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AN UPFER BOUND OF CLASS NUMBER OF CYCLOTOMIC FIELD $Q(\zeta_p)$

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Abstract

Let h_p be the class number of cyclotomic field $\mathbb{Q}(\zeta_p)$, where p is a prime number. Slavutski [4] proved that $h_p\leqslant 20\left(\frac{n}{12}\,p\right)^{(p-2)/2}$. The author improves it by proving $h_p\leqslant 10\left(\frac{\pi}{16}\,p\right)^{(p-2)/2}$.

Let p be an odd prime number, h_p and h_p^+ the class numbers of $\mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\zeta_p+\bar{\xi}_p)$ respectively ($\zeta_p=e^{2\pi i/p}$). It is well-known that $h_p|h_p$. There are several works[1,2,3] on upper-bound of $h_p^-=h_p/h_p^+$. The best result is given by Feng[2]:

$$h_p^-=2p\left(\frac{p}{4\pi^2}\right)^{(p-1)/4}\prod_{\chi(-1)=-1}L(1,\ \chi)\leqslant 2p\left(\frac{p-1}{8}\right)^{(p-1)/4}\prod_{\chi(-1)=-1}\frac{1}{|\chi(2)-2|}$$

$$=\begin{cases}2p\left(\dfrac{p-1}{8(2^{l/2}+1)^{4/l}}\right)^{(p-1)/4}, & \text{if } 2|l,\\ 2p\left(\dfrac{p-1}{8(2l-1)^{2/l}}\right)^{(p-1)/4}, & \text{if } 2\nmid l,\end{cases}\tag{1}$$

$$2p\left(\frac{p-1}{31.997158\cdots}\right)^{(p-1)/4},$$

where $\chi(-1)=-1$ means that χ passes through all odd characters of module p, $L(s,\ \chi)$ is the Dirichlet L-function, l is the order of $2\ (\mathrm{mod}\ p)$. In 1986, Slavuski[4] proved the following upper-bound of h_p:

$$h_p<20\left(\frac{\pi}{12}\,p\right)^{(p-2)/2}\ (\text{for } p>110).$$

In this paper we improve it by giving the following result.

Theorem. $h_p<10\left(\frac{\pi}{16}\,p\right)^{(p-2)/2}$.

Proof. For $p\leqslant 67$, the result can be verified by checking the table of h_p at the end of Washington's book[5]. From now on we suppose that $p\geqslant 71$. The start point is the class number formula (see, for example, [5], p. 37):

$$h_pR_p=2p(2\pi)^{-(p-1)/2}\sqrt{|d|}\prod_{\chi\neq\chi_0}|L(1,\ \chi)|,\tag{2}$$

where R_p is the regulator of $\mathbb{Q}(\zeta_p)$, $d=(-1)^{(p-1)/2}p^{p-2}$ is the discriminent of $\mathbb{Q}(\zeta_p)$,

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From formula (1) we know that

$$\prod_{\chi(-1)=-1}|L(1,\ \chi)|\leqslant\left(\frac{\pi^2(p-1)}{2p}\right)^{(p-1)/4}\prod_{\chi(-1)=-1}\frac{1}{|\chi(2)-2|}. \tag{3}$$

Now we estimate $\prod_{\chi(-1)=1,\chi\neq\chi_0}|L(1,\ \chi)|$. For non-trivial even character χ mod p wehavs (see, for example, [5], Theorem 4.9)

$$|L(1,\ \chi)|=\frac{1}{\sqrt{p}}\sum_{r=}^{p-1}\bar{\chi}(r)\ln|1-\zeta^r|,\ \zeta=\zeta_p.$$

Let $B=\sum_{r=1}^{p-1}\bar{\chi}(r)\ln|1-\zeta^r|=\sum_{r=1}^{p-1}\bar{\chi}(2r)\ln|1-\zeta^{2r}|$. Then

$$(2-\chi(2))B=\sum_{r=1}^{p-1}\bar{\chi}(r)\ln\left|\frac{1-\zeta^r}{1+\zeta^r}\right|=\sum_{r=1}^{p-1}\bar{\chi}(r)\ln\left|\operatorname{tg}\frac{\pi r}{p}\right|\overset{\text{def.}}{=\!=}\tilde{L}(1,\ \chi).$$

Thus

$$\prod_{\chi(-1)=1,\chi\neq\chi_0}|L(1,\ \chi)|=p^{-(p-3)/4}\prod_{(\chi-1)=1,\chi\neq\chi_0}\frac{1}{|\chi(2)-2|}\prod_{\chi(-1)=1,\chi\neq\chi_0}|\tilde{L}(1,\ \chi)|. \tag{4}$$

But

$$\sum_{\chi(-1)=1,\chi\neq\chi_0}|\tilde{L}(1,\ \bar{\chi})|^2=\sum_{\chi(-1)=1,\chi\neq\chi_0}\sum_{s,r=1}^{p-1}\bar{\chi}(r)\chi(s)\ln\left|\operatorname{tg}\frac{\pi r}{p}\right|\ln\left|\operatorname{tg}\frac{\pi s}{p}\right|$$

$$=\sum_{s,r}\ln\left|\operatorname{tg}\frac{\pi r}{q}\right|\ln\left|\operatorname{tg}\frac{\pi s}{p}\right|\sum_{\chi(-1)=1}\chi(s/r)-1)$$

$$=2\frac{p-1}{2}\sum_{r=1}^{p-1}\left(\ln\left|\operatorname{tg}\frac{\pi r}{p}\right|\right)^2-\sum_{r=1}^{p-1}\ln\left|\operatorname{tg}\frac{\pi r}{p}\right|\Big)^2,$$

where we use the orthogonal relation of charcters

$$\sum_{\chi(-1)=1}\chi(a)=\begin{cases}\frac{p-2}{2}, & \text{if } a\equiv\pm1(\operatorname{mod} p),\\ 0, & \text{otherwise.}\end{cases}$$

Since $\prod_{r=1}^{p-1}\left|\operatorname{tg}\frac{\pi r}{p}\right|=p$, we get

$$\sum_{(-1)=,1\chi\pm\chi_0}\tilde{L}(1,\ \chi)^2=2(p-1)S-\ln^2 p,$$

where $S=\sum_{r=1}^{q}\left(\ln\operatorname{tg}\frac{\pi r}{p}\right)^2$, $q=\frac{p-1}{2}$. Therefore

$$\Big(\prod_{\chi(1)=1,\chi\neq\chi_0}|\tilde{L}(1,\ \chi)|\Big)^{2/(p-3)}\leqslant\left(\frac{\sum_{\chi(-1)=1,\chi\neq\chi_0}|L(1,\ \chi)|^2}{(p-\chi)/2}\right)^{1/2}=\left(\frac{4(p-1)S-2\ln^2 p}{p-3}\right)^{1/2}.$$

Thus

$$\prod_{\chi(-1)=1,\chi\neq\chi_0}|\tilde{L}(1,\ \chi)|\leqslant\left(\frac{4(p-1)S-2\ln^2 p}{p-3}\right)^{(p-3)/4}. \tag{5}$$

We hare to estimate S, for doing that we need

Lemma (Euler-Maclaurin formula). *Suppose that $f(x)$ has derivative of secone order $f''(x)$ in $[a,\ b]$, $\rho(x)=\frac{1}{2}-\{x\}$ ($\{x\}$ is the fraction part of x),*

$$\sigma(x)=\int_0^x\rho(y)dy.$$

Then

$$\sum_{a\le n\le b} f(n)=\int_a^b f(x)dx+\rho(b)f(b)-\rho(a)f(a)+\sigma(a)f'(a)$$
$$-\sigma(b)f'(b)+\int_a^b \sigma(x)f''(x)dx.$$

Taking $f(x)=\left(\ln \operatorname{tg}\frac{\pi x}{p}\right)^2=\left(\ln \operatorname{ctg}\frac{\pi x}{p}\right)^2$, $a=\frac{1}{2}$, $b=q=\frac{p-1}{2}$, in above lemma we get

$$S=\int_{1/2}^{q}\left(\ln \operatorname{tg}\frac{\pi x}{p}\right)^2 dx+\frac{1}{2}\left(\ln \operatorname{tg}\frac{\pi q}{p}\right)^2+\frac{1}{8}\cdot 2\frac{\sec\frac{2\pi}{2p}}{\operatorname{tg}\frac{\pi}{2p}}\cdot\frac{\pi}{p}\ln \operatorname{tg}\frac{\pi}{2p}$$
$$+\int_{1/2}^{q}\sigma(x)\left[\left(\ln \operatorname{tg}\frac{\pi x}{p}\right)^2\right]''dx$$
$$=\frac{p}{\pi}\int_0^{\pi/2}(\ln \operatorname{tg} y)^2 dy-\frac{p}{\pi}\left(\int_0^{\pi/4}+\int_{\pi q/p}^{\pi/2}\right)(\ln \operatorname{tg} y)^2 dy$$
$$+\frac{1}{2}\left(\ln \operatorname{tg}\frac{\pi}{2p}\right)^2+\frac{\pi}{2p}\frac{1}{\sin\frac{\pi}{p}}\ln \operatorname{tg}\frac{\pi}{2p}+\int_{1/2}^{q}\sigma(x)\left[\left(\ln \operatorname{tg}\frac{\pi x}{p}\right)^2\right]''dx$$
$$=S_1+S_2+S_3+S_4+S_5.$$

Since

$$\int_0^{\pi/2}(\operatorname{tg}\varphi)^c d\varphi=\frac{\pi}{2\cos\frac{c\pi}{2}}\ (c \text{ is a variant}),$$

we get

$$\left[\int_0^{\pi/2}(\operatorname{tg}\varphi)^2 d\varphi\right]''=\left(\frac{\pi}{2\cos\frac{c\pi}{2}}\right)''.$$

Letting $c=0$, we have

$$\int_0^{\pi/2}(\ln \operatorname{tg}\varphi)^2 d\varphi=\frac{\pi^3}{8}.$$

So

$$S_1=\frac{p}{\pi}\int_0^{\pi/2}(\ln \operatorname{tg} y)^2 dy=\frac{p\pi^2}{8}, \tag{6}$$

$$S_2=-\frac{p}{\pi}\left(\int_0^{\pi/2p}+\int_{\pi q/p}^{\pi/2}\right)(\ln \operatorname{tg} y)^2 dy=-\frac{2p}{\pi}\int_0^{\pi/2p}(\ln \operatorname{tg} y)^2 dy$$
$$=-\frac{2p}{\pi}\left\{\left(\frac{\pi}{2p}\left(\ln \operatorname{tg}\frac{\pi}{2p}\right)^2+2\int_0^{\pi/2p}\frac{y\operatorname{ctg} y}{\cos^2 y}\ln \operatorname{ctg} y\, dy\right\}$$
$$=-\left(\ln \operatorname{tg}\frac{\pi}{2p}\right)^2-\frac{4p}{\pi}\int_0^{\pi/2p}\frac{y\operatorname{ctg} y}{\cos^2 y}\ln \operatorname{ctg} y\, dy. \tag{7}$$

Since $\ln \operatorname{ctg} y=-\ln \operatorname{tg} y>0$ for $0\le y\le\frac{\pi}{2p}$ and $y\operatorname{ctg} y<1$, we have

$$\int_0^{\pi/2p}\frac{y\operatorname{ctg} y}{\cos^2 y}\ln \operatorname{ctg} y\, dy\le\frac{1}{\cos^2\frac{\pi}{6}}\int_0^{\pi/2p}\ln \operatorname{ctg} y\, dy$$

$$=\frac{4}{3}\left[\frac{\pi}{2p}\ln\operatorname{ctg}\frac{\pi}{2p}+\int_0^{\pi/2p}\frac{y}{\sin y\cos y}dy=\frac{2\pi}{3p}\ln\operatorname{ctg}\frac{\pi}{2p}+\frac{4}{3}\int_0^{\pi/2p}\frac{y\operatorname{ctg}y}{\cos^2 y}dy\right.$$

$$\leqslant\frac{2\pi}{3p}\ln\operatorname{ctg}\frac{\pi}{2p}+\frac{4}{3}\frac{4}{3}\frac{\pi}{2p}.$$

From formula (7) we know that

$$S_2+S_3=-\frac{1}{2}\left(\ln\operatorname{ctg}\frac{\pi}{2p}\right)^2+R,$$

$$|R|\leqslant\frac{8}{3}\ln\operatorname{ctg}\frac{\pi}{2p}+\frac{32}{9}.$$

It is easy to verify by using $y\operatorname{ctg}y<1$ that

$$0<\ln\operatorname{ctg}\frac{\pi}{2p}<\ln p.$$

Noticing that $\left(\ln\operatorname{tg}\frac{\pi}{2p}\right)^2=(\ln p)^2+\theta_1\ln p,\ |\theta_1|<1,$ we have

$$S_2+S_3=-\frac{1}{2}\ln^2 p+\theta\ln^2 p+\theta',\quad |\theta|\leqslant\frac{8}{3}+\frac{1}{2}<\frac{10}{3},\quad |\theta'|\leqslant\frac{32}{9}. \tag{8}$$

Since $\sin x\geqslant\frac{\sqrt{3}}{2}x$ for $0\leqslant x\leqslant\frac{\pi}{6}$, we get

$$|S_4|=\left|\frac{\pi}{2p}\frac{1}{\sin\frac{\pi}{p}}\ln\operatorname{ctg}\frac{\pi}{2p}\right|\leqslant\frac{\pi}{2p}\frac{2}{\sqrt{3}}\frac{p}{\pi}\ln p=\frac{\sqrt{3}}{3}\ln p. \tag{9}$$

The last S_4 is

$$|S_5|=\left|\int_{1/2}^q\sigma(x)\left[\left(\ln\operatorname{ctg}\frac{\pi x}{p}\right)^2\right]''dx\right|$$

$$=\left|\int_{1/2}^q\sigma(x)\left(\frac{4\pi}{p}\frac{1}{\sin\frac{2\pi x}{p}}\ln\operatorname{ctg}\frac{\pi x}{p}\right)'dx\right|.$$

Since

$$0\leqslant\sigma(x)\leqslant\frac{1}{8}$$

and

$$\left(\frac{1}{\sin\frac{2\pi x}{p}}\ln\operatorname{ctg}\frac{\pi x}{p}\right)'<0,$$

we have

$$S_5|\leqslant\frac{\pi}{2p}\left|\int_{1/2}^q\left(\frac{1}{\sin\frac{2\pi x}{p}}\ln\operatorname{ctg}\frac{\pi x}{p}\right)'dx\right|$$

$$=\frac{\pi}{2p}\left|\frac{1}{\sin\frac{2\pi q}{p}}\ln\operatorname{ctg}\frac{\pi q}{p}-\frac{1}{\sin\frac{\pi}{p}}\ln\operatorname{ctg}\frac{\pi}{2p}\right|$$

$$=\frac{\pi}{p}\frac{1}{\sin\frac{\pi}{p}}\ln\operatorname{ctg}\frac{\pi}{2p}\leqslant\frac{2\sqrt{3}}{3}\ln p. \tag{10}$$

Putting formulas (6), (8), (9) and (10) together, we get

$$S=\frac{\pi^2}{8}p-\frac{1}{2}\ln^2 p+\theta\ln p+\theta', \ |\theta|\leqslant\frac{10}{3}+\sqrt{3}, \ |\theta'|\leqslant\frac{32}{9}. \tag{11}$$

From formulas (2), (3), (4), (5) and (11) we obtain

$$h_p R_p\leqslant 2p(2\pi)^{-(p-1)/2}p^{(p-2)/2}\left(\frac{\pi^2(p-1)}{2p}\right)^{(p-1)/4}p^{(p-3)/4}\prod_{\chi\neq\chi_0}\frac{1}{|\chi(2)-2|}$$

$$\left(\frac{4(d-1)\left[\frac{\pi^2}{8}p-\frac{1}{2}\ln^2 p+\theta\ln p+\theta^4\right]-2\ln^2 p}{p-3}\right)^{(p-3)/4}$$

$$=2p^{p/2}\left(\frac{p-1}{8p}\right)^{(p-1)/4}\left(\frac{\frac{\pi^2}{2}p(p-1)-2p\ln^2 p+4(p-1)\theta\ln p+4(p-1)\theta'}{p(p-3)}\right)^{(p-3)/4}$$

$$\cdot\prod_{\chi\neq\chi_0}\frac{1}{|\chi(2)-2|}. \tag{12}$$

Let l be the order of 2 mod p. Then

$$\prod_{\chi\neq\chi_0}\frac{1}{|\chi(2)-2|}=\prod_{\chi}\frac{1}{|\chi(2)-2|}=(2^l-1)^{-(p-1)/l}. \tag{13}$$

For R_p we have the following bound (cf. [6] or [2]).

Let R_k be the regulator of an algebraic field k. Then

$$\frac{2R_k}{W_k}\geqslant 0.04\exp(0.46r_1+0.1r_2),$$

where W_k is the number of roots of 1 in k, r_1 and $2r_2$ are numbers of real and imaginary imbedding of k into the complex field C. For cyclotomic field $k=\mathbb{Q}(\zeta_p)$, $W_k=2p, r_1=0, r_2=(p-1)/2$. Thus

$$R_p\geqslant 0.04p\exp(0.05(p-1)). \tag{14}$$

From (12), (13) and (14) we get

$$h_p\leqslant 50p^{(p-2)/2}\left(\frac{(p-1)e^{-0.2}}{8p(2^l-1)^{4/l}}\right)^{(p-1)/4}$$

$$\left(\frac{\frac{\pi^2}{2}p(p-1)-2p\ln^2 p+4(p-1)\theta\ln p+4(p-1)\theta'}{p(p-3)}\right)^{(p-3)/4}. \tag{15}$$

It is easy to see that if $p>127$, then $l\geqslant 11$ and

$$(2^l-1)^{2/l}\geqslant(2^{11}-1)^{2/11}=3.9996448\cdots.$$

From (15) we get

$$h_p\leqslant 10\left(\frac{\pi}{16}p\right)^{(p-2)/2} \quad (p>127).$$

For $71\leqslant p\leqslant 127$, we can verify it directly.

So

$$h_p\leqslant 10\left(\frac{\pi}{16}p\right)^{(p-2)/2} \quad \text{if } p\geqslant 71.$$

This completes the proof of the theorem.

References

[1] Carlitz, L., A generalization of Maillet's Determinat and a Bound for the First Factor of the Class Number, *Proc. Math. Amer. Soc.*, **12**(1961), 256—261.

[2] Feng Keqin, On the First Factor of the Class Number of a Cyclotomic Field, *Proc. Math. Amea. Soc.*, **84**(1982), 479—482.

[3] Metsänkyla, T., Class Number and μ-invariants of Cyclotomic Fields, *Proc. Math. Amer. Soc.*, **43** (1974), 199—200.

[4] Slavutski, I. Sh., Mean Value of L-functions and the class number of a Cyclotomic Field (in Russian), *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Steklov.* (LOMI), **154**(1986), 136—143.

[5] Washington, L. C., Introduction to Cyclotomic Fields, Springer-Verlag, New York Inc., 1982.

[6] Zimmert, R., Ideale Kleiner Norm in Idealklassen und eine Regulator abschätzung, *Invent. Math.*, **62**(1981), 367—390.