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CLASSICAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS OF CARLEMAN EQUATION

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Abstract

Applying the exponential formula in nonlinear semigroup, the author proves the existence of global classical solutions to the nonhomogeneous boundary value problems of Carleman equation, and discusses the asymptotic behaviour of the solutions.

§ 0. Introduction

The purpose of this paper is to study the existence and asymptotic nature of the classical solutions of initial boundary value problems for Carleman equation:

$$\begin{aligned}&\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial x}+u_1^2-u_2^2=0,\\&\frac{\partial u_2}{\partial t}-\frac{\partial u_2}{\partial x}+u_2^2-u_1^2=0,\end{aligned}\qquad t>0,\ 0<x<1, \tag{0.1}$$

$$u_1(t,\ 0)=g_1(t),\ u_2(t,\ 0)=g_2(t), \tag{0.2}$$

$$u_1(0,\ x)=\varphi_1(x),\ u_2(0,\ x)=\varphi_2(t). \tag{0.3}$$

The pure initial problems of the Carleman equation (0.1) have been studied by several authors. In 1963, Kolodner [7] proved that the initial value problem of Carleman equation (0.1) admits a classical solutson for nonnegative data in $C^1(\boldsymbol{R})$. In [6], Kaper and Leaf associated a nonlinear semigroup in $L^1(\boldsymbol{R})$ with solutions to equation (0.1) and showed that the results of Kolodner can be recovered from the abstract results. Up to now, there are few results on the initial boundary value problems for Carleman equation. Recently, Fitzgibbon [4] associated a nonlinear evolution system in $L^1(0,\ 1)$, generated by a family of nonlinear accretive operators of varying domain, with Carleman equation (0.1) and boundary conditions (0.2), so he obtained a mild solution of the boundary value problem (0.1)-(0.3), represented by the exponential formula of Crandall and Paqy[2] for nonnegative data in L^1 (0, 1). However, in which sense does the mild solution satisfy the boundary value problem (0.1)-(0.3)? And if the nonnegative data are in $C^1([0,\ 1])$, is the mild solution a classical solution of the

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boundary value problem (0.1)-(0.3)? There is no discussion about these important problems in [4]. One of the aims of this paper is to give an answer to these problems.

First, we show that the mild solution for bounded nonnegative data φ_1, φ_2, given by the exponential formula, satisfies a system of integral equations obtained from integrating the Carleman system along the characteristic curves. Then, we prove that this solution must be classical solution of boundary value problem (0.1)-(0.3) if the data $\varphi_1, \varphi_2 \in C^1([0, 1])$. Finaly, we discuss the asymptotic behaviour of the solutions, and show that if $g_1(t) \equiv g_2(t) \equiv a$, then the classical solution of the boundary value problem (0.1)-(0.3) approaches to (a, a) exponentially as $t \to \infty$.

§1. Main Results

Set

$$L^1 = L^1(0, 1) \times L^1(0, 1),$$

with the norm of $u = (u_1, u_2) \in L^1$,

$$\|u\|_{L^1} = \|u_1\|_{L^1(0,1)} + \|u_2\|_{L^1(0,1)}. \tag{1.1}$$

Clearly L^1 is a Banach space. Let L^1_+ denote the positive cone of L^1, i. e.,

$$L^1_+ = \{u = (u_1, u_2) \in L^1;\ u_1 \geq 0,\ u_2 \geq 0\}.$$

We define the operator $A(t)$ as following

$$A(t)u = \left(\frac{du_1}{dx} + u_1^2 - u_2^2,\ -\frac{du_2}{dx} + u_2^2 - u_1^2\right),\ \forall u = (u_1, u_2) \in D(A(t)), \tag{1.2}$$

where

$$D(A(t)) = \{u = (u_1, u_2) \in L^1_+;\ u_1, u_2 \in W^{1,1}(0, 1),\ u_1(0) = g_1(t),\ u_2(1) = g_2(t)\}.$$

Now the boundary value problem (0.1)-(0.3) can be written as an abstract Cauchy problem

$$\frac{du(t)}{dt} + A(t)u(t) = 0,\quad t > 0, \tag{1.3}$$

$$u(0) = \varphi,\ \varphi = (\varphi_1, \varphi_2). \tag{1.4}$$

From the results in [4], we have the following theorems.

Theorem A. *The family of operators $\{A(t);\ t \in [0, T]\}$ defined by* (1.2) *is a family of accretive operators in Banach space L^1, which satisfies*

(A. 1) $\overline{D(A(t))} = L^1_+$, *is independent of t.*

(A. 2) $R(I + \lambda A(t)) \supset L^1_+$, $\forall \lambda > 0$.

(A. 3) *There exists a continuous function f: $[0, \infty) \to L^1$ and a monotone increasing function L: $[0, \infty) \to [0, \infty)$ such that*

$$\|J_\lambda(t_1)u - J_\lambda(t_2)\|_{L^1} \leq \lambda \|f(t_1) - f(t_2)\|_{L^1} L(\|u\|_{L^1}),$$

where $\lambda>0$, $J_\lambda(t)=(I+\lambda A(t))^{-1}$.

Theorem B. *Let* g_1, g_2: $[0, T]\to[0, \infty)$ *be continuous. Then for any* $\varphi=(\varphi_1, \varphi_2)\in L^1_+$, *there exists a unique function* u: $[0, T]\to L^1_+$ *satisfying*

$$u(t)=\lim_{n\to\infty}\prod_{i=1}^{n}\Big(I+\frac{t}{n}A\Big(\frac{i}{n}t\Big)\Big)^{-1}\varphi. \tag{1.5}$$

Moreover, if a strong solution to (1.3) (1.4) *exists, it can be represented by* (1.5).

As the domain of $A(t)$ depends on t, we shall make use of the generalized domain introduced by Crbndall in [1]. Set

$$\hat{D}(A(t))=\{u\in\mathscr{D}(t);\ \lim_{\lambda\to0^+}\|A_\lambda(t)u\|_{L^1}<+\infty\}, \tag{1.6}$$

where $A_\lambda(t)=\lambda^{-1}(I-J_\lambda(t))$, $\mathscr{D}(t)=\bigcup_{K>0}(\bigcap_{0<\lambda<K}D_\lambda(t))$, $D_\lambda(t)=R(I+\lambda A(t))$. By using condition (A. 3), it is not difficult to show that $\hat{D}(A(t))$ is independent of t(see [1]). We set $\hat{D}=\hat{D}(A(t))$ for $0\leqslant t\leqslant T$. Because

$$D(A(t))\subset\hat{D}, \overline{D(A(t))}=L^1_+,$$

clearly $\hat{D}\cap L^1_+$ is dense in L^1_+.

Now, we state the main results in this paper.

Theorem 1. *Let* g_1, g_2 *be nonnegative continuous functions in*$[0, T]$, *and*

$$\varphi=(\varphi_1, \varphi_2)\in L^1_+\cap(L^\infty(0, 1))^2,$$

i. e., there exists a constant $M>0$ *such that*

$$\|\varphi_1\|_{L^\infty(0,1)}, \|\varphi_2\|_{L^\infty(0,1)}\leqslant M. \tag{1.7}$$

Then u, *defined by* (1.5), *belongs to* $C([0, T]; L^1_+)$ *and satisfies the system of integral equations*:

$$u_1(t, x)=\begin{cases}\varphi_1(x-t)+\int_0^t(u_2^2-u_1^2)(s, x+s-t)ds, & t\leqslant x,\\ g_1(t-x)+\int_{t-x}^t(u_2^2-u_1^2)(s, x+s-t)ds, & t>x,\end{cases} \tag{1.8}$$

$$u_2(t, x)=\begin{cases}\varphi_2(x+t)+\int_0^t(u_1^2-u_2^2)(s, x-s+t)ds, & t\leqslant 1-x,\\ g_2(x+t-1)+\int_{x+t-l}^t(u_1^2-u_2^2)(s, x-s+t)ds, & t>1-x.\end{cases} \tag{1.9}$$

Moreover, if we assume that $\varphi_1, \varphi_2\in C([0, 1])$ *and satisfy the compatibility conditions*

$$\varphi_1(0)=g_1(0), \varphi_2(1)=g_2(0), \tag{1.10}$$

then $u=(u_1, u_2)$ *defined by* (1.5) *is a continuous solution of the system* (1.8) (1.9), *i. e.*, $u_1, u_2\in C([0, T]\times[0, 1])$.

Theorem 2. *Let* $g_1, g_2\in C^1([0, T])$, $\varphi_1, \varphi_2\in C^1([0, 1])$ *be non-negative and satisfy the compatibility conditions in addition to* (1.10)

$$g_1'(0)=-\varphi_1'(0)-\varphi_1^2(0)+\varphi_2^2(0),\ g_2'(0)=\varphi_2^1(1)-\varphi_2^2(1)+\varphi_1^2(1). \tag{1.11}$$

Then $u=(u_1, u_2)$, *defined by* (1.5), *is the unique classical solution of the boundary* *value problem* (0.1)-(0.3).

Theorem 3. *Let $g_1 \equiv g_2 \equiv a$, where $a \geqslant 0$ is a constant, $\varphi = (\varphi_1, \varphi_2) \in L^1_+ \cap (L^\infty(0, 1))^2$. Then the mild solution $u = (u_1, u_2)$ of the boundary value problem* (0.1)-(0.3), *defined by* (1.5), *approaches to (a, a) exponentially as $t \to \infty$ in the following sense: there are constants $C \geqslant 0$ and $\lambda > 0$ such that*

$$\|u(t, \cdot) - (a, a)\|^2_{L^2} \leqslant Ce^{-\lambda t}\|\varphi - (a, a)\|^2_{L^2}, \tag{1.12}$$

where $\|u\|^2_{L^2} = \|u_1\|^2_{L^2(0,1)} + \|u_2\|^2_{L^2(0,1)}$ *for* $u = (u_1, u_2)$.

§ 2. Proof of Theorem 1

Let $T > 0$ be fixed.

Lemma 2. 1. *Let $g_1, g_2 \in C([0, T])$ be nonnegative, $\varphi = (\varphi_1, \varphi_2) \in L^1_+$ and*

$$\|\varphi_i\|_{L^\infty(0,1)}, \ \|g_i\|_{L^\infty(0,T)} \leqslant M, \ i = 1, 2. \tag{2.1}$$

For $\lambda > 0$, let

$$v(t) = (v_1(t), v_2(t)) = J_\lambda(t)\varphi. \tag{2.2}$$

Then $v \in L^1_+$ and

$$\|v_i(t)\|_{L^\infty(0,1)} \leqslant M, \ i = 1, 2, \ \forall t \in [0, T]. \tag{2.3}$$

Proof We need only to show (2.3). From (2.2) and the definition of $J_\lambda(t)$, we have

$$\begin{aligned} &v_1 + \lambda\left(\frac{dv_1}{dx} + v_1^2 - v_2^2\right) = \varphi_1, \\ &v_2 + \lambda\left(-\frac{dv_2}{dx} + v_2^2 - v_1^2\right) = \varphi_2, \end{aligned} \tag{2.4}$$

which are equivalent to

$$\begin{aligned} &\lambda \frac{dv_1}{dx} + kv_1 = (k-1)v_1 - \lambda(v_1^2 - v_2^2) + \varphi_1, \\ &-\lambda \frac{dv_2}{dx} + kv_2 = (k-1)v_2 - \lambda(v_2^2 - v_1^2) + \varphi_2. \end{aligned} \tag{2.5}$$

The constant k is chosen large enough. We write (2.5) as

$$Pv = Qv$$

or

$$v = Bv, \tag{2.6}$$

where $B = P^{-1}Q$.

As Kaper and Leaf did in [6], we can prove that if $v \in L^1_+$ and

$$\|v_i\|_{L^\infty(0,1)} \leqslant M, \ i = 1, 2, \tag{2.7}$$

then

$$Qv \in L^1_+ \text{ and } \|(Qv)_i\|_{L^\infty(0,1)} \leqslant kM, \ i = 1, 2. \tag{2.8}$$

Using (2.5) and (2.8) we obtain

$$(Bv)_1(x)=e^{-\frac{k}{\lambda}x}g_1(x)+\frac{1}{\lambda}\int_0^x e^{-\frac{k}{\lambda}(x-\xi)}(Qv)_1(\xi)d\xi\leqslant M.$$

We have a similar estimate for $(Bv)_2$. Therefore, if (2.8) holds, then

$$Bv\in L_+^1 \text{ and } \|(Bv)_i\|_{L^\infty(0,1)}\leqslant M,\ i=1,2. \tag{2.9}$$

Besides, if v, w both satisfy condition (2.8), then we have

$$\begin{aligned}\|Bv-Bw\|_{L^1}=&\frac{1}{\lambda}\int_0^1\left|\int_0^x e^{-\frac{k}{\lambda}(x-\xi)}((Qv)_1-(Qw)_1)(\xi)d\xi\right|dx\\&+\frac{1}{\lambda}\int_0^1\left|\int_x^1 e^{\frac{k}{\lambda}(x-\xi)}((Qv)_2-(Qw)_2)(\xi)d\xi\right|dx\\&\leqslant\frac{1}{\lambda}\int_0^1\int_\xi^1 e^{-\frac{k}{\lambda}(x-\xi)}|(Qv)_1-(Qw)_1|(\xi)dxd\xi\\&+\frac{1}{\lambda}\int_0^1\int_0^\xi e^{\frac{k}{\lambda}(x-\xi)}|(Qv)_2-(Qw)_2|(\xi)dxd\xi\\&\leqslant\frac{1}{k}\int_0^1(|k-1-\lambda(v_1+w_1)||v_1-w_1|+\lambda(v_2+w_2)|v_2-w_2|\\&+|k-1-\lambda(v_2+w_2)||v_2-w_2+\lambda(v_1+w_1)|v_1-w_1|)d\xi.\end{aligned}$$

Take $k>1+2\lambda M$, the abovo estimate gives

$$\|Bv-Bw\|_{L^1}\leqslant\left(1-\frac{1}{k}\right)\|v-w\|_{L^1}. \tag{2.10}$$

It follows immediately from (2.9) and (2.10) that system (2.4) admits a solution $v\in L_+^1$ and estimate (2.3) holds. The proof is complete.

As a corollary to Lemma 2.1 we have at once

Lemma 2. 2. *Under the assumptions of Lemma 2.1, the mild solution of the boundary value problem (0.1)-(0.3), defined by (1.5), belongs to $(L^\infty(0,1))^2$ and satisfies*

$$\|u_i(t)\|_{L^\infty(0,1)}\leqslant M,\ i=1,2. \tag{2.21}$$

Lemma 2. 3. *Let $\varphi\in L_+^1$, $\lambda=\frac{t}{n}$. Then the following hold for $1\leqslant k\leqslant n$:*

$$\begin{aligned}\left(\prod_{i=1}^k J_\lambda(i\lambda)\varphi\right)_1(x)=&\sum_{i=0}^{k-1}\frac{1}{i!}\left(\frac{x}{\lambda}\right)^i e^{-\frac{x}{\lambda}}g_1((k-i)\lambda)\\&+\frac{\lambda^{-k}}{(k-1)!}\int_0^x(x-\xi)^{k-1}e^{-\frac{1}{\lambda}(x-\xi)}\varphi_1(\xi)d\xi\\&+\sum_{i=0}^{k-1}\frac{\lambda^{-i}}{i!}\int_0^x(x-\xi)^i e^{-\frac{1}{\lambda}(x-\xi)}f_1\left(\prod_{j=1}^{k-i}J_\lambda(j\lambda)\varphi\right)(\xi)d\xi,\end{aligned} \tag{2.12}$$

$$\begin{aligned}\left(\prod_{i=1}^k J_\lambda(i\lambda)\varphi\right)_2(x)=&\sum_{i=0}^{k-1}\frac{1}{i!}\left(\frac{1-x}{\lambda}\right)^i e^{-\frac{1-x}{\lambda}}g_2((k-i)\lambda)\\&+\frac{\lambda^{-k}}{(k-1)!}\int_x^1(\eta-x)^{k-1}e^{-\frac{1}{\lambda}(\eta-x)}\varphi_2(\eta)d\eta\\&+\sum_{i=0}^{k-1}\frac{\lambda^{-i}i}{i!}\int_x(\eta-x)^i e^{-\frac{1}{\lambda}(\eta-x)}f_2\left(\prod_{j=1}^{k-i}J_\lambda(j\lambda)\varphi\right)(\eta)d\eta,\end{aligned} \tag{2.13}$$

where $f_1(u)=-u_1^2+u_2^2$, $f_2(u)=u_1^2-u_1^2$.

Proof For $k=1$, (2.12) and (2.13) follow at once from the definition of

$J_\lambda(t)$. Using inductive method, we can prove (2.12) and (2.13) for $1<k\leqslant n$. The detail is omitted.

Lemma 2.4. *Let* $h\in C([0,1])$. *Then*

$$\lim_{n\to\infty}\sum_{i=0}^{n-i}\frac{1}{i!}(n\sigma)^i e^{-n\sigma}h\left(\frac{i}{n}\right)=\begin{cases}h(\sigma), & 0\leqslant\sigma<1,\\ 0, & \sigma>1.\end{cases} \tag{2.14}$$

Proof It is known that

$$\lim_{n\to\infty}e^{-n\sigma}\sum_{i=0}^{\infty}h\left(\frac{i}{n}\right)\frac{(n\sigma)^i}{i!}=h(\sigma),\ \forall\sigma\geqslant 0. \tag{2.15}$$

holds for any bounded $h\in C([0,\infty))$ (see [3]).

We first assume that $h(x)\geqslant 0$, $\forall x\in[0,1]$. If $0\leqslant\sigma<1$, let $h_1(x)$ and $h_2(x)$ be nonnegative bounded continuous functions on $[0,\infty)$ such that

$$h_1(x)=\begin{cases}h(x), & 0\leqslant x\leqslant\sigma,\\ \leqslant h(x), & \sigma<x<1,\\ 0, & x\geqslant 1.\end{cases}$$

and $h_2(x)=h(x)$ for $0\leqslant x\leqslant 1$. Then we have

$$e^{-n\sigma}\sum_{i=0}^{\infty}h_1\left(\frac{i}{n}\right)\frac{(n\sigma)^i}{i!}\leqslant e^{-n\sigma}\sum_{i=0}^{n-1}h\left(\frac{i}{n}\right)\frac{(n\sigma)^i}{i!}\leqslant e^{-n\sigma}\sum_{i=0}^{\infty}h_2\left(\frac{i}{n}\right)\frac{(n\sigma)^i}{i!}. \tag{2.16}$$

Both the limits of right and left hand sides in (2.16), by (2.15) and the definition of h_1 and h_2, are $h(\sigma)$ as $n\to\infty$, so the lemma holds for this case.

For $\sigma>1$, let $h_3(x)$ be a nonnegative bounded continuous function on $[0,\infty)$ such that

$$h_3(x)=\begin{cases}h(x), & 0\leqslant x\leqslant 1,\\ 0, & x\geqslant\sigma.\end{cases}$$

We have

$$e^{-n\sigma}\sum_{i=0}^{n-1}h\left(\frac{i}{n}\right)\frac{(n\sigma)^i}{i!}\leqslant e^{-n\sigma}\sum_{i=0}^{\infty}h_3\left(\frac{i}{n}\right)\frac{(n\sigma)^i}{i!}. \tag{2.17}$$

The limit of right hand side of (2.17) is 0 as $n\to\infty$, so is that of left hand side of (2.17).

If $h(x)$ is not a nonnegative function, then there exists a constant $M>0$ such that $h(x)+M\geqslant 0$, $\forall x\in[0,1]$. From the result for nonnegative functions, we have

$$\lim_{n\to\infty}e^{-n\sigma}\sum_{i=0}^{n-1}\left(h\left(\frac{i}{n}\right)+M\right)\frac{(n\sigma)^i}{i!}=\begin{cases}h(\sigma)+M, & 0\leqslant\sigma<1\\ 0, & \sigma>1.\end{cases} \tag{2.18}$$

Noticing

$$e^{-n\sigma}\sum_{i=0}^{n-1}\frac{(n\sigma)^i}{i!}=\begin{cases}1, & 0\leqslant\sigma<1,\\ 0, & \sigma>1\end{cases}$$

(see [5]), we obtain the result of the lemma from (2.18).

Lemma 2.5. *Let* $g_1, g_2\in C([0,T])$. *Then*

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{i!}\left(\frac{x}{\lambda}\right)^i e^{-\frac{x}{\lambda}}g_1((n-i)\lambda)=\begin{cases}g_1(t-x), & x<t,\\ 0, & x>t,\end{cases} \tag{2.19}$$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{i!}\left(\frac{1-x}{\lambda}\right)^{-\frac{1-x}{\lambda}}g_2((n-i)\lambda)=\begin{cases}g_2(x+t-1), & x>1-t,\\ 0, & x<1-t,\end{cases}\tag{2.20}$$

for $t\in(0, T]$, $x\in[0, 1]$, *where*

$$\lambda=\frac{t}{n}.$$

Lemma 2.6. *Let* $\varphi\in L^1$, $\lambda=\frac{t}{n}$ *and* $t\in(0, T]$. *Then*

$$\lim_{n\to\infty}\frac{\lambda^{-n}}{(n-1)!}\int_0^x(x-\xi)^{n-1}e^{-\frac{1}{\lambda}(x-\xi)}\varphi_1(\xi)d\xi=\begin{cases}0, & x<t,\\ \varphi_1(x-t), & x>t,\end{cases}\tag{2.21}$$

$$\lim_{n\to\infty}\frac{\lambda^{-n}}{(n-1)!}\int_x^1(\eta-x)^{n-1}e^{-\frac{1}{\lambda}(\eta-x)}\varphi_2(\eta)d\eta=\begin{cases}0, & x>1-t,\\ \varphi_2(x+t-1), & x<1-t\end{cases}\tag{2.22}$$

in $L^1(0, 1)$.

Poof Set

$$(J_{1\lambda}\varphi_1)(x)=\lambda^{-1}\int_0^x e^{-\frac{1}{\lambda}(x-\xi)}\varphi_1(\xi)d\xi.$$

It is easy to verify that

$$\frac{\lambda^{-n}}{(n-1)!}\int_0^x(x-\xi)^{n-1}e^{-\frac{1}{\lambda}(x-\xi)}\varphi_L(\xi)d\xi=(J_{1\lambda}^n\varphi_1)(x).\tag{2.23}$$

Let A_1 be an operator defined by

$$D(A_1)=\{\psi\in W^{1,1}(0, 1),\ \psi(0)=0\},$$
$$A_1\psi=\frac{d\psi}{dx},\ \forall\psi\in D(A_1).$$

It is not difficult to verify that A_1 is an m-accretive operator in L^1 (0, 1). Thus A_1 generates a semigroup $S_1(t)$ in $L^1(0, 1)$. From the exponential formula of linear semigroup, we know that

$$S_1(t)\varphi_1=\lim_{n\to\infty}J_{1\lambda}^n\varphi_1\tag{2.24}$$

in $L^1(0, 1)$ and $S_1(t)\varphi_1$ is a strong solution of problem

$$\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial x}=0,$$
$$u(t, 0)=0,\ u(0, x)=\varphi_1(x)$$

if $\varphi_1\in D(A_1)$, that is

$$(S_1(t)\varphi_1)(x)=\begin{cases}0, & x<t,\\ \varphi_1(x-t), & x>t.\end{cases}\tag{2.25}$$

(2.23), (2.24) and (2.25) imply that (2.21) holds for $\varphi_1\in L^1(0, 1)$. Similarly, we can prove (2.22). The proof is complete.

Lemma 2.7. *Let* $h_1, h_2\in L^1(0, 1)$, $\lambda>0$ *and* $x\in[0, 1]$. *Then*

$$\left\|\frac{\lambda^{-i}}{i!}\int_0^x(x-\xi)^i e^{-\frac{1}{\lambda}(x-\xi)}[h_1(\xi)-h_2(\xi)]d\xi\right\|_{L^1(0,1)}\leqslant\lambda\|h_1-h_2\|_{L^1(0,1)},\tag{2.26}$$

$$\left\|\frac{\lambda^{-i}}{i!}\int_x^1(\eta-x)^i e^{-\frac{1}{\lambda}(\eta-x)}[h_1(\eta)-h_2(\eta)]d\eta\right\|_{L^1(0,1)}\leqslant\lambda\|h_1-h_2\|_{L^1(0,1)}.\tag{2.27}$$

Proof Noticing

$$\int_\xi^1 (x-\xi)^i e^{-\frac{1}{\lambda}(x-\xi)} dx \leqslant i!\lambda^{i+1},$$

we have

$$\begin{aligned}&\left\| \frac{\lambda^{-i}}{i!}\int_0^x (x-\xi)^i e^{-\frac{1}{\lambda}(x-\xi)} [h_1(\xi)-h_2(\xi)]d\xi \right\|_{L^1(0,1)} \\ &\leqslant \frac{\lambda^{-i}}{i!}\int_0^1\int_\xi^1 (x-\xi)^i e^{-\frac{1}{\lambda}(x-\xi)} dx |h_1(\xi)-h_2(\xi)|d\xi \\ &\leqslant \lambda\|h_1-h_2\|_{L^1(0,1)}.\end{aligned}$$

Similarly, we can prove (2.27).

Lemma 2.8. *Let the assumptions of Lemma 2.1 hold and $\varphi\in\hat{D}$. Then*

$$\lim_{n\to\infty}\left\|\sum_{i=0}^{n-1}\frac{\lambda^{-i}}{i!}\int_0^x (x-\xi)^i e^{-\frac{1}{\lambda}(x-\xi)}\left[f_1\Big(\prod_{j=1}^{n-i}J_\lambda(j\lambda)\varphi\Big)(\xi)-f_1(u((n-i)\lambda))(\xi)\right]d\xi\right\|_{L^1(0,1)} =0 \tag{2.28}$$

and

$$\lim_{n\to\infty}\left\|\sum_{i=0}^{n-1}\frac{\lambda^{-i}}{i!}\int_x^1 (\eta-x)^i e^{-\frac{1}{\lambda}(\eta-x)}\left[f_2\Big(\prod_{j=1}^{n-i}J_\lambda(j\lambda)\varphi\Big)(\eta)-f_2(u((n-i)\lambda))(\eta)\right]d\eta\right\|_{L^1(0,1)} =0 \tag{2.29}$$

hold for $t\in(0,T]$, $\lambda=\frac{t}{n}$, where $f_1(u)=-u_1^2+u_2^2$, $f_2(u)=u_1^2-u_2^2$.

Proof We verify (2.28). By using Lemmas 2.1, 2.2 and 2.7 it follows that

$$\begin{aligned}&\left\|\sum_{i=0}^{n-1}\frac{\lambda^{-i}}{i!}\int_0^x (x-\xi)^i e^{-\frac{1}{\lambda}(x-\xi)}\left[\Big(f_1\prod_{j=1}^{n-i}J_\lambda(j\lambda)\varphi\Big)(\xi)-f_1(u((n-i)\lambda))(\xi)\right]d\xi\right\|_{L^1(0,1)} \\ &\leqslant\sum_{i=0}^{n-1}\lambda\left\|f_1\Big(\prod_{j=1}^{n-i}J_\lambda(j\lambda)\varphi\Big)-f_1(u((n-i)\lambda))\right\|_{L^1(0,1)} \\ &\leqslant 2\lambda M\sum_{i=0}^{n-1}\left\{\left\|\Big(\prod_{j=1}^{n-i}J_\lambda(j\lambda)\varphi\Big)_2-u_2((n-i)\lambda)\right\|_{L^1(0,1)}\right. \\ &\qquad\left.+\left\|\Big(\prod_{j=1}^{n-i}J_\lambda(j\lambda)\varphi\Big)_2-u_2((n-i)\lambda)\right\|_{L^1(0,1)}\right\} \\ &\leqslant 2\lambda MK\sum_{i=0}^{n-1}\frac{(n-i)t}{n}\{(n-i)^{-\frac{1}{2}}+\rho((n-i)^{\frac{3}{4}}n^{-1}t)\} \\ &\leqslant 2MKt^2\sum_{i=0}^{n-1}(n^{-\frac{3}{4}}+n^{-1}\rho(tn^{-\frac{1}{4}})),\end{aligned} \tag{2.30}$$

where K is a constant and $\rho(r)$: $[0, T]\to[0, \rho(T)]$ is a non-decreasing function satisfying $\lim\limits_{r\to 0}\rho(r)=\rho(0)=0$. In the estimate of (2.30) we used Proposition 2.5 of [2]. Now (2.28) follows immediately from (2.30). In the same way we can prove (2.29).

Lemma 2.9. *Under the assumptions of Lemma 2.1, (2.28) and (2.29) hold for $t\in(0,T]$ and $\lambda=\frac{t}{n}$.*

Proof Since $\hat{D}\cap L^1_+$ is dense in L^1_+, it is not difficult to show that there exist $\varphi^{(k)}=(\varphi_1^{(k)},\varphi_2^{(k)})\in\hat{D}\cap L^1_+ (k=1, 2, \cdots)$ such that

$$\|\varphi_i^{(k)}\|_{L^\infty(0,1)}\leqslant \widetilde{M},\ i=1,\ 2;\ k=1,\ 2,\ \cdots, \tag{2.31}$$

$$\|\varphi^{(k)}-\varphi\|_{L^1}\to 0,\ \text{as } k\to\infty, \tag{2.32}$$

where $\widetilde{M}$ is a constant.

Let

$$u^{(k)}(t)=\lim_{n\to\infty}\prod_{i=1}^{n}\left(I+\frac{t}{n}A\left(\frac{i}{n}t\right)\right)^{-1}\varphi^{(k)}.$$

Then the nonexpansiveness of the resolvents $J_\lambda(t)$ implies that

$$\|u^{(k)}(t)-u(t)\|_{L^1}\leqslant\|\varphi^{(k)}-\varphi\|_{L^1}. \tag{2.32}$$

Using Lemma 2.1, 2.2 and 2.7, we can obtain

$$\left\|\sum_{i=0}^{n-1}\frac{\lambda^{-i}}{i!}\int_0^x(x-\xi)^ie^{-\frac{1}{\lambda}(x-\xi)}\left[f_1\left(\prod_{j=1}^{n-i}J_\lambda(j\lambda)\varphi^{(k)}\right)(\xi)-f_1\left(\prod_{j=1}^{n-i}J_\lambda(j\lambda)\varphi\right)(\xi)\right]d\xi\right\|_{L^1(0,1)}$$
$$\leqslant(M+\widetilde{M})t\|\varphi^{(k)}-\varphi\|_{L^1(0,1)}, \tag{2.34}$$

$$\left\|\sum_{i=0}^{n-i}\frac{\lambda^{-i}}{i!}\int_0^x(x-\xi)^ie^{-\frac{1}{\lambda}(x-\xi)}[f_1(u^{(k)}((n-i)\lambda))(\xi)-f_1(u((n-i)\lambda))(\xi)]\right\|_{L^1(0,1)}$$
$$\leqslant(M+\widetilde{M})t\|\varphi^{(k)}-\varphi\|_{L^1}. \tag{2.35}$$

Moreover, (2.34) and (2.35) imply that

$$\left\|\sum_{i=0}^{n-1}\frac{\lambda^{-i}}{i!}\int_0^x(x-\xi)^ie^{-\frac{1}{\lambda}(x-\xi)}\left[f_1\left(\prod_{j=1}^{n-i}J_\lambda(j\lambda)\varphi\right)(\xi)-f_1(u((n-i))(\xi)\right]d\xi\right\|_{L^1(0,1)}$$
$$\leqslant 2(M+\widetilde{M})t\|\varphi^{(k)}-\varphi\|_{L^1}+\left\|\sum_{i=0}^{n-1}\frac{\lambda^{-i}}{i!}\int_0^x(x-\xi)^ie^{-\frac{1}{\lambda}(x-\xi)}\right.$$
$$\left.\left[f_1\left(\prod_{j=1}^{n-i}J_\lambda(j\lambda)\varphi^{(k)}\right)(\xi)-f_1(u^{(k)}((n-i)\lambda))\right]d\xi\right\|_{L^1}. \tag{2.36}$$

Now (2.28) follows at once from Lemma 2.8 and (2.36). Similarly, we can prove (2.29).

Lemma 2.10. *Let the assumptions of Lemma 2.1 hold, $t\in(0, T]$ and*

$$\lambda=\frac{t}{n}.$$

Then

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{\lambda^{-i}}{i!}\int_0^x(x-\xi)^ie^{-\frac{1}{\lambda}(x-\xi)}f_1(u((n-i)\lambda))(\xi)d\xi$$
$$=\begin{cases}\displaystyle\int_0^t f_1(u(s))(x-t+s)ds,\ t<x,\\ \displaystyle\int_{t-x}^t f_1(u(s))(x-t+s)ds,\ t>x\end{cases} \tag{2.37}$$

and

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{\lambda^{-i}}{i!}\int_x^1(\eta-x)^ie^{-\frac{1}{\lambda}(\eta-x)}f_2(u((n-i)\lambda))(\eta)d\eta$$
$$=\begin{cases}\displaystyle\int_0^t f_2(u(s))(x+t-s)ds,\ t<1-x,\\ \displaystyle\int_{t+x-1}^t f_2(u(s))(x+t-s)ds,\ t>1-x\end{cases} \tag{2.38}$$

hold in $L^1(0, 1)$.

Proof Set

$$\psi(t, x) = f_1(u(t))(x).$$

By the definition of f_1 and Lemma 2.2,

$$\psi(t, \cdot) \in C([0, T]; L^1(0, 1)).$$

We first assume that $\psi(t, x) \in C([0, T] \times [0, 1])$. Write the right hand side of (2.37) as

$$\sum_{i=0}^{n-1} \frac{\lambda^{-i}}{i!} \int_0^x (x-\xi)^i e^{-\frac{1}{\lambda}(x-\xi)} f_1(u((n-i)\lambda))(\xi)\, d\xi = t\int_0^{\frac{x}{t}} h_n(\sigma) d\sigma, \tag{2.39}$$

where

$$h_n(\sigma) = \sum_{i=0}^{n-1} \frac{(n\sigma)^i}{i!} e^{-n\sigma} \psi\Big(t - \frac{i}{n} t, x - \sigma t\Big). \tag{2.40}$$

It is easy to verify that

$$|h_n(\sigma)| \leqslant \max_{[0,T]\times[0,1]} |\psi(t, x)|$$

and

$$\lim_{n\to\infty} h_n(\sigma) = \begin{cases} \psi(t-\sigma t, x-\sigma t), & 0\leqslant\sigma<1, \\ 0, & \sigma>1, \end{cases}$$

by Lemma 2.4, for $\sigma \in \left[0, \frac{x}{t}\right]$. Then we, from Lebesgue theorem, have

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\lambda^{-i}}{i!} \int_0^x (x-\xi)^i e^{-\frac{1}{\lambda}(x-\xi)} \psi((n-i)\lambda, \xi) d\xi$$
$$= \begin{cases} \int_0^t \psi(s, x-t+s) ds, & t<x, \\ \int_{t-x}^t \psi(s, x-t+s) ds, & t>x. \end{cases} \tag{2.41}$$

Noticing

$$\left| \sum_{i=0}^{n-1} \frac{\lambda^{-i}}{i!} \int_0^x (x-\xi)^i e^{-\frac{1}{\lambda}(x-\xi)} \psi((n-i)\lambda, \xi) d\xi \right| \leqslant t \max_{[0,T]\times[0,1]} |\psi(t, x)|,$$

it follows, by Lebesgue theorem again, that (2.41) holds in $L^1(0, 1)$.

Finaly, a density argument proves that (2.41) holds in L^1 (0, 1) for $\psi \in C([0, T]; L^1(0, 1))$. This completes the proof of (2.37). Similarly we can prove (2.38).

Proof of Theorem 1 As a consequence of Lemma 2.3, 2.5, 2.6, 2.9 and 2.10, we have immediately the results of Theorem 1. The proof of Theorem 1 is complete.

§ 3. Existence of The Classical Solutions

This section is devoted to the proof of Theorem 2. Let $u(t, x) \in (C([0, T] \times [0, 1]))^2$ be the solution of the system (1.8) (1.9) of integral equations given by (1.5). Consider the following system of integral equations

$$v_1(t,x)=\begin{cases}\psi_1(x-t)+2\int_0^t(u_2v_2-u_1v_1)(s,x-t+s)ds, & t\le x,\\ g_1'(t-x)+2\int_{t-x}^t(u_2v_2-u_1v_1)(s,x-t+s)ds, & t>x,\end{cases}\tag{3.1}$$

$$v_2(t,x)=\begin{cases}\psi_2(x+t)+2\int_0^t(u_1v_1-u_2v_2)(s,x-t+s)ds, & t\le 1-x,\\ g_2'(x+t-1)+2\int_{t+x-1}^t(u_1v_1-u_2v_2)(s,x-t+s)ds, & t>1-x\end{cases}\tag{3.2}$$

with unknown $v=(v_1,v_2)$, where

$$\psi_1(x)=-\varphi_1'(x)-\varphi_1^2(x)+\varphi_2^2(x),\quad \psi_2(x)=\varphi_2'(x)+\varphi_1^2(x)-\varphi_2^2(x).\tag{3.3}$$

By the compatibility conditions (1.11) and the expressions (3.3) of ψ_1 and ψ_2, we have $\psi_1(0)=g_1'(0)$ and $\psi_2(1)=g_2'(0)$. Using the Picard's iterative method, we can prove the following

Lemma 3. 1. *Under the assumptions of Theorem 2, the system (3.1) (3.2) of integral equations admits a unique continuous solution $v(t,x)=(v_1(t,x),v_2(t,x))$.*

Lemma 3. 2. *Let $v=(v_1(t,x),v_2(t,x))$ be a continuous solution of the system (3.1) (3.2) and*

$$w_i(t,x)=\varphi_i(x)+\int_0^t v_i(\tau,x)d\tau,\ i=1,2.\tag{3.4}$$

Then

$$w_i(t,x)\equiv u_i(t,x),\ \forall(t,x)\in[0,T]\times[0,1],\ i=1,2.\tag{3.5}$$

Proof Without loss of generality, we assume that $T\le 1$. Let M_1 be a constant such that

$$|v_i(t,x)|\le M_1,\ |w_i(t,x)|\le M_1,\ \forall(t,x)\in[0,T]\times[0,1],\ i=1,2.\tag{3.6}$$

When $x<t$, noticing

$$\int_0^x\psi_1(x-\tau)d\tau=\varphi_1(0)-\varphi_1(x)+\int_0^x[\varphi_2^2(x-\tau)-\varphi_1^2(x-\tau)]d\tau,$$

$$\begin{aligned}\int_0^x[\varphi_2^2(x-\tau)-\varphi_1^2(x-\tau)]d\tau&=\int_0^x(w_2^2-w_1^2)(\sigma+t-x,\sigma)d\sigma\\&\quad-\int_0^x\Big[\int_x^t+\int_{x-\sigma}^x\Big]2(w_2v_2-w_1v_1)(\tau+\sigma-x,\sigma)d\tau\,d\sigma\\&=\int_{t-x}^t(w_2^2-w_1^2)(s,x-t+s)\,ds-\int_x^t\int_{\tau-x}^{\tau}2(w_2v_2-w_1v_1)(s,x-\tau+s)dsd\tau\\&\quad-\int_0^x\int_0^\tau 2(w_2v_2-w_1v_1)(s,x-\tau+s)ds\,d\tau\end{aligned}$$

and the compatibility conditions, from (3.4) and (3.1) we can obtain

$$\begin{aligned}w_1(t,x)&=g_1(t-x)+\int_{t-x}^t(w_2^2-w_1^2)(s,x-t+s)ds\\&\quad+\int_0^x\int_0^\tau 2[v_2(u_2-w_2)-v_1(u_1-w_1)](s,x-\tau+s)ds\,d\tau\\&\quad+\int_x^t\int_{\tau-x}^\tau 2[v_2(u_2-w_2)-v_1(u_1-w_1)](s,x-\tau+s)ds\,d\tau.\end{aligned}\tag{3.7}$$

By using Lemma 2.2 and (3.6), from (1.8) and (3.7) it follows that

$$\int_0^t |u_1(t,x)-w_1(t,x)|dx \leqslant (M+M_1)\int_0^t\int_{t-x}^t (|u_1-w_1|+|u_2-w_2|)(s, x-t+s)\,ds\,dx$$

$$+2M_1\int_0^t\int_0^x\int_0^\tau (|u_1-w_1|+|u_2-w_2|)(s, x-\tau+s)\,ds\,d\tau\,dx$$

$$+2M_1\int_0^t\int_x^t\int_{\tau-x}^\tau (|u_1-w_1|+|u_2-w_2|)(s, x-\tau+s)\,ds\,d\tau\,dx$$

$$=(M+M_1)\int_0^t\int_{t-s}^t (|u_1-w_1|+|u_2-w_2|)(s, x-t+s)\,dx\,ds$$

$$+2M_1\int_0^t\int_0^{t-x}\int_s^{s+x} (|u_1-w_1|+|u_2-w_2|)(s, x-\tau+s)\,d\tau\,ds\,dx$$

$$+2M_1\int_0^t\int_{t-x}^t\int_s^t (|u_1-w_1|+|u_2-w_2|)(s, x-\tau+s)\,d\tau\,ds\,dx$$

$$\leqslant C(T)\int_0^t \|u(s)-w(s)\|_{L^1}ds, \tag{3.8}$$

where $C(T)$ is a constant depending on T.

For the case where $x \geqslant t$, similarly we can obtain

$$w_1(t,x)=\varphi_1(x-t)+\int_0^t (w_2^2-w_1^2)(s, x-t+s)\,ds$$

$$+\int_0^t\int_0^\tau 2[v_2(u_2-w_2)-v_1(u_1-w_1)](s, x-\tau+s)\,ds\,d\tau. \tag{3.9}$$

From (1.8) and (3.9) we have the following estimate

$$\int_t^1 |u_1(t,x)-w_1(t,x)|dx \leqslant C(T)\int_0^t \|u(s)-w(s)\|_{L^1}ds. \tag{3.10}$$

By combining (3.8) and (3.10), it follows that

$$\|u_1(t)-w_1(t)\|_{L^1(0,1)} \leqslant C(T)\int_0^t \|u(s)-w(s)\|_{L^1}ds. \tag{3.11}$$

Similarly, we have

$$\|u_2(t)-w_2(t)\|_{L^1(0,1)} \leqslant C(T)\int_0^t \|u(s)-w(s)\|_{L^1}ds. \tag{3.12}$$

The estimates (3.11) and (3.12) imply that

$$\|u(t)-w(t)\|_{L^1} \leqslant C(T)\int_0^t \|u(s)-w(s)\|_{L^1}ds. \tag{3.13}$$

The assertion (3.5) now is a consequence of inequality (3.13). The proof is complete.

Proof of Theorem 2. For the existence of classical solution it suffices to prove that $u(t,x)$ has continuous derivatives with respect to t and x. This is a consequence of Lemma 3.1, 3.2 and the fact that $u(t,x)$ is a solution of the system (1.8) (1.9) of integral equations. By using an estimate similar to that in Lemma 3.2, it is not difficult to prove the uniqueness. The proof is complete.

§ 4. Asymptotic Behaviour of Solutions

In this section we discuss the asypmtotic behaviour, as $t\to\infty$, of the solution for boundary problem (0.1)-(0.3) with $g_1(t)\equiv g_2(t)\equiv a$, where a is a nonnegative constant

Set

$$w_1=u_1-a,\ w_2=u_2-a. \tag{4.1}$$

Then boundary value problem (0.1)-(0.3) is transformed into the following form

$$\frac{\partial w_1}{\partial t}+\frac{\partial w_1}{\partial x}+2a(w_1-w_2)+w_1^2-w_2^2=0, \tag{4.2}$$

$$\frac{\partial w_2}{\partial t}-\frac{\partial w_2}{\partial x}+2a(w_2-w_1)+w_2^2-w_1^2=0, \tag{4.3}$$

$$w_1(t,\ 0)=w_2(t,\ 1)=0, \tag{4.4}$$

$$w_1(0,\ x)=\eta_1(x),\ w_2(0,\ x)=\eta_2(x), \tag{4.5}$$

where $\eta_1(x)=\varphi_1(x)-a$, $\eta_2(x)=\varphi_2(x)-a$.

For above boundary value problem, we have

Lemma 4.1. *Let* $\eta_1,\ \eta_2\in L^\infty(0,\ 1)$,

$$-a\leqslant\eta_i(x)\leqslant K,\ \forall x\in[0,\ 1],\ i=1,\ 2, \tag{4.6}$$

where K *is a nonnegative constant, and* $w(t,\ x)=(w_1(t,\ x),\ w_2(t,\ x))$ *be solution of the boundary value problem* (4.2)-(4.5). *Then there exist constants* C *and* $\lambda>0$ *such that*

$$\|w(t)\|_{L^2}^2\leqslant Ce^{-\lambda t}\|\eta\|_{L^2}^2. \tag{4.7}$$

Proof Let $\eta_1,\ \eta_2\in C^1([0,\ 1])$ and satisfy

$$\eta_1(0)=\eta_2(1)=0, \tag{4.8}$$

$$\eta_1'(0)-(2a+\eta_2(0))\eta_2(0)=0, \tag{4.9}$$

$$\eta_2'(1)+(2a+\eta_1(1))\eta_1(1)=0. \tag{4.10}$$

From Theorem 2, under above coditions the boundary value problem (4.2)-(4.5) admits a unique classical solution $w=(w_1,\ w_2)\in(C^1([0,\ \infty)\times[0,\ 1]))^2$.

Multiplying both sides of (4.2) and (4.3) by w_1 and w_2, respectively, and integrating the sum of resulted expressions over interval [0, 1] with respect to x, we can get

$$\frac{1}{2}\frac{d}{dt}\int_0^1(w_1^2+w_2^2)(t,\ x)dx+\frac{1}{2}(w_1^2(t,\ 1)+w_2^2(t,\ 0))$$
$$+\int_0^1(w_1-w_2)^2(2a+w_1+w_2)(t,\ x)dx=0. \tag{4.11}$$

From (4.6) and Lemma 2.2, we have

$$-a\leqslant w_i(t,\ x)\leqslant K,\ \forall(t,\ x)\in[0,\ \infty)\times[0,\ 1],\ i=1,\ 2. \tag{4.12}$$

Therefore, taking into account (4.12), (4.11) implies that

$$\frac{d}{dt}\int_0^1 (w_1^2+w_2^2)(t, x)dx \leqslant -(w_1^2(t, 1)+w_2^2(t, 0)). \tag{4.13}$$

As in [8], set

$$G(w)(x)=\int_{1-x}^{2+x}(w_1^2+w_2^2)(t, x)dt. \tag{4.14}$$

Noticing the equations (4.2) and (4.3) satisfied by w_1 and w_2, differentiating (4.14) with respect to x gives

$$\frac{dG(w)(x)}{dx}=2w_1^2(1-x, x)+2w_2^2(2+x, x)+2\int_{1-x}^{2+x}(w_2^2-w_1^2)(w_1+w_2+2a)(t, x)dt. \tag{4.15}$$

By use of (4.12), it follows from (4.15) that

$$\frac{dG(w)(x)}{dx} \geqslant -4(K+a)G(w)(x). \tag{4.16}$$

Thus

$$G(w)(x) \leqslant e^{4(K+a)}G(w)(1). \tag{4.17}$$

Set $R=[1, 2]\times[0, 1]$, then

$$\iint_R (w_1^2+w_2^2)(t, x)dtdx \leqslant \int_0^1 G(w)(x)dx \leqslant e^{4(K+a)}\int_0^3 w_1^2(t, 1)dt. \tag{4.18}$$

Set

$$E(w)(t)=\int_0^1 (w_1^2+w_2^2)(t, x)dx.$$

By integrating inequality (4.13) from 1 to 3 with respect to t, it follows that

$$E(w)(3)-E(w)(0)=-\int_0^3 w_1^2(t, 1)dt. \tag{4.19}$$

Since $E(w)(t)$, by (4.13), is monotone decreasing, using (4.18) we have

$$E(w)(3) \leqslant \int_1^2 E(w)(t)dt \leqslant \iint_R (w_1^2+w_2^2)(t, x)dtdx$$
$$\leqslant e^{4(K+a)}\int_0^3 w_1^2(t, 1)dt. \tag{4.20}$$

Combining (4.19) and (4.20) gives

$$E(w)(3)-E(w)(0) \leqslant -e^{-4(K+a)}E(w)(3),$$

that is,

$$E(w)(3) \leqslant (1+e^{-4(K+a)})^{-1}E(w)(0). \tag{4.21}$$

Taking into account the monotonicity of $E(w)(t)$, from (4.21), we see that there exist constants C and $\lambda>0$ such that

$$E(w)(t) \leqslant Ce^{-\lambda t}E(w)(0). \tag{4.22}$$

For the general case where $\eta_1, \eta_2 \in L^\infty(0.1)$ and satisfy (4.6), it is not difficult to see that there exists a sequence of functions $\eta^{(k)}(x)=(\eta_1^{(k)}(x), \eta_2(x)) \in (C^1([0, 1]))^2$, $k=1, 2, \cdots$, satisfying (4.6) and (4.8)-(4.10), such that

$$\lim_{k\to\infty}\|\eta^{(k)}-\eta\|_{L^1}=0. \tag{4.23}$$

Let $w^{(k)}(t, x)=(w_1^{(k)}(t, x), w_2^{(k)}(t, x))$ be the solution of boundary value problem (4.2)-(4.5) with initial data $\eta_1^{(k)}(x)$ and $\eta_2^{(k)}(x)$. So estimate (4.22) holds for $w=w^{(k)}$, that is,

$$\|w^{(k)}(t)\|_{L^2}^2 \leqslant Ce^{-\lambda t}\|\eta^{(k)}\|_{L^2}^2. \tag{4.24}$$

Moreover, the nonexpansiveness of the evolution operators gives

$$\|w^{(k)}(t)-w(t)\|_{L^1}\leqslant\|\eta^{(k)}-\eta\|_{L^1}, \ \forall t\in[0, \infty). \tag{4.25}$$

Taking into account the uniformly boundedness of $w_1^{(k)}$ and $w_2^{(k)}$ in $L^\infty(0, 1)$, by the application of estimate (4.25), we see that the limit of (4.24), as $k\to\infty$, gives (4.7). The proof is complete.

Proof of Theorem 3 Theorem 3 is a simple consequence of Lemma 4.1.

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