ and a positive real number $m$, we define a hyperplane $H_{\alpha,k}=H_{-\alpha,-k}=\{v\in E;\ \langle v,\alpha^\vee\rangle=k\}$ and a stripe $H^m_{\alpha,k}=H^m_{-\alpha,-k}=\{v\in E;\ k\leqslant\langle v,\alpha^\vee\rangle\leqslant k+m\}$.

Let $\mathscr{F}=\{H_{\alpha,n};\ \alpha\in\Phi,\ n\in\mathbf{Z}\}$ and $\mathfrak{A}=$ the set of the closure of the connected components of $E-\bigcup_{H\in\mathscr{F}}H$. The elements of $\mathfrak{A}$ are called (closed) alcoves. It is well-known that for any $A\in\mathfrak{A}$, there is a $|\Phi|$-tuple $(k_\alpha)_{\alpha\in\Phi}$ over $\mathbf{Z}$ such that $A=\bigcap_{\alpha\in\Phi^+}H^1_{\alpha,k_\alpha}$ and $k_\alpha$'s satisfy

(1) $k_{-\alpha}=-k_\alpha$, for $\alpha\in\Phi$;

(2) $|\alpha|^2k_\alpha+|\beta|^2k_\beta+1\leqslant|\alpha+\beta|^2(k_{\alpha+\beta}+1)$

$$\leqslant|\alpha|^2k_\alpha+|\beta|^2k_\beta+|\alpha|^2+|\beta|^2+|\alpha+\beta|^2-1$$

for $\alpha,\beta\in\Phi$ with $\alpha+\beta\in\Phi^+$ (see [6, Theorem 5.2]).

Let $\mathscr{E}_\Delta$ denote the set of $|\Phi|$-tuples $K=(k_\alpha)_{\alpha\in\Phi}$, which satisfies (1) and (2). Then the map $A\to(k_\alpha)_{\alpha\in\Phi}$ gives a bijection from $\mathscr{U}$ to $\mathscr{E}_\Delta$ and we call $K$ the coordinate form of $A$.

Let $W$ (resp. $W_a$) be the Weyl (resp. affine Weyl) group determined by $\Phi$. Then $W$ is generated by the reflections $s_\alpha$ on $E$ for $\alpha\in\Phi$, and $W_a$ is the semi-direct product $W\propto D$ where $D$ denotes the group consisting of all translations $T_\lambda$, $\lambda\in\mathbf{Z}\Phi$ on $E$. Let $-\alpha_0$ be the highest short root of $\Phi$ and $s_0=s_{\alpha_0}\cdot T_{-\alpha_0}$, $s_i=s_{\alpha_i}(1\leqslant i\leqslant l)$. Then $W_a$ can be regarded as a Coxeter group with generator set $S=\{s_i;\ 0\leqslant i\leqslant l\}$. If $z=$

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$xy$ and $l(z)=l(x)+l(y)$, then we write $z=x\cdot y$.

By [6] there exists a bijection between $W_a$ and $\mathfrak{A}$ such that if $w\in W_a$ corresponds to $A_w=(k(\alpha, w))_{\alpha\in\Phi}$, then $k(\alpha, w)=\langle\lambda, \alpha^\vee\rangle+k(\alpha, \bar{w})$, where $w=\bar{w}T_\lambda$ for $\bar{w}\in W$ and $\lambda\in\mathbf{Z}\Phi$, and

(3) $k(\alpha, s_j w)=k(\alpha, w)+k((\alpha_j)\bar{w}^{-1}, s_j)$, $0\leqslant j\leqslant l$.

We shall identify $W_a$ with $\mathfrak{A}$ or $\mathscr{E}_\Delta$ as a set in subsequent discussion.

For any subset $X\subset W$ we denote by $\langle X\rangle$ the union of all alcoves in $E$ corresponding to the elements of $X$. In [5], G. Lusztig conjectured that if $L$ is a left cell, then $\langle L\rangle$ is a contractible polyhedron. In this paper we prove the following result:

**Theorem A.** *Let $L$ be a left cell, tnen there are finitely many contractible polyhedrons $\langle L_i\rangle$ such that $L=\bigcup_i L_i$.*

As a by-product of the proof of Theorem A, we get a geometric explanation of the finiteness of cells in an affine Weyl group (Theorem B).

## § 2. Weyl Chambers and Sign Types

Let $\mathscr{C}^+$ be the (closed) dominant Weyl chamber of $E$ with respect to $\Delta$, that is,

$$\mathscr{C}^+=\{v\in E;\ \langle v, \alpha^\vee\rangle\geqslant 0 \text{ for } \alpha\in\Phi^+\}.$$

Then there exists a bijection $w\to w\mathscr{C}^+$ between $W$ and the set of Weyl chambers of $E$ and the chamber $w\mathscr{C}^+$ is dominant with respect to the basis $w\Delta$ of $\Phi$. Thus if $y\in W_a$ corresponds to an alcove $A\subset w\mathscr{C}^+$, then, for any $\alpha\in\Phi^+$,

(4) $k(\alpha, y)\begin{cases}\geqslant 0 & \text{if } \alpha\in w\Phi^+\cap\Phi^+,\\ <0 & \text{otherwise.}\end{cases}$

Let $\mathscr{S}$ be the set of sign types of $W_a$ (see [7]). We regard $X\in\mathscr{S}$ as a subset of $W_a$. Thus for any $\alpha\in\Phi^+$ the signs of $k(\alpha, z)$ for all $z\in X$ are the same. (Note that if $k(\alpha, z)=0$, the sign of $k(\alpha, z)$ is defined to be the zero sign) and if $y\notin X$ then there exists at least one root $\beta\in\Phi^+$ such that the signs of $k(\beta, z)$ and $k(\beta, y)$ are different.

For $X\in\mathscr{S}$ there is a unique $w\in W$ such that $\langle X\rangle\subset\mathscr{C}=w\mathscr{C}^+$. We say the function $k(\alpha, -): W_a\to\mathbf{Z}$ is bounded over $X$, if the image of $X$ is bounded. Let

$$\Phi_1=\{\alpha\in\Phi^+;\ k(\alpha, -) \text{ is bounded over } X\},\ \Delta'=w\Delta.$$

$$\Phi_0=\{\alpha\in\Phi^+;\ k(\alpha, -) \text{ is zero over } X\}.$$

**Lemma 2.1.** (a) *$\Phi'=\Phi_1\cup(-\Phi_1)$ is a root subsystem of $\Phi$.*

(b) *For any $\alpha\in\Phi_2=w\Phi^+-\Phi'$, $k(\alpha, y)$ tends to $\infty$ as $l(y)$ tends to $\infty$.*

(c) *For any $\alpha\in w\Phi^+$ there are $\gamma\in\Delta'$ and a sequence in $w\Phi^+$: $\gamma_0=\gamma, \gamma_1, \cdots, \gamma_n=\alpha$ sc that $\gamma_i-\gamma_{i-1}\in\Delta'$ for any $i$, $1\leqslant i\leqslant n$.*

(d) $\Phi_0\subset w\Phi^+$.

*Proof* (a) If $\alpha$, $\beta\in\Phi_1$ with $\alpha+\beta\in\Phi$, then $k(\alpha+\beta, -)$ is bounded over $X$ by (2). Hence $\alpha+\beta\in\Phi_1$. Therefore $\Phi'$ is a subsystem of $\Phi$.

(b) Since $\alpha\notin\Phi'$, it follows from (4) that $k(\alpha, y)\to\infty$ as $l(y)\to\infty$, $y\in X$.

(c) See [1, Lemma 2.3.1].

(d) Since $\langle X\rangle$ is the closure of one of the connected components of $E-\bigcup_{H\in\mathscr{F}} H$, where $\mathscr{F}=\{H_{\alpha,k};\ \alpha\in\Phi^+$ and $k=0, 1\}$, it follows that

$$\langle X\rangle\subset(\bigcap_{\alpha\in\Phi_0} H^1_{\alpha,0}).$$

Hence we get the result.

Let $w\in W$. For $K=(k_\alpha)_{\alpha\in\Phi}\in\mathscr{E}_\Delta$, we put $K'=(k'_\alpha)_{\alpha\in\Phi}$, where $k'_\alpha=k_\alpha$ if $\alpha\in w\Phi^+\cap\Phi^+$ and $k'_\alpha=k_\alpha-1$ if $\alpha\in w\Phi^+\cap(-\Phi^+)$. Then we have

**Lemma 2.2.** *For any $\alpha$, $\beta\in w\Phi^+$ with $\alpha+\beta\in w\Phi^+$, inequality* (2) *holds for $k'_\alpha$ and $k'_\beta$.*

*Proof* By [6, Theorem 5.2], $(k_\alpha)_{\alpha\in\Phi^+}$ determines an alcove $A$ and is the coordinate form of $A$ with respect to $\Phi^+$. If we choose $\Delta'$ as the basis of $\Phi$, then $w\Phi^+$ is the corresponding set of positive roots and $(k'_\alpha)_{\alpha\in w\Phi^+}$ is the coordinate form of $A$ with respect to $w\Phi^+$. Again by using [6, Theorem 5.2], we get the lemma.

Thus we have defined a bijection $f_w$: $\mathscr{E}_\Delta\to\mathscr{E}_{\Delta'}$ by $f_w(K)=K'$, and denote $f_w(K_y)$ by $K'_y=(k'(\alpha, y))_{\alpha\in\Phi}$ for any $y\in W_a$.

We know that $\mathscr{C}$ is a union of boxes

$$B=\{v\in E;\ 0\leqslant b_\alpha\leqslant\langle v, \alpha^\vee\rangle\leqslant b_\alpha+1,\ b_\alpha\in\mathbf{Z},\ \alpha\in\Delta'\},$$

and if $A\subset\mathscr{C}$ is an alcove and $A=\bigcap_{\alpha\in\Phi} H^1_{\alpha,k_\alpha}$ then there exists a unique box $B=\bigcap_{\alpha\in\Delta} H^1_{\alpha,k_\alpha}$ such that $A\subset B$.

If $H\in\mathscr{F}$, the complement $E-H$ has two components. We denote their closures by $H^+$ and $H^-$ such that $H^+$ meets any translation of $\mathscr{C}$ in $E$.

Let $b=(b_\alpha)_{\alpha\in\Delta'}$ be a $|\Delta|$-tuple over $\mathbf{Z}^+$. If $\Lambda\subset\Delta'$, we denote by $\overline{\Lambda}$ the complement $\Delta'-\Lambda$, and call

$$P(\Lambda, b)=\langle p(\Lambda, b)\rangle=(\bigcap_{\alpha\in\Lambda} H^1_{\alpha,b_\alpha})\cap(\bigcap_{\alpha\in\overline{\Lambda}} H^+_{\alpha,b_\alpha}).$$

a SPECIAL POLYHEDRON. If $P=\langle Y\rangle$, $Y\subset\mathscr{C}$, we write

$$\dim P=\{\alpha\in\Delta';\ k(\alpha, -) \text{ is bounded over } Y\},$$

called the dimension of $P$. Clearly, $\dim P(\Lambda, b)=|\Lambda|$.

**Lemma 2.3.** *Let $X\in\mathscr{S}$ and $x\in X$ is the shortest element of $X$. Then*

(a) *$\Delta_1=\Phi'\cap\Delta'$ is a basis of $\Phi'$ and $|k(\alpha, x)|=1$ for any $\alpha\in\overline{\Delta}_1$.*

(b) *$\Phi'=\mathbf{Z}\Phi_0\cap\Phi$ and $\langle X\rangle\subset\bigcap_{\alpha\in\Delta_1} H^1_{\alpha,k(\alpha,x)}$.*

*Proof* (a) For $\beta\in\Phi'$, $\beta=\sum_{\gamma\in\Delta'} a_\gamma\gamma$, since $k'(\alpha, y)\geqslant 0$ for any $\alpha\in w\Phi^+$, $\langle y\rangle\subset\mathscr{C}$ and $k'(\beta, -)$ is bounded over $X$, it follows from 2.1 (c) and (2) that $k'(\gamma, -)$ is bounded over $X$ for $\gamma\in\Delta'$ with $a_\gamma\neq 0$. This means that $\Delta_1$ is a basis of $\Phi'$.

Suppose that there is an $\gamma\in\bar{\Delta}_1$ such that $k=|k(\gamma, x)|>1$. We choose $y\in X$ such that $y$ has a (codim 1) face $s$ lying in the hyperplane $H_{\gamma,k}$ if $\gamma\in\Phi^+$ or $H_{\gamma,-k+1}$ if $\gamma\in-\Phi^+$. Clearly, $sy\in X$, but we have $0<|k(\gamma, sy)|<|k(\gamma, x)|$. This contradicts [7, Proposition 7.2].

(b) Let $\Phi'_0=\mathbf{Z}\Phi_0\cap\Phi$. Then we have clearly $\Phi'_0\subset\Phi'$ by (2). Let $\Lambda$ be a basis of $\Phi'_0$ and $\Lambda'\supset\Lambda$ is a basis of $\Phi'$. Suppose that $\Lambda'\neq\Lambda$ and $\alpha\in\Lambda'-\Lambda$. Since $\langle X\rangle$ is the closure of one of the connected components of $E-\bigcup_{H\in\Lambda}H$, it is easy to see that dim $\langle X\rangle=\dim(\bigcap_{\beta\in\Phi_0}H^1_{\beta,0})=|\Lambda|$. On the other hand, $k(\alpha, -)$ is bounded over $X$ and $k(\alpha, x)\neq 0$, so there is an integer $m\neq 0$ with $k(\alpha, x)m>0$ such that $\langle X\rangle\subset H^m_{\alpha,k(\alpha,x)}\cap(\bigcap_{\beta\in\Phi_0}H^1_{\beta,0})$. Since $\alpha\in\Lambda'-\Lambda$, it follows that

$$\dim[(\bigcap_{\alpha\in\Phi_0}H^1_{\beta,0})\cap H^m_{\alpha,k(\alpha,x)}]<|\Lambda|.$$

So we get a contradiction. This shows $\Lambda=\Lambda'$, hence $\Phi'_0=\Phi'$.

Now we suppose that there are $\alpha\in\Delta_1$ and $y\in X$ such that $k'(\alpha, y)>0$. By above discussion, there exists $\beta\in\Phi_0$ such that $\beta-\alpha\mathbf{Z}^+\Delta_1$. Thus 2.1 (c) and (2) imply $k(\beta, x)=k'(\beta, x)>0$, contrary to $\beta\in\Phi_0$. Hence $k(\alpha, y)=0$ if $\alpha\in\Delta_1\cap\Phi^+$ and $k(\alpha, y)=-1$ if $\alpha\in\Delta_1\cap(-\Phi^+)$.

**Remark 2.4.** Let $X\in\mathscr{S}$. Then $\langle X\rangle\subset P(\Delta_1, x)$ where $x_\alpha=k'(\alpha, x)$, $\alpha\in\Delta'$ and $x$ is the shortest element of $X$. By 2.3, $H_{\alpha,x_\alpha}\in\mathscr{F}$. This implies $p(\Delta_1, x)-X$ is a union of finitely many sign types and

$$\dim\langle X\rangle=\dim P(\Delta_1, x).$$

## § 3. Proof of Theorem A

From now on, we assume that $X\in\mathscr{S}$ is infinite. Let $\Lambda$ be a subset of $\Delta'$ such that $\Lambda\supset\Delta_1$. For $z\in X$, we put $b_\alpha=k'(\alpha, z)$ for $\alpha\in\Delta'$. Thus $P(\Lambda, b)$ is a special polyhedron contained in $P(\Delta_1, b)$.

**Lemma 3.1.** *Let $P=P(\Lambda, b)$ be as above. Then there is a sequenec in $P$: $y_1, y_2, \cdots$ such that for any $n>0$,*

(a) *$y_n=z_n\cdot y_{n-1}$ for some $z_n\in W_a$;*

(b) *$k'(\alpha, y_n)=k'(\alpha, y_{n-1})+1$ for $\alpha\in\bar{\Lambda}$;*

(c) *If $\Psi$ is the root subsystem of $\Phi$ generated by $\Lambda$ and $\Psi_2=w\Phi^+-\Psi$, then $k'(\alpha, y_n)$ tends to $\infty$ as $n$ tends to $\infty$ for any $\alpha\in\Psi_2$.*

(d) *$k'(\alpha, y_n)=k'(\alpha, y_1)$ for all $\alpha\in\Psi$.*

*Proof* Let $y_1=z$, $T=\bigcap_{\alpha\in\Psi}H^1_{\alpha,k(\alpha,z)}$ and $B_z$ is the box of $\mathscr{C}$ containing $z$. Then $y_1\in T\cap B_z=B'$.

We claim that $B'$ has a longest element $u$ and $u$ must have:

(5). There are $|\bar{\Lambda}|$ codim 1 faces which lie in the following hyperplanes:

$$H_{\alpha, k'(\alpha, y_1)+1}, \ \alpha \in \bar{\Lambda}.$$

In fact, if $B'$ has two elements $u$ and $u'$ with the same maximal length, then one of them, say $u'$, has a face $t \in S$ lying in a hyperplane $H = H_{\beta, k'(\beta, u')+1}$ with $\beta \in \Psi_2 - \bar{\Lambda}$, and $H$ separates $u$ and $u'$. This implies $tu \in B'$, but $l(tu') > l(u')$, contrary to the maximality of $l(u')$. So $B'$ has a longest element, denoted by $u$. On the other hand, since each $H_{\alpha, k'(\alpha, y_1)+1}$ $(\alpha \in \bar{\Lambda})$ intersects $T$, there is $y \in B'$ such that $y$ has (5). We choose $y$ satisfying (5) of maximal length. Suppose $s \in S$ is a face of $y$ such that $sy > y$ and $s \subset H = H_{\beta, k(\beta, y)+1}$ with $\beta \in w\Phi^+ - \bar{\Lambda} \cup \Psi$. Since $H^+ \cap B'$ is a convex set and its codim 1 faces lie in the hyperplane of forms: $H_{\gamma, k}$ for $\gamma \in \bar{\Lambda} \cup \Psi$, $k \in \mathbf{Z}$, it follows that $H^+ \cap B'$ is a union of alcoves and still has $|\bar{\Lambda}|$ codim 1 faces lying in the hyperplane of (5). So we have at least one alcove $y'$ in $H^+ \cap B'$ satisfying (5) and $l(y') > l(y)$. This is contrary to the choice of $y$. Indeed, we have proved that if $sw = s.\ w$ for $s \in S$, then $sw \notin B'$, hence $u = y$. The claim is proved.

By the above claim, there is $u_1 \in W_a$ such that $u = u_1 \cdot y_1$. Let $s$ be a face of $u$ and $s$ lies in $H_{\beta, k'(\beta, y)+1}$, $\beta \in \bar{\Lambda}$. It is easy to see that $su = s \cdot u$. Thus we get $y_{12} = su \in P$ such that $y_{12} = (su_1) \cdot u$, $k'(\alpha, y_{12}) = k'(\alpha, y_1)$ for $\alpha \in \Psi \cup \bar{\Lambda} - \{\beta\}$ and $k'(\beta, y_{12}) = k'(\beta, y_1) + 1$.

Replacing $y_1$ by $y_{12}$, we can proceed as above and get $u_2 \in W_a$ and $\gamma \in \bar{\Lambda} - \{\beta\}$ such that $y_{13} = u_2 \cdot y_{12} \in P$ and

$$k'(\alpha, y_{13}) = \begin{cases} k'(\alpha, y_1) & \text{if } \alpha \in \Psi \subset \bar{\Lambda} - \{\beta, \gamma\}, \\ k'(\alpha, y_1) + 1 & \text{if } \alpha = \beta \text{ or } \gamma. \end{cases}$$

Continuing this procedure, we can find an element $y_2 \in P$ such that $y_2 = z_1 \cdot y_1$ for some $z_1 \in W_a$ and $k'(\alpha, y_2) = k'(\alpha, y_1)$ if $\alpha \in \Psi$, $k'(\alpha, y_2) = k'(\alpha, y_1) + 1$ if $\alpha \in \bar{\Lambda}$.

By the same technique and replacing $y_1$ by $y_2$, we can find $y_3 \in P$ satisfying certain properties as above. Finally, we get a sequence in $P$: $y_1, y_2, \cdots$ such that (a), (b) and (d) hold. (c) follows from (2) and 2.1 (c).

**Corollary 3.2.** *There exists a contractible polyhedron* $P' = \langle L \rangle$ *such that* $\dim P' = \dim P$ *and* $L$ *is contained in a single left cell.*

*Proof* By 3.1 there is a sequence in $T \cap P$: $y_1, y_2, \cdots$ such that (a)—(d) in 3.1 hold. Since the function $a(-)$ ([3]) has an upper bound, there is $m \geq 1$ such that $a(y_p) = a(y_m)$ for any $p \geq m$. Hence $y_p \underset{L}{\sim} y_m$ for $p \geq m$ since $y_p \underset{L}{\leq} y_m$.

Now we show that $P' = T \cap (\bigcap_{\beta \in \bar{\Lambda}} H^+_{\beta, k'(\beta, y)+N})$ is contained in a left cell, where $N = N(\Phi)$ is a fixed positive number.

Assume $y \in P'$. We use induction on $h(\alpha)$ (with respect to $\Delta'$) to prove that $k'(\alpha, y) \geq k'(\alpha, y_m)$ for any $\alpha \in \Psi_2$. We have already the inequality for $h(\alpha) = 1$ (i.e., $\alpha \in \Delta'$) by the definition of $P'$. Suppose $h(\alpha) > 1$. By 2.1 there exist $\delta_0 \in \bar{\Lambda}$ and a sequence in $w\Phi^+$: $\delta_0=\gamma_0, \gamma_1, \cdots, \gamma_t=\alpha$ such that $\delta_i=\gamma_i-\gamma_{i-1}\in\Delta'$ $(1\leqslant i\leqslant t)$. Then, by (2),

$$\begin{aligned}|\alpha|^2(k'(\alpha, y)+1) &\geqslant |\delta_t|^2k'(\delta_t, y)+|\gamma_{t-1}|^2(k'(\gamma_{t-1}, y)+1)+(1-|\gamma_{t-1}|^2)\\ &\geqslant |\delta_t|^2k'(\delta_t, y)+|\delta_{t-1}|^2k'(\delta_{t-1}, y)\\ &\quad+|\gamma_{t-2}|^2(k'(\gamma_{t-2}, y)+1)+(2-|\gamma_{t-1}|^2-|\gamma_{t-2}|^2)\\ &\geqslant \sum_{i=0}^{t}|\delta_i|^2k'(\delta_i, y)+t-\sum_{i=1}^{t-1}|\gamma_i|^2\\ &\geqslant \sum_{i=0}^{t}|\delta_i|^2k'(\delta_i, y_m)+(t+|\gamma_0|^2N)-\sum_{i=1}^{t-1}|\gamma_i|^2.\end{aligned}$$

$$\begin{aligned}|\alpha|^2(k'(\alpha, y_m)+1) &\leqslant |\delta_t|^2k'(\delta_t, y_m)+|\gamma_{t-1}|^2(k'(\gamma_{t-1}, y_m)+1)+|\gamma_t|^2+|\delta_t|^2-1\\ &\leqslant \sum_{i=1}^{t}|\delta_i|^2k'(\delta_i, y_m)+|\gamma_0|^2(k'(\gamma_0, y_m)+1)\\ &\quad+\sum_{i=1}^{t}(|\delta_i|^2+|\gamma_i|^2)-t\\ &=\sum_{i=0}^{t}|\delta_i|^2k'(\delta_i, y_m)+\sum_{i=1}^{t}(|\gamma_i|^2+|\delta_i|^2)+|\gamma_0|^2-t.\end{aligned}$$

So we can choose $N$ such that $|\alpha|^2(k'(\alpha, y)+1)\geqslant|\alpha|^2(k'(\alpha, y_m)+1)$ for any $\alpha\in\Psi_2$. Thus we get $k(\alpha, y)\geqslant k(\alpha, y_m)$ for any $\alpha\in w\Phi^+$. This implies by (3) that there is a $w'\in W_a$ such that $y=w'\cdot y_m$.

On the other hand, by 3.1, there is $p\geqslant m$ such that $k'(\alpha, y_p)\geqslant k'(\alpha, y)$ for any $\alpha\in w\Phi^+$. So we have $w''\in W_a$ such that $y_p=w''\cdot y$. It follows that $a(y)=a(y_m)$. Hence $y\underset{L}{\sim}y_m$ and $H$ is contained in a left cell.

**Theorem 3.3.** *Let $P=P(\Lambda, b)$ be as in 3.1. Then*

(a) *Tnere is a special polyhedron $P'\subset P$ such that $P-P'$ is a union of finitely many special polyhedron $P_i$ with* $\dim P_i<\dim P$.

(b) *$P'$ is a union of finitely many contractible polydedron $P'_i$ each of which is contained in a single left cell.*

*proof* Let $y_{11}, y_{21}, \cdots, y_{n1}$ be the set

$$\{y;\ \langle y\rangle\subset(\bigcap_{\alpha\in\Lambda}H^1_{\alpha,k(\alpha,z)})\cap(\bigcap_{\alpha\in\Phi}H^1_{\alpha,k(\alpha,z)}).$$

Then there is, for any $i$, $1\leqslant i\leqslant n$, a sequence in $P$: $y_{i1}, y_{i2}, \cdots, y_{ip}, \cdots$ which satisfies (a)—(d) in 3.1, and $k'(\alpha, y_{1p})=k'(\alpha, y_{ip})$ for any $i$, $p>0$ and $\alpha\in\Delta'$. Thus there exists $m(i)\geqslant0$ such that $a(y_{ip})=a(y_{im(i)})$ for $p\geqslant m(i)$. Let $m=\max\{m(i);\ 1\leqslant i\leqslant n\}$. Then $y_{ip}\underset{L}{\sim}y_{im}$ for any $p\geqslant m$ and $i$, $1\leqslant i\leqslant n$.

Let $b_\alpha=k'(\alpha, y_{11})$, $c_\alpha=k'(\alpha, y_{1m})$ for $\alpha\in\Delta'$. For a subset $\Gamma$, $\Lambda\subset\Gamma\subset\Delta'$, put

$$D(\Gamma)=\{d=(d_\alpha)_{\alpha\in\Delta'};\ (d_\alpha)_{\alpha\in\Lambda}=(b_\alpha)_{\alpha\in\Lambda},\ (d_\alpha)_{\alpha\in\bar{\Gamma}}=(c_\alpha+N)_{\alpha\in\bar{\Gamma}}$$
$$\text{and } (d_\alpha)_{\alpha\in\Gamma-\Lambda}\in\prod_{\alpha\in\Gamma-\Lambda}[b_\alpha, c_\alpha+N]\},$$

where $[a, b]$ denotes the set of all numbers $k$ with $a\leqslant k\leqslant b$. Then we have

(i) $P(\Lambda, b)=\bigcup_{\substack{\Lambda\subset\Gamma\subset\Delta'\\ d\in D(\Gamma)}}P(\Gamma, d)$,

(ii) $\dim P(\Gamma, d) < \dim P(\Lambda, b)$ if $\Gamma \neq \Lambda$.

(iii) Let $P' = P(\Lambda, d)$ and $P'_i = (\bigcap_{\alpha \in \Psi} H^1_{\alpha, k(\alpha, y_{i1})}) \cap P'$. Then $P'_i$ is a contractible polyhedron contained in a single left cell by 3.2 and $P' = \bigcup_{i=1}^{n} P'_i$.

From (i)—(iii), the theorem is proved.

*Proof of Theorem A* By 2.4, we see that if $X \in \mathscr{S}$, then $P(\Delta_1, b) - X$ is a union of finitely many sign types. Therefore by using 3.3 and induction on dim $\langle X \rangle$, we get $P(\Delta_1, b)$ is a union of finitely many special polyhedron $P_i$ and $P_i$ is a union of finitely many contractible polyhedrons each of which is contained in a single left cell. Since $\mathscr{S}$ is finite, Theorem A is proved.

These arguments also imply the following result:

*Theorem B* $W_a$ has a finitely many left cells.

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