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ON REFLEXIVITY AND HYPERREFLEXIVITY FOR LINEAR SUBSPACE OF OPERATORS

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Abstract

It is proved that any Von Neumann type subspace is reflexive. Particularly, all weakly closed -algebras and their weakly closed ideals are reflexive. Some results concerning hyperreflexivity of Von Neumann type subspace and Von Neumand algebra are obtained.

§1. Preliminaries

The reflexivity and hyperreflexivity for subspace of operators on Hilbert space H have been studied by some mathematicians and have proved its worth[1,6,7,8].

In paper [3] J. A. Erdos gave a systematic account of generalized reflexivity in a new formulation which contains an object analogous to lattices. Many results concerning reflexivity for subspace and subspace map were obtained in [3]. The purpose of this paper is to give some results on reflexivity and hyperreflexivity for subspace which is closely related to Von Neumann algebras.

Most of the notations and terminology follow Erdose's paper[3]. Let H_i be Hilbert spaces, and $L(H_1, H_2)$ be the set of all bounded linear operators from H_1 into H_2. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ denote the set of all closed subspaces of H_1 and H_2 (or the set of all selfadjoint projections of $L(H_1)$ and $L(H_2)$) respectively. For convenience, we do not distinguish closed subspace from self-adjoint projection in this paper.

Let $\mathscr{A}$ be a subset of $L(H_1, H_2)$, the map of $\mathscr{A}$ is a map φ: $\mathscr{P}_1 \to \mathscr{P}_2$ given by

$$\varphi(P) = [\mathscr{A}P],$$

where $[\mathscr{A}P]$ denotes the closure of span $\{Ax: A \in \mathscr{A}, x \in P\}$. We write $\varphi = \text{Map}\,\mathscr{A}$. Given a map φ: $\mathscr{P}_1 \to \mathscr{P}_2$, we denote by Op φ the set $\{A \in L(H_1, H_2): AP \subseteq \varphi(P), \forall P \in \mathscr{P}_1\}$.

Definition 1.1. [3] *A subset $\mathscr{A}$ of $L(H_1, H_2)$ is said to be reflexive if $\mathscr{A} = \text{OpMap}\,\mathscr{A}$; A map φ: $\mathscr{P}_1 \to \mathscr{P}_2$ is called reflexive if $\varphi = \text{Map Op}\,\varphi$.*

Let φ be a join-continuous zero-preserving map from $\mathscr{P}_1$ to $\mathscr{P}_2$. Define the co-map ψ: $\mathscr{P}_2 \to \mathscr{P}_1$ of φ by

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$$\psi(P_2)=V\{P_1\in\mathscr{P}_1:\ \varphi(P_1)\leqslant P_2\}.$$

If the range of φ and ψ consist of commuting projections, we call φ a commutative subspace map.

Proposition 1.2. *If $\mathscr{A}$ is a reflexive subspace of $L(H_1, H_2)$, then*

(1) $\mathrm{Alg}\{\varphi(P):\ P\in\mathscr{P}_1\}=\{T\in L(H_2):\ T\mathscr{A}\subseteq\mathscr{A}\}$,

(2) $\mathrm{Alg}\{\psi(Q):\ Q\in\mathscr{P}_2\}=\{S\in L(H_1):\ \mathscr{A}S\subseteq\mathscr{A}\}$,

where $\varphi=\mathrm{Map}\ \mathscr{A}$ and $\psi=$co-map of φ.

Proof We only prove (2). Suppose $\mathscr{A}S\subseteq\mathscr{A}$. For any P such that $\varphi(P)\leqslant Q$, $[\mathscr{A}SP]\subseteq[\mathscr{A}P]\subseteq Q$. Hence $SP\subseteq\psi(Q)$, which implies $S\widehat{\psi}(Q)\subseteq\psi(Q)$ by the definition of co-map. Therefore S belongs to the left side of (2).

For the opposite inequality, let $S\psi(Q)\subseteq(\psi Q)$ for any $Q\in\mathscr{P}_2$. If $Q=\varphi(P)$, then $P\leqslant\psi(Q)$ and so $SP\subseteq\psi(Q)$. Hence $[\mathscr{A}SP]\subseteq Q=[\mathscr{A}P]$ for any $P\in\mathscr{P}_1$, which implies $\mathscr{A}S\subseteq\mathscr{A}$ by the reflexivity of $\mathscr{A}$.

The following proposition is an immediate consequence of the definition of reflexivity and Lemma 2.1 and Lemma 4.1 of [3].

Proposition 1.3. *Let $\mathscr{A}$ be a subset of $L(H_1, H_2)$. Then the following statements are equivalent:*

(1) *$\mathscr{A}$ is reflexive.*

(2) *$\mathscr{A}=\mathrm{Op}\ \varphi_1$ for some φ_1: $\mathscr{P}_1\to\mathscr{P}_2$*

(3) *$\mathscr{A}=\{T\in L(H_1, H_2):\ Tx\in[\mathscr{A}x]$ for any $x\in H_1\}$.*

(4) *$\mathscr{A}=\{T\in L(H_1, H_2):\ TP\subseteq\varphi(P),\ \forall P\in\mathrm{Range}\ \psi.\}$.*

(5) *$\mathscr{A}=\{T\in L(H,H):\ T\psi(Q)\subseteq Q,\ \forall Q\in\mathrm{Range}\ \varphi\}$,*

where $\varphi=\mathrm{Map}\ \mathscr{A}$ and ψ is the co-map of φ.

In [6] J. Kraus and D. Larson generalized the concepts of hyperreflexivity of unital algebras to general subspaces of operators on Hilbert space, and this was used to show that there exisit reflexive algebras which are not hyperreflexive. For subspace of $L(H_1, H_2)$, here we also give the following definition.

Definition 1.4. *A subset $\mathscr{A}$ of $L(H_1, H_2)$ will be called hyperreflexive if there exists a constant $K>0$ such that for any $T\in L(H_1, H_2)$*

$$d(T, \mathscr{A})\leqslant K\alpha(T, \mathscr{A}), \qquad (*)$$

where $\alpha(T, \mathscr{A})=\sup\{\|Q^{\perp}SP\|:\ Q\in\mathscr{P}_2,\ P\in\mathscr{P}_1,\ Q^{\perp}AP=0$ for $A\in\mathscr{A}\}$. The smallest K for which (*) holds is said to be the hyperreflexive constant for $\mathscr{A}$.

It is obvious that the hyperreflexive subspace is reflexive.

Proposition 1.5. *Let $\mathscr{A}$ be a reflexive subspace of $L(H_1, H_2)$, $\varphi=\mathrm{Map}\ \mathscr{A}$, ψ is the co-map of φ. Then*

(1) $\alpha(T, \mathscr{A})=\sup\{|\langle Tx, y\rangle|:\ \|X\|=\|y\|=1,\ \langle Ax, y\rangle=0,\ \forall A\in\mathscr{A}\}$.

(2) $\alpha(T, \mathscr{A})=\sup\{\|I-\varphi(P))TP\|:\ P\in\mathscr{P}_1\}$.

(3) $\alpha(T, \mathscr{A})=\sup\{\|(I-\varphi(P))TP\|:\ P\in\mathrm{Range}\ \psi\}$.

(4) $\alpha(T, \mathscr{A}) = \sup\{\|(I-Q)T\psi(Q)\|: Q\in \text{Range } \varphi\}$.

Proof We only prove (2) and (3). The others are left to the reader.

First it is obvious that $\alpha(T, \mathscr{A}) \geqslant \sup\{\|\varphi(P)^{\perp}TP\|: P\in \mathscr{P}_1\} \geqslant \sup\{\|\varphi(P)^{\perp}TP\|: P\in \text{Range } \psi\}$.

To provide the opposite inequality, let $Q\in \mathscr{P}_2$, $P\in \mathscr{P}_1$ and $Q^{\perp}AP=0$ for any $A\in \mathscr{A}$. Then $\varphi(P) = [\mathscr{A}P]\subseteq Q$. Thus

$$\|Q^{\perp}TP\| \leqslant \|(I-\varphi(P))TP\|,$$

which implies $\alpha(T, \mathscr{A}) \leqslant \sup\{\|I-\varphi(P))TP\|: P\in \mathscr{P}_1\}$ and therefore (2) holds.

For each $P\in \mathscr{P}_1$, let $P = \Lambda\{P'\in \text{Range } \psi: P'\geqslant P\}$. Since the range of ψ is complete with respect to meets, it follows that $P_1\in \text{Range } \psi$. By lifting theorem (Theorem 3.3 in[3]), $\varphi(P) = \varphi(P_1)$. Therefore $\|(I-\varphi(P))TP\| \leqslant \|(I-\varphi(P_1))TP_1\|$, which implies that

$$\sup\{\|\varphi(P)^{\perp}TP\|: P\in \mathscr{P}_1\} \leqslant \sup\{\|\varphi(P)^{\perp}TP\|: P\in \text{Range } \psi\}.$$

And hence (3) holds by (2).

Corollary 1. 6. *If φ is a nest map from $\mathscr{P}_1$ to $\mathscr{P}_2$, that is, one of Range φ and Range ψ (and hence both) is totally ordered, then* Op φ *is hyperreflexive.*

Proof This is an immediate consequence of Theorem 1 in [10] and (3) of above proposition.

Theorem 1. 6. *For any $A\in L(H_1, H_2)$, $\mathscr{A} = \{\lambda A\}$ is hyperreflexive with hyperreflexive constant* 1.

Proof Let $\tilde{\mathscr{A}} = \begin{pmatrix} 0 & 0 \\ \mathscr{A} & 0 \end{pmatrix} \in L(H_1\oplus H_2)$. Then $\mathscr{A}$ is hyperreflexive[7, 9] with hyperreflexive constant 1. Thus for any $T\in L(H_1, H_2)$, let T

$$\tilde{T} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

Then

$$d(T, \mathscr{A}) = d(\tilde{T}, \tilde{\mathscr{A}}) = \sup\{|\langle\tilde{T}\tilde{x}, \tilde{y}\rangle|: \|\tilde{x}\| = \|\tilde{y}\| = 1,\ \tilde{x}, \tilde{y}\in H_1\oplus H_2\ \langle S\tilde{x}, \tilde{y}\rangle = 0 \text{ for } S\in \tilde{\mathscr{A}}\}$$
$$= \sup\{\|\langle Vx, y\rangle\|: \|x\xi = \|y\| = 1,\ x\in H_1,\ y\in H_2 \langle Ax, y\rangle = 0\}.$$

This completes the proof.

§ 2. Von Neumann Type Subspaces

In this section we will discuss the reflexivity and hyperreflexivity of a class of subspaces which are called Von Neumann type subspaces.

Definitoin 2. 1. *A weakly closed subspace $\mathscr{M}$ of $L(H_1, H_2)$ is said to be a Von Neumann type subspace if $\mathscr{M}\mathscr{M}^*\mathscr{M}\subseteq \mathscr{M}$.*

A mup φ is said to be a Von Neumann type map if φ is reflexive and Op φ *is a*

Von Neumunn type subspace.

Example 1. If $U\in L(H_1, H_2)$ is a unitary operator, then $\{\lambda U\}$ is a Von Neumann type subspace.

Example 2. Any weakly closed *-algebra of $L(H)$ and its weakly closed ideals (left or right) are Von Neumann type subspaces.

Theorem 2. 2. *Any Von Neumann type subspace is reflexive.*

Proof Let $\mathscr{M}$ be a Von Neumann typesub space of $L(H_1, H_2)$. Define $\mathscr{A}$ and $\mathscr{B}$ to be the Von Neumann algebras generated by $\{\mathscr{M}^*\mathscr{M}, I\}$ and $\{\mathscr{M}\mathscr{M}^*, I\}$ respectively. It is easy to verify that

$$\tilde{\mathscr{M}}=\begin{pmatrix}\mathscr{A} & \mathscr{M}^*\\ \mathscr{M} & \mathscr{B}\end{pmatrix}$$

is a Von Neumann algebra of $L(H_1\oplus H_2)$, and hence $\tilde{\mathscr{M}}$ is reflexive.

Now suppose that $T\in L(H_1, H_2)$ such that $Tx\in[\mathscr{M}x]$ for any $x\in H_1$. Let

$$\tilde{T}=\begin{pmatrix}0 & 0\\ T & 0\end{pmatrix}\in L(H_1\oplus H_2).$$

Then for any $x\oplus y\in H_1\oplus H_2$, we have $\tilde{T}(x\oplus y)=(0\oplus Tx)\in[\tilde{\mathscr{M}}(x\oplus y)]$ and therefore $\tilde{T}\in\tilde{\mathscr{M}}$ which implies $T\in\mathscr{M}$. Thus $\mathscr{M}$ is reflexive by (3) of Proposition 1.3.

Corollary 2. 3. *Any weakly closed *-algebra of $L(H)$ and its weakly closed ideals are reflexive.*

Corollary 2. 4. *Let $\mathscr{A}$ be a Von Neumann algebra such that $\mathscr{A}'$ is commutative. Then for any weakly closed (two side) ideal $\mathscr{M}$ of $\mathscr{A}$ with property (*) (cf. [5]), we have $C(\mathscr{A}, \mathscr{M})=\text{Alg Lat }\mathscr{M}$. Particularly, $C(\mathscr{A},\mathscr{M})$ is reflexive, where $C(\mathscr{A},\mathscr{M})=\{T: TA-TA\in\mathscr{M}, \forall A\in\mathscr{A}\}$.*

Proof This is easy to prove by the proof of Theorem 11 in [5] and above Corollary 2.3.

Example 3. Let H_1 be a Hilbert space and $H_2=C$. It is easy to verify that any weakly closed subspace of $L(H_1, H_2)$ $(=H_1)$ is a Von Neumann type subspace and hence is reflexive. Using (1) of Proposition 1.5, we see that it is in fact hyperreflexive by a very simple computation.

Lemma 2. 5. *Let $\mathscr{M}$ be a Von Neumann type subspace of $L(H_1, H_2)$, $\mathscr{A}$, $\mathscr{B}$ and $\tilde{\mathscr{M}}$ be as in the proof of Theorem 2.2. If $\tilde{\mathscr{M}}$ is hyperreflexive with hyperreflexive constant K, then $\mathscr{M}$ is also hyperreflexive with hyperreflexive constant less than K.*

Proof For any $T\in L(H_1, H_2)$, let

$$\tilde{T}=\begin{pmatrix}0 & 0\\ T & 0\end{pmatrix}\in L(H_1\oplus H_2).$$

Since $\text{Lat }\tilde{\mathscr{M}}\subseteq\{P\oplus Q: \varphi(P)\leqslant Q, P\in\mathscr{P}_1, Q\in\mathscr{P}_2\}$, we obtain

$$\begin{aligned}d(T,\mathscr{M})&=d(\tilde{T},\tilde{\mathscr{M}})\leqslant K\sup\{\|E^{\perp}TE\|: E\in\text{Lat }\tilde{\mathscr{M}}\}\\ &\leqslant K\sup\{\|(P\oplus Q)^{\perp}T(P+Q)\|: \varphi(P)\leqslant Q, P\in\mathscr{P}, Q\in\mathscr{P}_2\}\end{aligned}$$

$$= K \sup\{\|Q^{\perp}TP\|: \varphi(P) \leqslant Q,\ P \in \mathscr{P}_1,\ Q \in \mathscr{P}_2\}$$
$$\leqslant K \sup\{\|(I-\varphi(P))TP\|: P \in \mathscr{P}_1\},$$

which completes the proof.

Corollary 2.6. *Let φ be a Von Neumann type commutative subspace map. Then If $I \in (\mathrm{Op}\,\varphi)^*(\mathrm{Op}\,\varphi)$, $\mathrm{Op}\,\varphi$ is hyperreflexive.*

Remark. This is analogous to Lemma 3.1 in [11].

Proof Let $\mathscr{M} = \mathrm{Op}\,\varphi$ and $\mathscr{U}$ be as in the proof of Theorem 2.2. Since φ is a commutative subspace map, it can be verified that $\mathscr{U}'$ is commutative. Thus $\mathscr{U}$ is hyperreflexive by Lemma 3.1 in [11]. By above lemma it follows that $\mathscr{M}$ is hyperreflexive.

For the hyperreflexivity, it is an outstanding open question whether any Von Neumann algebra is hyperreflexive. This is closely related to similar problem of representation and derivation problem of C^*-algebra (See 2.4).

The following theorem is an immediate corollary of Lemma 2.5.

Theorem 2.7. *The following statements are equivalent:*

(1) *All Von Neumann algebras are hyperreflexive.*

(2) *All Von Nemann type subspaces are hyperreflexive.*

(3) *All weakly closed *-algebras are hyperreflexive.*

(4) *All weakly closed ideals of Von Neumann algebras are hyperreflexive.*

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