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GENERALIZED CLIFFORD TORUS IN S^{n+1} AND PRESCRIBED MEAN CURVATURE FUNCTION

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Abstract

Given some function $H(X)$, one can find a compact hypersurface in S^{n+1}, which is homeomorphic to $S^m(1)\times S^{n-m}(1)$ and whose mean curvature is given by $H(X)$.

Introduction

The following problem is very interesting:

Let H be a real valued function on $S^{n+1}(n>2)$. Find suitable conditions on H to insure that one can find a closed hypersurface in S^{n+1} which is homeomorphic to Clifford torus $S^m\times S^{n-m}(2\leqslant m\leqslant n-1)$ and whose mean curvature is given by H.

In this paper, I obtain an existence theorem.

§ 1. Fundamental Equation

In this section, we shall derive an equation for the mean curvature of generalized Clifford torus in $S^{n+1}(n\geqslant 3)$.

For the unit sphere S^{n+1} in $R^{n+2}(n\geqslant 3)$, we know the Clifford torus

$$M_{m,n-m}=S^m\Big(\sqrt{\frac{m}{n}}\Big)\times S^{n-m}\Big(\sqrt{\frac{n-m}{n}}\Big)(2\leqslant m\leqslant n-1)$$

is a minimal hypersurface in S^{n+1}, where $S^P(r)$ denotes a P-dimensional sphere with radius r. The position vector field of $M_{m,n-m}$ can be written as

$$Y=\sqrt{\frac{m}{n}}\,\boldsymbol{r}+\sqrt{\frac{n-m}{n}}\,\boldsymbol{\rho}, \tag{1.1}$$

where $R^{n+2}=R^{m+1}\times R^{n-m+1}$, $\boldsymbol{r}$ is the position vector field of unit sphere $S^m(1)$ in $R^{m+1}(m\geqslant 2)$, and $\boldsymbol{\rho}$ is the position vector field of unit sphere $S^{n-m}(1)$ in R^{n-m+1}. $\langle,\rangle$ denotes the inner product of two vectors in R^{n+2}. At any fixed point, $\langle\boldsymbol{r},\boldsymbol{\rho}\rangle=0$.

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We introduce the generalized Clifford torus M, whose position vector field is given by

$$\boldsymbol{X}=\frac{e^u}{\sqrt{1+e^{2u}}}\boldsymbol{r}+\frac{1}{\sqrt{1+e^{2u}}}\boldsymbol{\rho}, \tag{1.2}$$

where u is a differentiable function on $S^m(1)$. M lies in S^{n+1} and is homeomorphic to $M_{m,n-m}$

We adapt a local orthogonal frame $\{e_1, e_2, \cdots, e_{n+2}\}$ in R^{n+2}, such that restricted to R^{m+1}, $e_1, \cdots, e_m$ are tangent to $S^m(1)$; e_{n+1} is the radical direction on $S^m(1)$, and restricted to R^{n-m+1}, $e_{m+1}, \cdots, e_n$ are tangent to $S^{n-m}(1)$, e_{n+2} is the radical direction on $S^{n-m}(1)$. Let $\{\omega_1, \omega_2, \cdots, \omega_{n+2}\}$ be the dual frame. The indexes i, j, k, l, $\cdots$ run over $1, \cdots, m$; α, β, $\cdots$ run over $m+1, \cdots, n$; A, B, $\cdots$ run over $1, \cdots, n$. $M=\boldsymbol{X}(S^m(1)\times S^{n-m}(1))$. At a fixed point $\boldsymbol{X}(P)$ of M, where $P\in S^m(1)\times S^{n-m}(1)$, we choose the normal coordinate. Then

$$\boldsymbol{r}_i(P)=e_i(P),\ \boldsymbol{r}_\alpha(P)=0,$$
$$\boldsymbol{\rho}_i(P)=0,\ \boldsymbol{\rho}_\alpha(P)=e_\alpha(P), \tag{1.3}$$

where the subscripts i, α express the covariant derivatives with respect to directions e_i, e_α, respectively. At point $\boldsymbol{X}(P)$,

$$\boldsymbol{X}_i=e^u(1+e^{2u})^{-3/2}[u_i\boldsymbol{r}+(1+e^{2u})\boldsymbol{r}_i-e^u u_i\boldsymbol{\rho}]. \tag{1.4}$$

By virtue of $u_\alpha=0$, we can see at point $\boldsymbol{X}(P)$,

$$\boldsymbol{X}_\alpha=(1+e^{2u})^{-1/2}\boldsymbol{\rho}_\alpha. \tag{1.5}$$

The metric tensor of M is

$$g_{ij}=\langle\boldsymbol{X}_i,\ \boldsymbol{X}_j\rangle=e^{2u}(1+e^{2u})^{-2}[u_iu_j+(1+e^{2u})\delta_{ij}],$$
$$g_{i\alpha}=g_{\alpha i}=\langle\boldsymbol{X}_\alpha,\ \boldsymbol{X}_i\rangle=0,$$
$$g_{\alpha\beta}=\langle\boldsymbol{X}_\alpha,\ \boldsymbol{X}_\beta\rangle=(1+e^{2u})^{-1}\delta_{\alpha\beta}. \tag{1.6}$$

At point $\boldsymbol{X}(P)$, the inverse metric tensor of M satisfies

$$g^{ij}=e^{-2u}(1+e^{2u})[\delta_{ij}-(1+e^{2u}+\sum_k u_k^2)^{-1}u_iu_j],$$
$$g^{i\alpha}=g^{\alpha i}=0,$$
$$g^{\alpha\beta}=(1+e^{2u})\delta_{\alpha\beta}. \tag{1.7}$$

The normal vector $\boldsymbol{n}$ of M in S^{n+1} satisfies

$$\langle\boldsymbol{n},\ \boldsymbol{X}\rangle=0,\ \langle\boldsymbol{n},\ \boldsymbol{X}_i\rangle=0,\ \langle\boldsymbol{n},\ \boldsymbol{X}_\alpha\rangle=0. \tag{1.8}$$

By a calculation, at point $\boldsymbol{X}(P)$, We can see

$$\boldsymbol{n}=(1+e^{2u}+\sum_i u_i^2)^{-1/2}(-\boldsymbol{r}+e^u\boldsymbol{\rho}+\sum_j u_j\boldsymbol{r}_j). \tag{1.9}$$

We calculate the mean curvature H of M at point $\boldsymbol{X}(P)$. By the definition and Weingarten formula,

$$\begin{aligned}nH&=-\sum_{A,B}g^{AB}\langle d\boldsymbol{n}(e_A),\ \boldsymbol{X}_B\rangle\\&=-\sum_{i,j}g^{ij}\langle d\boldsymbol{n}(e_i),\ \boldsymbol{X}_j\rangle-\sum_{\alpha,\beta}g^{\alpha\beta}\langle d\boldsymbol{n}(e_\beta),\ \boldsymbol{X}_\alpha\rangle.\end{aligned} \tag{1.10}$$

By a straight calculation, we obtain at point $X(P)$

$$dn=-\frac{1}{2}d\ln(1+e^{2u}+\sum_i u_i^2)n+(1+e^{2u}+\sum_i u_i^2)^{-1/2}\{e^u\sum_j u_j\omega_j\rho$$
$$+\sum_k[\sum_l u_{kl}\omega_l-\omega_k]r_k-\sum_j u_j\omega_j r+e^u\sum_\beta \omega_\beta\rho_\beta\}. \tag{1.11}$$

So, we have

$$\langle dn(e_i),\ X_j\rangle=e^u[(1+e^{2u})(1+e^{2u}+\sum_k u_k^2)]^{-1/2}\times(u_{ij}-u_iu_j-\delta_{ij}),$$
$$\langle dn(e_\beta),\ X_\alpha\rangle=e^u[(1+e^{2u})(1+e^{2u}+\sum_k u_k^2)]^{-1/2}\delta_{\alpha\beta}. \tag{1.12}$$

Inserting (1.7) and (1.12) into (1.10), we obtain

$$(1+e^{2u}+\sum_i u_i^2)\sum_j u_{jj}-\sum_{i,j}u_iu_ju_{ij}$$
$$=(1-n+m+me^{-2u})e^{2u}\sum_i u_i^2+(2m-n)e^{2u}+m-(n-m)e^{4u}$$
$$-ne^u(1+e^{2u})^{-1/2}(1+e^{2u}+\sum_i u_i^2)^{3/2}H^2\left(\frac{e^u}{\sqrt{1+e^{2u}}}r+\frac{1}{\sqrt{1+e^{2u}}}\rho\right).$$

In this paper, function $H\left(\frac{e^u}{\sqrt{1+e^{2u}}}r+\frac{1}{\sqrt{1+e^{2u}}}\rho\right)$ is assumed to be independent of $S^{n-m}(1)$; in other words, we can write

$$H\left(\frac{e^u}{\sqrt{1+e^{2u}}}r+\frac{1}{\sqrt{1+e^{2u}}}\rho\right)=H\left(\frac{e^u}{\sqrt{1+e^{2u}}}r,\right) \tag{1.3}$$

and the right side of (1.13) can be written as $B(x, u, \nabla u)$, where x is a point of $S^m(1)$. Equation (1.13) is a quasilinear elliptic equation, when function $H(X)\in C^{k,\alpha}(N)$, where $N=\{X\in R^{m+1}|0<|X|<1\}$ and integer $k\geqslant1$, $0<\alpha<1$. Of course $X=Sr$, $0<s<1$.

It is well known that if we could prove there exists a constant C_1 independent of t such that $\|u\|_{C^1(S^m(1))}\leqslant C_1$ holds for all $(u, t)\in C^{k,\alpha}(S^m(1))\times[0,1]$ satisfying

$$(1+e^{2u}+\sum_i u_i^2)\sum_j u_{jj}-\sum_{i,j}u_iu_ju_{ij}=tB(x, u, \nabla u)+(1-t)u, \tag{1.14}$$

then there is a solution u satisfying (1.13). It is enough to prove the above statement for $t\in(0, 1]$.

§ 2. Existence Theorem

In this section, u denotes a solution of (1.14).

Lemma 1. *For the given function $H(X)$ in $C^{k,\alpha}(N)$, if there are two constants S_1, S_2, where*

$$1>S_1\geqslant\sqrt{\frac{1}{2}}\geqslant S_2>0,$$

such that

$$H(S\mathfrak{r})<\frac{m-nS^2}{nS\sqrt{1-S^2}}$$

when $1>S>S_1$, *and*

$$H(S\mathfrak{r})>\frac{m-nS^2}{nS\sqrt{1-S^2}}$$

when $S_2>S>0$, *then*

$$\ln\frac{S_2}{\sqrt{1-S_2^2}}\leqslant u\leqslant\ln\frac{S_1}{\sqrt{1-S_1^2}}.$$

Proof Because $S^m(1)$ is compact, there exists a point $x_1\in S^m(1)$ such that

$$u(x_1)=\max_{x\in S^m(1)}(x). \tag{2.1}$$

Obviously

$$u_i(x_1)=0,\ \sum_{i,j}[(1+e^{2u}+\sum_l u_l^2)\delta_{ij}-u_iu_j)]u_{ij}(x_1)\leqslant 0. \tag{2.2}$$

If $u(x_1)>\ln\frac{S_1}{\sqrt{1-S_1^2}}\geqslant 0$, then $\frac{e^u}{\sqrt{1+e^{2u}}}(x_1)>S_1$. Using the hypothesis of Lemma 1, We can see

$$\begin{aligned}&[tB(x,u,\nabla u)+(1-t)u](x_1)\\&=t[(2m-n)e^{2u}+m-(n-m)e^{4u}-ne^u(1+e^{2u})H\Big(\frac{e^u}{\sqrt{1+e^{2u}}}\mathfrak{r}\Big)](x_1)\\&+(1-t)u(x_1)>(1-t)u(x_1)\geqslant 0.\end{aligned} \tag{2.3}$$

It is impossible because of equation (1.14) for $t\in(0,1]$. So, $\forall x\in S^m(1)$,

$$u(x)\leqslant\ln\frac{S_1}{\sqrt{1-S_1^2}}. \tag{2.4}$$

Similar to the above discussion, We have

$$u(x_2)=\min_{x\in S^m(1)}u(x)\geqslant\ln\frac{S_2}{\sqrt{1-S_2^2}}. \tag{2.5}$$

Secondly, We shall estimate $|\nabla u|^2=\sum_i u_i^2$.

Set function ([1] or [2])

$$\psi=e^{2u}\ln(|\nabla u|^2+1). \tag{2.6}$$

By successive differentiation (cf. [2])

$$\psi_i=2e^{2u}[u_i\ln(|\nabla u|^2+1)+(|\nabla u|^2+1)^{-1}\sum_k u_ku_{ki}], \tag{2.7}$$

$$\begin{aligned}\psi_{ij}=&4e^{2u}u_j[u_i\ln(|\nabla u|^2+1)+(|\nabla u|^2+1)^{-1}\sum_k u_ku_{ki}]\\&+2e^{2u}[u_{ij}\ln(|\nabla u|^2+1)+2(|\nabla u|^2+1)^{-1}u_i\sum_k u_ku_{kj}\\&+(|\nabla u|^2+1)^{-1}(\sum_k u_{kj}u_{ki}+\sum_k u_ku_{kij})\\&-2(|\nabla u|^2+1)^{-2}\sum_k u_ku_{ki}\sum_l u_lu_{lj}].\end{aligned} \tag{2.8}$$

Thus at the point $x_0\in S^m(1)$, where ψ attains its maximum, we have

$$\psi_i(x_0)=0,\ \sum_{i,j}[(1+e^{2u}+|\nabla u|^2)\delta_{ij}-u_iu_j]\psi_{ij}(x_0)\leqslant 0. \tag{2.9}$$

By (2.7), at point x_0, we can see

$$\sum_k u_k u_{ki} = -u_i(|\nabla u|^2+1)\ln(|\nabla u|^2+1). \tag{2.10}$$

From (2.10), we have

$$\sum_{k,i} u_k u_{ki} u_i = -|\nabla u|^2(|\nabla u|^2+1)\ln(|\nabla u|^2+1),$$

$$\sum_{i,j,k} u_k u_{ki} u_j u_{ji} = |\nabla u|^2(|\nabla u|^2+1)^2[\ln(|\nabla u|^2+1)]^2. \tag{2.11}$$

Using (2.8), (2.11) and the second formula of (2.9), we obtain

$$\begin{aligned}0 \geqslant &(1+e^{2u}+|\nabla u|^2)\ln(|\nabla u|^2+1)\sum_j u_{jj} \\ &+(|\nabla u|^2+1)^{-1}(1+e^{2u}+|\nabla u|^2)\sum_{i,j} u_{ij}^2 \\ &+(|\nabla u|^2+1)^{-1}[(1+e^{2u}+|\nabla u|^2)\times\sum_{i,j} u_i u_{ijj} - \sum_{i,j,k} u_i u_j u_k u_{kij}] \\ &-2(1+e^{2u})|\nabla u|^2\ln(|\nabla u|^2+1)[\ln(\nabla u|^2+1)+1].\end{aligned} \tag{2.12}$$

Not loss of generalization, we assume $|\nabla u|(x_0)>1$. For $|\nabla u|(x_0)\leqslant 1$, we have $\forall x \in S^m(1)$,

$$|\nabla u|^2(x) \leqslant e^{C_2}-1, \tag{2.13}$$

where

$$C_2 = \frac{S_1^2(1-S_2^2)}{S_2^2(1-S_1^2)}\ln 2.$$

Differentiating (1.14) and using (2.11), we can see

$$\begin{aligned}&(1+e^{2u}+|\nabla u|^2)\sum_{i,j} u_{jji}u_i - \sum_{i,j,k} u_i u_j u_k u_{ijk} \\ &= t\sum_k [B(x, u, \nabla u)]_k u_k + 2|\nabla u|^2(|\nabla u|^2+1)^2[\ln(|\nabla u|^2+1)]^2 \\ &\quad +2|\nabla u|^2[(|\nabla u|^2+1)\ln(|\nabla u|^2+1)-e^{2u}]\sum_j u_{jj} \\ &\quad +(1-t)|\nabla u|^2.\end{aligned} \tag{2.14}$$

Using the Ricci formula of $S^m(1)$, we can see

$$\sum_{i,j} u_i u_{ijj} = \sum_{i,j} u_i u_{jji} + (m-1)|\nabla u|^2. \tag{2.15}$$

Utilizing (1.14) and (2.11), we have

$$\begin{aligned}\sum_j u_{jj} = &(1+e^{2u}+|\nabla u|^2)^{-1}[tB(x, u, \nabla u)+(1-t)u \\ &-|\nabla u|^2(|\nabla u|^2+1)\ln(|\nabla u|^2+1)].\end{aligned} \tag{2.16}$$

By the Cauchy inequality, we can see

$$\begin{aligned}\sum_{i,j} u_{ij}^2 &= |\nabla u|^{-2}\sum_k u_k^2 \sum_{i,j} u_{ij}^2 \geqslant |\nabla u|^{-2}\sum_i\Big(\sum_k u_k u_{ki}\Big)^2 \\ &= (|\nabla u|^2+1)^2[\ln(|\nabla u|^2+1)]^2.\end{aligned} \tag{2.17}$$

By use of (2.14), (2.15), (2.16) and (2.17), (2.12) can be reduced to the following form

$$\begin{aligned}0 \geqslant &(1+e^{2u})|\nabla u|^2[\ln(|\nabla u|^2+1)]^2 + 3tB(x, u, \nabla u)\ln(|\nabla u|^2+1) \\ &+(|\nabla u|^2+1)^{-1}t\sum_k[B(x, u, \nabla u)]_k u_k - C_3|\nabla u|^2\ln(|\nabla u|^2+1),\end{aligned} \tag{2.18}$$

where all of the items are computed at point x_0.

From the expression of $B(x, u, \nabla u)$, we can see

$$3tB(x, u, \nabla u)\ln(|\nabla u|^2+1)$$

$$\geqslant -3tne^u(1+e^{2u})^{-1/2}(1+e^{2u}+|\nabla u|^2)^{3/2}H\ln(|\nabla u|^2+1)-C_4|\nabla u|^2\ln(|\nabla u|^2+1), \tag{2.19}$$

$$\begin{aligned}(|\nabla u|^2+1)^{-1}t\sum_k[B(x,\ u,\ \nabla u)]_k u_k \\ \geqslant -tne^u(1+e^{2u})^{-1/2}(1+e^{2u}+|\nabla u|^2)^{3/2}[(1+e^{2u})^{-1}H-3H\ln(|\nabla u|^2+1) \\ +(|\nabla u|^2+1)^{-1}\sum_k H_k u_k]-C_5|\nabla u|^2\ln(|\nabla u|^2+1),\end{aligned} \tag{2.20}$$

where $H=H\left(\frac{e^u}{\sqrt{1+e^{2u}}}\boldsymbol{r}\right)$. Set $P=\{\boldsymbol{X}=S\boldsymbol{r}\in N\,|\,S_2\leqslant S\leqslant S_1\}$. C_3, C_4, C_5 are positive constants. They rely on S_1, S_2 and $\max\limits_{\boldsymbol{X}\in P}|H(\boldsymbol{X})|$, and they are independent of t.

By a straight calculation, we have

$$-\sum_k H_k u_k \geqslant -\frac{\partial}{\partial S}H(S\boldsymbol{r})\Big|_{S=\frac{e^u}{\sqrt{1+e^{2u}}}} e^u(1+e^{2u})^{-3/2}|\nabla u|^2 - C_6|\nabla u|. \tag{2.21}$$

Inserting (2.19), (2.20) and (2.21) into (2.18), we can see

$$\begin{aligned}0\geqslant &(1+e^{2u})|\nabla u|^2[\ln(|\nabla u|^2+1)]^2 \\ &-tne^u(1+e^{2u})^{-3/2}(1+e^{2u}+|\nabla u|^2)^{3/2}\left[H+\frac{\partial}{\partial S}H(S\boldsymbol{r})\Big|_{S=\frac{e^u}{\sqrt{1+e^{2u}}}}\frac{e^u}{\sqrt{1+e^{2u}}}\right] \\ &-C_7|\nabla u|^2\ln(|\nabla u|^2+1),\end{aligned} \tag{2.22}$$

where C_6, C_7 are positive constants, relying only on S_1, S_2, $\max\limits_{\boldsymbol{X}\in P}|H(\boldsymbol{X})|$ and $\max\limits_{\boldsymbol{X}\in P}|\nabla H(\boldsymbol{X})|$, and independent of t.

Now, we establish the following theorem.

Theorem. *Set*

$$N=\{\boldsymbol{X}=S\boldsymbol{r}\in R^{m+1}\,|\,0<S<1\},$$

where $\boldsymbol{r}$ is the position vector field of unit sphere $S^m(1)\subset R^{m+1}$ $(2\leqslant m\leqslant n-1)$. Suppose that the function $H(\boldsymbol{X})\in C^{k,\alpha}(N)$ $(K\geqslant 1$, an integer and $0<\alpha<1)$ satisfies the following two conditions:

(1) There are two constants S_1, S_2, where $1>S_1\geqslant\sqrt{\frac{1}{2}}\geqslant S_2>0$, such that

$$H(S\boldsymbol{r})<\frac{m-nS^2}{nS\sqrt{1-S^2}}$$

when $1>S>S_1$, and

$$H(S\boldsymbol{r})>\frac{m-nS^2}{nS\sqrt{1-S^2}}$$

when $S_2>S>0$.

(2) Set $P=\{S\boldsymbol{r}\in N\,|\,S_2\leqslant S\leqslant S_1\}$, $\frac{\partial}{\partial S}[SH(S\boldsymbol{r})]\leqslant 0$, in P.

Then, there exists a compact hypersurface $M_1\times M_2$ in the $(n+1)$-dimensional unit sphere S^{n+1} whose mean curvature is given by $H(\boldsymbol{X})$ and depends only on M_1, where M_1 is homeomorphic to $S^m(1)$, M_2 is homeomorphic to $S^{n-m}(1)$.

Proof By the condition (1) in Theorem, all of the above arguments are valid, and

$$S_2 \leqslant \frac{e^u}{\sqrt{1+e^{2u}}}(x_0) \leqslant S_1. \tag{2.23}$$

Using condition (2), we can see

$$-\left[H(S\boldsymbol{r}) + S\frac{\partial}{\partial S}H(S\boldsymbol{r})\right]\Bigg|_{s=\frac{e^u}{\sqrt{1+e^{2u}}}(x_0)} \geqslant 0. \tag{2.24}$$

By (2.22) and (2.24), at once we have

$$|\nabla u|^2(x_0) \leqslant C_8, \tag{2.25}$$

where C_8 is a positive constant, and independent of t. Then $\forall x \in S^m(1)$,

$$|\nabla u|^2(x) \leqslant e^{C_9} - 1, \tag{2.26}$$

where $C_9 = \frac{S_1^2(1-S_2^2)}{S_2^2(1-S_1^2)}\ln(C_8+1)$.

Remark. There are many functoins satisfying the two conditions in Theorem. For example

$$H(S\boldsymbol{r}) = \frac{m-nS^2}{nS^{k+3}(1-S^2)^{(k+1)/2}} + f(\boldsymbol{r}),$$

where constants

$$m > \frac{2}{3}n, \quad 0 < k \leqslant \frac{3m-2n}{3n-2m},$$

$f(\boldsymbol{r})$ is a smooth function on $S^m(1)$, and

$$f(\boldsymbol{r}) \leqslant \frac{1}{n}(k+1)(n-m)2^{k+3}.$$

References

[1] Huang Xuanguo, Generalized torus in R^3 with prescribed mean curvature, *Chin. Ann. of Math.*, **9B:1** (1988), 123-133.

[2] Huang Xuanguo, Generalized torus in S^3 with prescribed mean curvature, *Chin, Ann. of. Math.*, **9A:2** (1988), 119—129.