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ON THE APPROXIMAMATION OF DIFFERENTIABLE FUNCTIONS BY EULER MEANS

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Abstract

Let $f\in C_{2\pi}^r$. Denote by $\mathscr{E}_n(f, x)$ the n-th Euler mean of $f(x)$. This paper gives the asymptotic representations of the deviation $\mathscr{E}_n(f, x)-f(x)$ and the quantity

$$\sup_{f\in W^rH_\omega}\max_x|\mathscr{E}_n(f, x)-f(x)|.$$

Additionally, some applications of these asymptotic representations are obtained.

§1. Introduction

Let $C_{2\pi}$ be the class of continuous functions of period 2π. For $f\in C_{2\pi}$, denote by

$$\mathscr{E}_n(f, x)=2^{-n}\sum_{k=0}^{n}C_n^k s_k(f, x)$$

the n-th Euler mean of $f(x)$, where

$$C_n^k=\frac{n!}{(n-k)!k!},$$

$s_k(f, x)$ is the k-th partial sum of Fourier series of $f(x)$. Let $\omega(f, \delta)$ be the modulus of continuity of $f(x)$. For given modulus of continuity $\omega(\delta)$, define the subclasses of $C_{2\pi}$ as follows:

$$H_\omega=\{f\colon f\in C_{2\pi} \text{ and } \omega(f, \delta)\leqslant\omega(\delta),\ 0\leqslant\delta\leqslant\pi\},$$

$$W^rH_\omega=\{f\colon f\in C_{2\pi},\ f^{(r)}\in H_\omega\},$$

where $r=0, 1, 2,\cdots, C_{2\pi}^r$ is the subset of $C_{2\pi}$——the class of all functions, which have r-th continuous derivatives.

On the approximation of functions by Euler means, recently there are some new results. In 1983, C. K. Chui and A. S. B. Holland[1] proved that, if $f\in \mathrm{Lip}\,\alpha$ $(0<\alpha<1)$ and it is valid uniformly on x that

$$\int_{\pi}^{\pi}\frac{\left|\varphi_x\left(t+\frac{2\pi}{n}\right)-\varphi_x(t)\right|}{t}\,dt\leqslant Mn^{-\alpha},$$

then

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$$\|\mathscr{E}_n(f)-f\|=O(n^{-\alpha}),$$

where $\|\cdot\|$ is the norm in the space $C_{2\pi}$, $\|\cdot\|=\max|\cdot|$, and

$$\varphi_x(t)=\frac{1}{2}\{f(x+t)-2f(x)+f(x-t)\}.$$

Our work[2] also got more general result by another different method. Recently under the enlightenment of S. M. Nikolski[4], Efimov[3] established the following theorems:

Theorem A. *If $f\in C_{2\pi}$, then*

$$\mathscr{E}_n(f, x)-f(x)=\frac{1}{2\pi^2}\sum_{m=0}^{\sqrt{n}}\frac{n}{m+1}\int_{\frac{4m+1}{2n}\pi}^{\frac{4m+5}{2n}\pi}(f(x+2z)-2f(x)+f(x-2z))\sin zdz$$
$$+O\left(\omega\left(f,\frac{1}{n}\right)\right). \tag{1.1}$$

Theorem B. *Let $\omega(\delta)$ be a concave modulus of continuity. Then*

$$\sup_{f\in H_\omega}\|\mathscr{E}_n(f)-f\|=\frac{\ln n}{\pi^2}\int_0^{\frac{\pi}{2}}\omega\left(\frac{4z}{n}\right)\sin zdz+O\left(\omega\left(\frac{1}{n}\right)\right). \tag{1.2}$$

It is well-known that, on the approximation to differentiable functions by the partial sum of Fourier series, A. V. Efimov came to a famous conclusion (cf. [5]):

$$\sup_{f\in W^rH_\omega}\|s_n(f)-f\|=\frac{2\ln n}{(n+1)^r\pi^2}\int_0^{\frac{\pi}{2}}\omega\left(\frac{2z}{n}\right)\sin zdz+O\left(\frac{1}{(n+1)^r}\,\omega\left(\frac{1}{n+1}\right)\right). \tag{1.3}$$

Therefore, it is natural to ask: If $f\in C_{2\pi}^r$, what asymptotic representations have the deviation $\mathscr{E}_n(f, x)-f(x)$ and the quantity $\sup_{f\to H^rW_\omega}\|\mathscr{E}_n(f)-f\|$? The purpose of this paper is to answer this question, the conclusion is that

$$\mathscr{E}_n(f, x)-f(x)=\frac{2^r}{n^r\pi^2}\sum_{k=1}^{[\sqrt{n}]}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{2}}\left(f^{(r)}\left(2+2t_k+\frac{2t}{n}\right)-f^{(r)}\left(x+2t_k-\frac{2t}{n}\right)\right.$$
$$\left.+f^{(r)}\left(x-2t_k'-\frac{2t}{n}\right)-f^{(r)}\left(x-2t_k'+\frac{2t}{n}\right)\right)\sin tdt$$
$$+O\left(\frac{1}{n^r}\,\omega\left(f^{(r)},\frac{1}{n}\right)\right), \tag{1.4}$$

where

$$t_k=\frac{k\pi}{n}-\frac{r'\pi}{2n},\ t_k'=\frac{k\pi}{n}+\frac{r'\pi}{2n},\ r'=4\left\{\frac{r}{4}\right\}.$$

In addition

$$\sup_{f\in W^rH_\omega}\|\mathscr{E}_n(f)-f\|=\frac{2^r\ln n}{n^r\pi^2}\int_0^{\frac{\pi}{2}}\omega\left(\frac{4z}{n}\right)\sin zdz+O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right). \tag{1.5}$$

In § 2, several lemmas will be established. In § 3, (1.4) and (1.5) will be proved. § 4 will give some applications of (1.4) and (1.5), and raise some open problems.

§ 2. Lemmas

To prove (1.4), we need some lemmas.

Lemma 1. *For natural numers n and s, we have*

$$2^{-n}\sum_{k=0}^{n}(k+1)^{-s}\leqslant 2^{s}S!\,(n+1)^{-s}.$$

Lemma 2. *For natural numbers n and $k=0, 1, \cdots, n$, we have*

$$C_n^k\frac{1}{(k+1)^r}=C_{n+r}^{k+r}\frac{1}{(n+1)^r}+O\left(C_n^k\frac{1}{(n+1)(k+1)^r}\right).$$

The proofs of Lemmas 1 and 2 are not difficult, we omit them here.

Lemma 3. *Let $f\in C_{2\pi}^r$. Then it is valil uniformly on x and $k(0\leqslant k\leqslant n)$ that*

$$s_k(f,x)-f(x)=\frac{1}{(k+1)^r\pi}\int_{\Delta_n}(f^{(r)}(x+t)-f^{(r)}(x))\frac{\sin\left(\frac{2k+1}{2}t+\frac{r\pi}{2}\right)}{2\sin\frac{t}{2}}dt$$

$$+O\left(\frac{n+1}{(k+1)^{r+1}}\omega\left(f^{(r)},\frac{1}{k+1}\right)\right),$$

where

$$\Delta_n=\left[-\pi,-\frac{4\pi}{n}\right]\cup\left[\frac{4\pi}{n},\pi\right].$$

Proof It is known that (cf. [6] or [7]), when$f\in C_{2\pi}^r$,

$$s_k(f,x)-f(x)=\frac{1}{(k+1)^r\pi}\int_{\Delta_k}(f^{(r)}(x-t)-f^{(r)}(x))\frac{\sin\left(\frac{2k+1}{2}t+\frac{r\pi}{2}\right)}{2\sin\frac{t}{2}}dt$$

$$+O\left(\frac{1}{(k+1)^r}\omega\left(f^{(r)},\frac{1}{k+1}\right)\right).$$

It is easy to see, for $0\leqslant k<n$, that

$$\int_{\Delta_n-\Delta_k}(f^{(r)}(x+t)-f^{(r)}(x))\frac{\sin\left(\frac{2k+1}{2}t+\frac{r\pi}{2}\right)}{2\sin\frac{t}{2}}dt=O\left(\omega f^{(r)},\frac{1}{k+1}\int_{\frac{4\pi}{n}}^{\frac{4\pi}{k+1}}\frac{dt}{t}\right)$$

$$=O\left(\omega\left(f^{(r)},\frac{1}{k+1}\right)\ln\frac{n+1}{k+1}\right).$$

Hence Lemma 3 is established.

Lemma 4. *It is valid uniformly on t that*

$$2^{-n}\sum_{k=0}^{n}C_{n+r}^{k+r}\sin\left(\frac{2k+1}{2}t+\frac{r\pi}{2}\right)=2^r\cos^{n+r}\frac{t}{2}\sin\frac{(n+1-r)t+r\pi}{2}+O\left(\frac{1}{n}\right).$$

Proof Since

$$2^{-n}\sum_{j=0}^{r-1}C_{n+r}^{j}=O\left(\frac{1}{n}\right),$$

we have

$$2^{-n}\sum_{k=0}^{n} C_{n+r}^{k+r}\sin\left(\frac{2k+1}{2}t+\frac{r\pi}{2}\right)$$
$$=2^{-n}\sum_{k=0}^{n+r} C_{n+r}^{k}\sin\frac{(2k-2r+1)t+r\pi}{2}+O\left(\frac{1}{n}\right).$$

Noticing that

$$1+e^{it}=2\cos\frac{t}{2}e^{it/2}$$

we can easily get

$$\sum_{k=0}^{n+r} C_{n+r}^{k}\sin\frac{(2k-2r+1)t+r\pi}{2}=2^{n+r}\cos^{n+r}\frac{t}{2}\sin\frac{(n+1-r)t+r\pi}{2},$$

so Lemma 4 is proved.

§ 3. Main Results

Theorem 1. *Let $f\in C_{2\pi}^{r}$. Then it is valid uniformly on x that*

$$\mathscr{E}_n(f,x)-f(x)=\frac{2^r}{n^r\pi^2}\sum_{k=1}^{[\sqrt{n}]}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{2}}\Big(f^{(r)}\Big(x+2t_k+\frac{2t}{n}\Big)-f^{(r)}\Big(x+2t_k-\frac{2t}{n}\Big)$$
$$+f^{(r)}\Big(x-2t_k'-\frac{2t}{n}\Big)-f^{(r)}\Big(x-2t_k'+\frac{2t}{n}\Big)\Big)\sin t\,dt$$
$$+O\left(\frac{1}{n^r}\omega\Big(f^{(r)},\frac{1}{n}\Big)\right),$$

where

$$t_k=\frac{k\pi}{n}-\frac{r'\pi}{2n},\quad t_k'=\frac{k\pi}{n}+\frac{r'\pi}{2n},\quad r'=4\left\{\frac{r}{4}\right\}.$$

Proof Without lossing generality, assume that $x=0$ and write

$$A_k=\frac{1}{\pi}\int_{\Delta_n}\left(f^{(r)}(t)-f^{(r)}(0)\right)\frac{\sin\dfrac{(2k+1)t+r\pi}{2}}{2\sin\dfrac{t}{2}}dt.$$

From the definition of $\mathscr{E}_n(f,0)$ and Lemma 3

$$\mathscr{E}_n(f,0)-f(0)=2^{-n}\sum_{k=0}^{n}C_n^k(s_k(f,0)-f(0))$$
$$=2^{-n}\sum_{k=0}^{n}C_n^k\frac{1}{(k+1)^r}A_k+O\left(2^{-n}\sum_{k=0}^{n}C_n^k\frac{n+1}{(k+1)^{r+1}}\omega\Big(f^{(r)},\frac{1}{k+1}\Big)\right).\qquad(3.1)$$

Applying Lemma 2, we can get

$$2^{-n}\sum_{k=0}^{n}C_n^k\frac{1}{(k+1)^r}A_k$$
$$=2^{-n}\sum_{k=0}^{n}C_{n+r}^{k+r}\frac{1}{n^r}A_k+O\left(2^{-n}\sum_{k=0}^{n}C_n^k\frac{1}{(k+1)^{r+1}}|A_k|\right),\qquad(3.2)$$

and applying Lemma 4

$$2^{-n}\sum_{k=0}^{n}C_{n+r}^{k+r}\frac{1}{n^r}A_k$$

$$=\frac{2^r}{n^r}\frac{1}{\pi}\int_{\Delta_n}\frac{f^{(r)}(t)-f^{(r)}(0)}{2\sin\frac{t}{2}}\cos^{n+r}\frac{t}{2}\sin\frac{(n+1-r)t+r\pi}{2}dt+O\left(\frac{1}{n^r}\omega\left(f^{(r)},\frac{1}{n}\right)\right),\tag{3.3}$$

where we use an obvious estimate

$$\int_{\Delta_n}\frac{|f^{(r)}(t)-f^{(r)}(0)|}{|t|}dt\leqslant 2\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(f^{(r)},t)}{t}dt=O\left(n\omega\left(f^{(r)},\frac{1}{n}\right)\right).$$

Since $A_k=O\left(n\omega\left(f^{(r)},\frac{1}{n}\right)\right)$, from (3.1), (3.2) and (3.3), it is follows that

$$\mathscr{E}_n(f,0)-f(0)=\frac{2^r}{n^r\pi}\int_{\Delta_n}\frac{f^{(r)}(t)-f^{(r)}(0)}{2\sin\frac{t}{2}}\cos^{n+r}\frac{t}{2}\sin\frac{(n+1-r)s+r\pi}{2}dt$$
$$+O\left(2^{-n}\sum_{k=0}^{n}C_n^k\frac{n+1}{(k+1)^{r+1}}\omega\left(f^{(r)},\frac{1}{k+1}\right)+\frac{1}{n^r}\omega\left(f^{(r)},\frac{1}{n}\right)\right).\tag{3.4}$$

However by the property of the modulus of continuity

$$\omega\left(f^{(r)},\frac{1}{k+1}\right)\leqslant 2\frac{n}{k+1}\omega\left(f^{(r)},\frac{1}{n}\right)\quad(k=0,1,\cdots,n-1)$$

it follows that

$$2^{-n}\sum_{k=0}^{n}C_n^k\frac{n+1}{(k+1)^{r+1}}\omega\left(f^{(r)},\frac{1}{k+1}\right)\leqslant 2^{-n+1}\sum_{k=0}^{u}C_n^k\frac{(n+1)^2}{(k+1)^{r+2}}\omega\left(f^{(r)},\frac{1}{n}\right)$$
$$=O\left(\frac{1}{n^r}\omega\left(f^{(r)},\frac{1}{n}\right)\right).$$

Hence (3.4) implies

$$\mathscr{E}_n(f,0)-f(0)=\frac{2^r}{n^r\pi}\int_{\Delta_n}\frac{f^{(r)}(t)-f^{(r)}(0)}{2\sin\frac{t}{2}}\cos^{n+r}\frac{t}{2}\sin\frac{(n+1-r)t+r\pi}{2}dt$$
$$+O\left(\frac{1}{n^r}\omega\left(f^{(r)},\frac{1}{n}\right)\right).\tag{3.5}$$

To deduce Theorem 1 from (3.5), write

$$I_1=\int_{\frac{4\pi}{n}}^{\pi}\frac{f^{(r)}(t)-f^{(r)}(0)}{2\sin\frac{t}{2}}\cos^{n+r}\frac{t}{2}\sin\frac{(n+1-r)t+r'\pi}{2}dt,$$

$$I_2=\int_{-\pi}^{-\frac{4\pi}{n}}\frac{f^{(r)}(t)-f(0)}{2\sin\frac{t}{2}}\cos^{n+r}\frac{t}{2}\sin\frac{(n+1-r)t+r'\pi}{2}dt,$$

where

$$r'=4\left\{\frac{r}{4}\right\}.$$

Therefore (3.5) becomes

$$\mathscr{E}_n(f,0)-f(0)=\frac{2^r}{n^r\pi}(I_1+I_2)+O\left(\frac{1}{n^r}\psi\left(f^{(r)},\frac{1}{n}\right)\right).\tag{3.6}$$

Because

$$\frac{\sin\frac{(n+1-r)t+r'\pi}{2}}{2\sin\frac{t}{2}}-\frac{\sin\frac{nt+r'\pi}{2}}{t}$$

$$=\sin\frac{nt+r'\pi}{2}\left(\frac{\cos\frac{(r-1)t}{2}}{2\sin\frac{t}{2}}-\frac{1}{t}\right)+\cos\frac{nt+r'\pi}{2}\ \frac{\sin\frac{(r-1)t}{2}}{2\sin\frac{t}{2}},$$

and from the monotone of the function $\cos^{n+r}\frac{t}{2}$ in $[0, \pi]$ and the differentiability of functions

$$\frac{\cos\frac{(r-1)t}{2}}{2\sin\frac{t}{2}}-\frac{1}{t}\quad\text{and}\quad\frac{\sin\frac{(r-1)t}{2}}{2\sin\frac{t}{2}}$$

in $(0, \pi]$, it follows by integration mean value theorem and common calculations that

$$I_1=\int_{\frac{4\pi}{n}}^{\pi}\frac{f^{(r)}(t)-f^{(r)}(0)}{t}\cos^{n+r}\frac{t}{2}\sin\frac{nt+r'\pi}{2}\,dt+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\right)$$

$$=\int_{\frac{2\pi}{n}}^{\frac{\pi}{2}}\frac{f^{(r)}(2t)-f^{(r)}(0)}{t}\cos^{n+r}t\sin\left(nt+\frac{r'\pi}{2}\right)dt+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\right).$$

So

$$I_1=\int_{\frac{2\pi}{n}}^{\frac{\pi}{2}}\frac{f^{(r)}\left(2u-\frac{r'\pi}{n}\right)-f^{(r)}(0)}{u-\frac{r'\pi}{2n}}\cos^{n+r}\left(u-\frac{r'\pi}{2n}\right)\sin nu\,du+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\right)$$

$$=\sum_{k=3}^{\left[\frac{n-1}{2}\right]}(-1)^k\int_{-\frac{\pi}{2n}}^{\frac{\pi}{2n}}\frac{f^{(r)}(2u+2t_k)-f^{(r)}(0)}{u+t_k}\cos^{n+r}(u+t_k)\sin nu\,du+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\right)$$

where

$$t_k=\frac{k\pi}{n}-\frac{r'\pi}{2n}.$$

Since in $\left[-\frac{\pi}{2n},\frac{\pi}{2n}\right]$

$$|\cos^{n+r}(u+t_k)-\cos^{n+r}t_k|\leqslant\cos^{n+r}\left(t_k-\frac{\pi}{2n}\right)-\cos^{n+r}\left(t_k+\frac{\pi}{2n}\right),$$

$$\frac{f^{(r)}(2u+2t_k)-f^{(r)}(0)}{v+t}=O\left(n\omega\left(f^{(r)},\frac{1}{n}\right)\right),$$

we have

$$I_1=\sum_{k=3}^{\left[\frac{n-1}{2}\right]}(-1)^k\cos^{n+r}t_k\int_{-\frac{\pi}{2n}}^{\frac{\pi}{2n}}\frac{f^{(r)}(2u+2t_k)-f^{(r)}(0)}{u+t_k}\sin nu\,du$$

$$+O(1)\left(\sum_{k=3}^{[\frac{n-1}{2}]}\left(\cos^{n+r}\left(t_k-\frac{\pi}{2n}\right)-\cos^{n+r}\left(t_k+\frac{\pi}{2n}\right)\right)\omega\left(f^{(r)},\frac{1}{n}\right)+\omega\left(f^{(r)},\frac{1}{n}\right)\right).$$

Hence

$$I_1=\sum_{k=3}^{[\frac{n-1}{2}]}\frac{(-1)^k}{n}\cos^{n+r}t_k\int_{-\pi/2}^{\pi/2}\frac{f^{(r)}\left(\frac{2u}{n}+2t_k\right)-f^{(r)}(0)}{\frac{u}{n}+t_k}\sin u du+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\right). \tag{3.7}$$

It is not difficult to verify

$$0<-\cos^{n+r}t_k\int_{-\pi/2}^{\pi/2}\frac{\sin u}{\frac{u}{n}+t_k}du<\frac{10n}{k^2}$$

and it decreases as k increases. By Abel transform,

$$\sum_{k=3}^{[\frac{n-1}{2}]}\frac{(-1)^k}{n}\cos^{n+r}t_k\int_{-\pi/2}^{\pi/2}\frac{f^{(r)}(2t_k)-f^{(r)}(0)}{\frac{u}{n}+t_k}\sin u du=O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\right),$$

so (3.7) implies

$$I_1=\sum_{k=3}^{[\frac{n-1}{2}]}\frac{(-1)^k}{n}\cos^{n+r}t_k\int_{-\pi/2}^{\pi/2}\frac{f^{(r)}\left(\frac{2u}{n}+2t_k\right)-f^{(r)}(2t_k)}{\frac{u}{n}+t_k}\sin u du+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\right).$$

Noticing that

$$\frac{1}{\frac{u}{n}+t_k}-\frac{n}{k\pi}=O\left(\frac{n}{k^2}\right)$$

we can get

$$I_1=\sum_{k=3}^{[\frac{n-1}{2}]}\frac{(-1)^k}{k\pi}\cos^{n+r}t_k\int_{-\pi/2}^{\pi/2}\left(f^{(r)}\left(\frac{2u}{n}+2t_k\right)-f^{(r)}(2t_k)\right)\sin u du+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\right). \tag{3.8}$$

By some direct calculations (cf. [3])

$$\sum_{k=[\sqrt{n}]}^{[\frac{n-1}{2}]}\frac{1}{k}\cos^{n+r}t_k+\sum_{k=3}^{[\sqrt{n}]}\frac{1}{k}|\cos t_k-1|=O(1),$$

so (3.8) implies

$$I_1=\sum_{k=3}^{[\sqrt{n}]}\frac{(-1)^k}{k\pi}\int_{-\pi/2}^{\pi/2}\left(f^{(r)}\left(\frac{2u}{n}+2t_k\right)-f^{(r)}(2t_k)\right)\sin u du+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\right). \tag{3.9}$$

Similarly

$$I_2=\sum_{k=3}^{[\sqrt{n}]}\frac{(-1)^k}{k\pi}\int_{-\pi/2}^{\pi/2}\left(f^{(r)}\left(-\frac{2u}{n}-2t_k'\right)-f^{(r)}(-2t_k')\right)\sin u du+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\right), \tag{3.10}$$

where

$$t'_k=\frac{k\pi}{n}+\frac{r'\pi}{2n}.$$

Combining (3.6), (3.9) and (3.10), we have

$$\mathscr{E}_n(f,\ 0)-f(0)=\frac{2^r}{n^r\pi^2}\sum_{k=3}^{[\sqrt{n}]}\frac{(-1)^k}{k}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(f^{(r)}\left(\frac{2u}{n}+2t_k\right)-f^{(r)}(2t_k)\right.$$
$$\left.+f^{(r)}\left(-\frac{2u}{n}-2t'_k\right)-f^{(r)}(-2t'_k)\sin u du+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\frac{1}{n^r}\right).\tag{3.11}$$

In other words

$$\mathscr{E}_n(f,\ 0)-f(0)=\frac{2^r}{n^r\pi^2}\sum_{k=1}^{[\sqrt{n}]}\frac{(-1)^k}{k}\int_0^{\pi/2}\left(f^{(r)}\left(\frac{2u}{n}+2t_k\right)-f^{(r)}\left(2t_k-\frac{2u}{n}\right)\right.$$
$$\left.+f^{(r)}\left(-\frac{2u}{n}-2t'_k\right)-f^{(r)}\left(\frac{2u}{n}-2t'_k\right)\right)\sin u du+O\left(\omega\left(f^{(r)},\frac{1}{n}\right)\frac{1}{n^r}\right).\tag{3.12}$$

Thus the proof of Theorem 1 is completed.

From (3.12) one can see that, if $f\in W^rH_\omega$, then

$$\|\mathscr{E}_n(f)-f\|\leqslant\frac{2^r\ln n}{n^r\pi^2}\int_0^{\pi/2}\omega\left(\frac{4u}{n}\right)\sin u du+O\left(\frac{1}{n^r}\ \omega\left(\frac{1}{n}\right)\right).$$

Using well-known methods, for the given modulus of continuity $\omega(\delta)$, we can construct easily a function $f_\omega\in W^rH_\omega$ such that

$$\|\mathscr{E}_n(f_\omega)-f_\omega\|\geqslant\frac{2^{r-1}\ln n}{n^r\pi^2}\int_0^{\pi/2}\omega\left(\frac{4u}{n}\right)\sin u du+O\left(\frac{1}{n^r}\ \omega\left(\frac{1}{n}\right)\right),$$

and if $\omega(\delta)$ is a concave modulus of continuity, then there exists $f^*_\omega\in W^rH_\omega$ such that

$$\|\mathscr{E}_n(f^*_\omega)-f^*_\omega\|=\frac{2^r\ln n}{n^r\pi^2}\int_0^{\pi/2}\omega\left(\frac{4u}{n}\right)\sin u du+O\left(\frac{1}{n^r}\ \omega\left(\frac{1}{n}\right)\right).$$

Thus follows

Theorem 2. *For any given integer $r\geqslant 0$ and modulus of continuity $\omega(\delta)$ we have*

$$\sup_{f\in W^rH_\omega}\|\mathscr{E}_n(f)-f\|=\frac{\theta_n 2^r\ln n}{n^r\pi^2}\int_0^{\frac{\pi}{2}}\omega\left(\frac{4u}{n}\right)\sin u du+O\left(\frac{1}{n^r}\ \omega\left(\frac{1}{n}\right)\right),$$

where $\theta_n\in\left[\frac{1}{2},1,\right]$ and $\theta_n=1$ if $\omega(\delta)$ is a concave function.

§ 4 Corollaries and Problems

In detail, the conclusion of Theorem 1 is

$$\mathscr{E}_n(f,\ x)-f(x)=\frac{2^r}{n^r\pi^2}\sum_{k=1}^{[\sqrt{n}]}\frac{(-1)^k}{k}\int_0^{\pi/2}\left\{f^{(r)}\left(x-\frac{r'\pi}{n}+\frac{2k\pi+2u}{n}\right)\right.$$

$$-f^{(r)}\left(x-\frac{r'\pi}{n}+\frac{2k\pi-2u}{n}\right)+f^{(r)}\left(x-\frac{r'\pi}{n}-\frac{2k\pi+2u}{n}\right)-f^{(r)}\left(x-\frac{r'\pi}{n}-\frac{2k\pi-2u}{n}\right)\Big\}$$

$$\sin u du+O\left(\frac{1}{n^r}\,\omega\left(f^{(r)},\frac{1}{n}\right)\right).$$

comparing it with the case $r=0$, we get

Corollary 1. *If $f\in C^r_{2\pi}$, then*

$$\mathscr{E}_n(f,x)-f(x)=\frac{2^r}{n^r}\left(\mathscr{E}_n^{(r)}\left(f,x-\frac{r'\pi}{n}\right)-f^{(r)}\left(x-\frac{r'\pi}{n}\right)\right)+O\left(\frac{1}{n^r}\,\omega\left(f^{(r)},\frac{1}{n}\right)\right).$$

Corollary 2. *For any given modulus of continuity $\omega(\delta)$*

$$\sup_{f\in W^rH_\omega}\|\mathscr{E}_n(f)-f\|=\frac{2^r}{n^r}\sup_{f\in H_\omega}\|\mathscr{E}_n(f)-f\|+O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right).$$

Corollary 3. *Let $f\in C^r_{2\pi}$. Then*

$$\|\mathscr{E}_n(f)-f\|=o\left(\frac{1}{n^r}\right)$$

if and only if

$$\|\mathscr{E}_n^{(r)}(f)-f^{(r)}\|=o(1).$$

We see that the discussion above is under the restriction that r is a natural number. Hence we raise the following

Problem 1. Are there any results similar to Theorems 1 and 2 for the derivatives in Weyl meaning?

We also know the (general definition of Euler means is

$$\mathscr{E}_{n,q}(f,x)=\frac{1}{(1+q)^n}\sum_{v=0}^{n}C_n^v q^{n-v}S_v(f,x),$$

where q is a given positive number. Using a well-known estimate

$$S_v(f,x)-f(x)=O\left(\frac{\ln(v+1)}{(v+1)^r}\,\omega\left(f^{(r)},\frac{1}{v+1}\right)\right)$$

for $f\in W^rH_\omega$ we have

$$\|\mathscr{E}_{n,q}(f)-f\|=O\left(\frac{\ln n}{n^r}\,\omega\left(\frac{1}{n}\right)\right).$$

Therefore we raise the following

Probem 2 For $q\geqslant 0$, what asymptotic representation has the quantity

$$\sup_{f\in W^rH_\omega}\|\mathscr{E}_{n,q}(f)-f\|?$$

Instead of W^rH_ω, considering the class of functions defined by Stechkin

$$W^r_\beta H_\omega=\left\{f\colon f(x)=\frac{a_0}{2}+\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi_+(x+t)\sum_{k=1}^{\infty}\frac{\cos\left(kt+\frac{\beta\pi}{2}\right)}{k^r}\,dt,\ \varphi_+\in H_\omega\right\}$$

where $r>0$, β is a given real number. By the same method we can establish the following theorem:

Theorem 3. *Let $r>0$, $f\in W^r_\beta H_\omega$. Then*

$$\mathscr{E}_n(f,x)-f(x)=\frac{2^r}{n^r}\left(\mathscr{E}_n\left(\varphi_+,x-\frac{\beta'\pi}{n}\right)-\varphi_+\left(x-\frac{\beta'\pi}{n}\right)\right)+O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right)$$

uniformly on β *and* x, *where* $\beta' = 4\left\{\frac{\beta}{4}\right\}$.

From Theorem 3 it follows

Corollary 4. *Let* $r>0$. *Then*

$$\sup_{f\in W^rH_\omega}\|f-\mathscr{E}_n(f)\| = \frac{2^r}{n^r}\sup_{f\in H_\omega}\|f-\mathscr{E}_n(f)\| + O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right).$$

In particular, if $\beta=\frac{\pi}{2}$, we get the theorem about the conjugate function.

Theorem 4. *Let* $r>0$, $f\in W^rH_\omega$. *Then*

$$\tilde{\mathscr{E}}_n(f, x)-\tilde{f}(x) = \frac{2^r}{n^r}\left(\mathscr{E}_n\left(f^{(r)}, x-\frac{r''\pi}{n}\right)-f^{(r)}\left(x-\frac{r''\pi}{n}\right)\right)+O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right),$$

where $\tilde{f}(x)$ *is the conjugate function of* $f(x)$,

$$r'' = 4\left\{\frac{r+1}{4}\right\}.$$

Therefore

$$\sup_{f\in W^rH_\omega}\|\tilde{\mathscr{E}}_n(f)-\tilde{f}\| = \frac{2^r}{n^r}\sup_{a\in H_\omega}\|\mathscr{E}_n(f)-f\| + O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right).$$

About the case $r=0$, we have

Theorem 5.

$$\tilde{\mathscr{E}}_n(f, x)-\tilde{f}_{\frac{\pi}{n}}(x) = \mathscr{E}_n\left(f, x-\frac{\pi}{n}\right)-f\left(x-\frac{\pi}{n}\right)+O\left(\omega\left(f, \frac{1}{n}\right)\right)$$

where

$$\tilde{f}_{\frac{\pi}{n}}(x) + -\frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi}\frac{f(x+t)-f(x-t)}{2\,\mathrm{tg}\,\frac{t}{2}}\,dt.$$

Hence

$$\sup_{f\in H_\omega}\|\tilde{\mathscr{E}}_n(f, x)-\tilde{f}_{\frac{\pi}{n}}(x)\| = \sup_{f\in H_\omega}\|\mathscr{E}_n(f)-f\| + O\left(\omega\left(\frac{1}{n}\right)\right).$$

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