Chin. Ann. of Math.
12B:1(1991), 96—105.

THE CHUNG-SMIRNOV LAW FOR THE PRODUCT-LIMIT ESTIMATOR UNDER RANDOM CENSORSHIP

GU MINGGAO(顾鸣高)*

Abstract

The maximal deviation of the product limit estimator on the whole line is investigated. The analogue of Chung-Smirnov law of iterated logarithm is proved under very mild conditions on censoring. An improved convergence rate is found and shown to be best possible. The result is proved by an i.i.d. representation scheme of the product limit estimator on the whole line. Improved rates of convergence for the i. i. d. representation on compact set are also derived.

§1. Introduction and Main Results

Let X_1^0, X_2^0, $\cdots$ be a sequence of independent real random variables with common continuous distribution function F^0. Another sequence, independent of the $\{X_j^0\}$, Y_1, Y_2, $\cdots$ of independent random variables with common (left continuous) distribution function H censors on the right the preceding one, so that the observation available to us at the n^{th} stage consist of the pairs (X_j, δ_j), $1\leqslant j\leqslant n$, where $X_j=\min(X_j^0, Y_j)$ and δ_j is the indicator of the event $\{X_j=X_j^0\}$. The Kaplan-Meier[20] type product-limit estimator $\hat{F}_n^0$ of F^0 is defined by

$$1-\hat{F}_{n,K}^0(t)=\begin{cases}\prod\limits_{1\leqslant j\leqslant n:X_j\leqslant t}(1-1/m(X_j))^{\delta_j}, & t\leqslant X_{n:n},\\ 0, & t>X_{n:n},\end{cases}$$

where $X_{n:n}=\max(X_1, \cdots, X_n)$ and $m(s)=\sum\limits_{1\leqslant j\leqslant n}1(X_j\geqslant s)$. Variants of the Kaplan-Meier estimate are Prentice's[25] moment estimator defined by

$$1-\hat{F}_{n,P}^0(t)=\prod_{1\leqslant j\leqslant n:X_j\leqslant t}(1-1/(m(X_j)+1))^{\delta_j},$$

and Altshuler's[3] estimate

$$1-\hat{F}_{n,A}^0(t)=\exp(-\sum_{1\leqslant j\leqslant n:X_j\leqslant t}\delta_j/m(X_j)).$$

The properties of $\hat{F}_n^0$ were investigated notably by Breslow and Crowly[4], Aalen[1,2], Susarla and Van Ryzin[27,28], Földes, Rejtö and Winter[15], Földes and

Manuscript received March 9, 1988. Revised August 1, 1988.
*McGill University, Montreal, Canada.

Rejtö[13,14], Phadia and Van Ryzin[24], Burke et al.[5], Gill[17,18], Ghorai et al.[16], Csörgö and Horváth[7], Lo and Singh[22], Cuzick[8], Wang[29] and many others.

Let F be the distribution function of X and

$$U_n(t)=\sqrt{n}\,(\hat{F}_n^0-F^0), \tag{1}$$

where in place of $\hat{F}_n^0$, we may put $\hat{F}_{n,K}^0$, $\hat{F}_{n,P}^0$ or $\hat{F}_{n,A}^0$. Most of the previously mentioned work deals with the convergence property of $U_n(t)$ for $t\leqslant T$ where T is a point such that $1-F(T)=(1-F^0(T))(1-H(T))>0$. For any distribution G set $T_G=\inf\{t:\ G(t)=1\}$. A recent paper by Wang[29] studied the uniform consistency on $[-\infty,\ T_F]$ for $\hat{F}_n$. Gill[18] has proved that U_n converges to a time changed Brownian bridge on $[-\infty,\ T_F]$ under the condition that

$$\int_{-\infty}^{T_F}\frac{dF^0(s)}{1-H(s)}<\infty. \tag{2}$$

Földes[11] has proved that

$$\limsup_{n\to\infty}\ \sup_{-\infty\leqslant t\leqslant T_F}|U_n(t)|/\sqrt{\log\log n}\leqslant C \text{ a.s.} \tag{3}$$

for some constant C if $T_{F^0}<T_H$ or $T_{F^0}=T_H$ and $H(T_H)<1$. The result of Csörgö and Horváth[7] implied that the constant C can be taken as $11.536478/(1-H(T_F))$. On the other hand, it is known that if $T_{F^0}>T_H$, the $\hat{F}_n^0$ are not even consistent (See [7] or [29]). Theorem 2 below gives the smallest possible constant C, $C_{F^0,H}$ in the log log statement of (3). Therefore the equality holds in (3) with the smallest constant. It is not clear in the literature whether a statement like (3) will continue to be true in the case $T_{F^0}=T_H$ and $H(T_H)=1$. Ghorai et al[16] and Csörgö and Horváth[8] obtained a weaker rate than (3). Theorem 2 can also answer that question and gives the smallest possible constant C under the condition that H does not grow too rapidly to 1 near the point T_F.

Actually we prove a stronger, Finkelstein[26] (p. 513) type result:

Theorem 1. *Suppose that there exist constants s_0, $k>0$ and $0\leqslant\alpha<1$ such that $k(1-F^0(s))^\alpha\leqslant1-H(s)$ for $s\in(s_0,\ T_F)$, then the sequence $\{(1/2\log\log n)^{1/2}U_n(\cdot)\}$ is almost surely relatively compact in the supremum norm of functions over $[-\infty,\ T_F]$, and its set of limit points is*

$$g=\{(1-F^0(\cdot))h(d(\cdot)):\ h\in S\}, \tag{4}$$

where S is a set of absolutely continuous functions:

$$S=\left\{h\,|\,h:\ [0,\ \infty]\mapsto R,\ h(0)=0,\ \int_0^\infty\left(\frac{dh(x)}{dx}\right)^2dx\leqslant1\right\}$$

and d: $(-\infty,\ T_F]\mapsto[0,\ \infty]$,

$$d(t)=\int_{-\infty}^{t}(1-F^0(s))^{-2}(1-H(s))^{-1}dF^0(s).$$

Theorem 1 reduces to Finkelstein's Theorem[26] when there is no censoring ($H=0$).

Theorem 2. *Under the same conditions as Theorem 1, we have*

$$\limsup_{n\to\infty}\left(\frac{1}{2\log\log n}\right)^{1/2}\sup_{-\infty\leqslant t\leqslant\infty}|U_n(t)|=C_{F^0,H}, \tag{5}$$

where

$$C_{F^0,H}=\sup_{-\infty\leqslant t\leqslant\infty}(1-F^0(t))\sqrt{d(t)}. \tag{6}$$

When there is no censoring ($H(t)=0$ for all t), simple calculus shows that $C_{F^0,H}=1/2$, so Theorem 2 reduces to the Chung-Smirnov[6] law of the iterated logarithm for empirical distributions.

§ 2. A Extended i.i.d. Representation of $\hat{F}_n^0-F^0$

In the process of proving Theorem 1, we have developed Theorem 3, which improves upon the work on i.i.d. representation of the Kaplan-Meier estimator by Lo and Singh[22]. Let us first denote $\overline{G}=1-G$ for any distribution function G, $\widetilde{F}(t)=\int_{-\infty}^{t}\overline{H}(s)dF^0(s)$ and

$$A(t)=\int_{-\infty}^{t}\overline{F}^0(s)^{-1}dF^0(s)=\int_{-\infty}^{t}\overline{F}(s)^{-1}d\widetilde{F}(s),$$

$$N_n(t)=\sum_0^n 1(X_j\leqslant t,\ \delta_j=1)=n\widetilde{F}_n(t),$$

$$A_n(t)=\int_{-\infty}^{t}\frac{dN_n(s)}{m(s)}=-\log\hat{F}^0_{n,A}(t),$$

$$M_n(t)=N_n(t)-\int_{-\infty}^{t}m(s)dA(s),$$

$$\widetilde{B}_n(t)=\int_{-\infty}^{t}1(m(s)\geqslant n\overline{F}(s)/2)\frac{dM_n(s)}{m(s)}$$

$$=A_n(t)-A(t)+\int_{-\infty}^{t}1(m(s)<n\overline{F}(s)/2)\frac{dM_n(s)}{m(s)}+\int_{-\infty}^{t}1(m(s)=0)dA(s),$$

$$B_n^*(t)=\int_{-\infty}^{t}1(m(s)\geqslant n\overline{F}(s)/2)\frac{dM_n(s)}{n\overline{F}(s)},$$

$$B_n(t)=\int_{-\infty}^{t}\frac{dM_n(s)}{n\overline{F}(s)},$$

where $\widetilde{F}_n$ is the empirical distribution of $\widetilde{F}$ and the equality for A_n and B_n is obvious. It is easy to see that B_n is a sum of i.i.d. processes. In the following we use the notation $\|f\|_a^b$ to denote the supremum of fanction f over $[a, b]$.

Theorem 3. *Assume* (2). *Let a_n be such that $\overline{F}(a_n)=\gamma n^{-\alpha}(\log n)^{\beta}$, where $\gamma>0$, $0\leqslant\alpha\leqslant 1$ and for $\alpha=0$, $-\infty<\beta\leqslant 0$, for $0<\alpha<1$, $-\infty<\beta\leqslant\infty$ and for $\alpha=1, 1<\beta<\infty$. Then for any $\delta>0$ and n large enough*

$$\overline{F}^0(t)(A_n(t)-A(t))=\overline{F}^0(t)B_n(t)+R_n(t),$$

where

$$P(\|R_n(t)\|_{-\infty}^{a_n}\geqslant 4M^{1/2}(1+\delta)^2(\log n)^{1-\beta/2}/n^{1-\alpha/2})\leqslant 3n^{-(1+\delta)}\quad\text{a.s.} \tag{7}$$

Corollary 1. *Under the conditions of Theorem 3,*

$$\sup_{-\infty<t\leq a_n} |R_n(t)| = O((\log n)^{1-\beta/2}/n^{1-\alpha/2}) \quad a.s.$$

Proof Use Borel–Cantelli Lemma and Theorem 3.

With Corollary 1, by taking $\alpha=0$, $\beta=0$ and using the argument of Lo and Singh[22], or the argument we use in the proof of Theorem 4 below, it is easy to see a strong result on i.i.d. representation of the Kaplan–Meier estimator.

Corollary 2. *Let T be such that $F(T)<1$. Then*

$$\hat{F}_n^0(t) - F^0(t) = \bar{F}^0(t) B_n(t) + R_n^*(t),$$

$$\sup_{-\infty<t\leq T} |R_n^*(t)| = O(\log n/n) \quad a.\ s.$$

In most applications the order $o(n^{-1/2})$ for R_N^* would be enough (as in the case of Theorem 1), in which case if H does not grow too rapidly to 1 near the end point T_F, we have the following i.i.d. representation of the Product-Limit Estimator on the whole real line.

Theorem 4. *Under the conditions of Theorem 1, we have*

$$\hat{F}_n^0(t) - F^0(t) = \bar{F}^0(t) B_n(t \wedge b_n) + R_n^*(t),$$

$$\sup_{-\infty<t<\infty} |R_n^*(t)| = o(n^{-1/2}) \quad a.\ s. \tag{8}$$

where b_n is such that $\bar{F}^0(b_n) = n^{-1/2}(\log n)^{-1}$.

To show the strength of Theorem 4, we state an easy consequence[18] without proof.

Corollary 3. *Under the conditions of Theorem 1,*

$$U_n(\cdot) \xrightarrow{W} \bar{F}^0(\cdot) W(d(\cdot)),$$

where W is the Brownian Motion process and d is the transformation defined in Theorem 1.

§ 3. Proofs

For the sake of simplicity, we will assume that H is continuous in the following proofs. Otherwise we may define an H^* by stretching out H a finite amout so that H^* is continuous and define F^{0*} by stretching out F^0 accordingly. Then we deal with random variables $X_j^{0*} = (F^{0*})^{-1}(U_j)$ and $Y_j^* = (H^*)^{-1}(V_j)$, $j=1, 2, \cdots$, where U_j and V_j are mutually independent uniform [0, 1] distributed random variables. It can be shown that the distribution of the functionals we deal with in this paper are the same for (X_j^0, Y_j) and (X_j^{0*}, Y_j^*).

Lemma 1.[1,2] *Let $\mathscr{F}_t$ be the σ-algebra including all the information up to t, then $M(t)$, $\tilde{B}_n(t)$, $B_n^*(t)$ and $B_n(t)$ are quadratic martingales with respect to $\mathscr{F}_t$.*

Lemma 2.[18] *Let h be a continuous, nonnegative and nonincreasing function and* *let Z be a semimartingale, zero at time zero. Then for all r*

$$\sup_{0\leq t\leq r} h(t)\,|Z(t)|\leq 2\sup_{0\leq t\leq r}\left|\int_0^t h(s)dZ(s)\right| \tag{9}$$

Lemma 3 (Martingale exponential ineqaulity)[26] *If M is a mean zero local martingale with $\|\Delta M\|\leq c$, then for all $t>0$, $\lambda>0$, and $\tau>0$ we have*

$$P(\|M\|_0^t\geq\lambda,\ \langle M\rangle(t)\leq r)\leq 2\exp\Big(-\frac{\lambda^2}{2r}\,\psi\Big(\frac{\lambda c}{r}\Big)\Big), \tag{10}$$

where $\psi(x)\to 1$ if $x\to 0$, and $x\psi(x)\sim\log x$ as $x\to\infty$.

Lemma 4.[21,30] *For any $\lambda>0$ and $-\infty<a\infty$*

$$P\Big(\Big\|\frac{m(\cdot)}{n\overline{F}(\cdot)}-1\Big\|_0^a\geq\lambda\Big)\leq 2\exp\Big(-n\overline{F}(a)\,\frac{\lambda^2}{2}\,\psi(-\lambda)\Big), \tag{11}$$

where ψ is the same as in Lemma 3.

Lemma 5.[21,30] *If $\overline{F}(a_n)=n^{-\alpha}(\log n)^{\beta}$, where α and β are as in Theorem 3, then*

$$\limsup_{n\to\infty}(n\overline{F}(a_n)/2\log\log n)^{1/2}\left|\frac{m(\cdot)}{n\overline{F}(\cdot)}-1\right|_0^{a_n}=1\quad a.\ s. \tag{12}$$

Proof of Theorem 3 We first notice that $\|m/n\overline{F}-1\|$ is $o(1)$ by Lemma 5 and the choices of α and β, therefore for $t\in[0,\ a_n]$, $m(t)>n\overline{F}(t)/2$ a.s. for n large enough. In other words, $A_n-A=\widetilde{B}_n$ a.s. for n large enough. So we can replace A_n-A by $\widetilde{B}_n$ and B_n by B_n^* in the sequel.

Define

$$E_n=\left\{\left\|\int_0 \overline{F}^0\Big(\frac{1}{m(s)}-\frac{1}{n\overline{F}(s)}\Big)1(m>n\overline{F}/2)\,dM_n(s)\right\|_0^{a_n}\geq\lambda_n/2\right\}$$

and

$$I_n=\left\{\int_0^{a_n}\frac{(m(s)-n\overline{F}(s))^2}{m(s)n^2\overline{F}(s)}\,\frac{dF^0(s)}{\overline{H}(s)}\geq\tau_n\right\},$$

where λ_n and τ_n will be specified later. By Lemma 2 and Lemma 3 we have

$$P(\|\overline{F}^0(\widetilde{B}_n-B_n^*)\|_0^{a_n}\geq\lambda_n)\leq P(E_n)\leq 2\exp\Big(-\frac{\lambda_n^2}{8\tau_n}\,\psi\Big(\frac{\lambda_n c_n}{2\tau_n}\Big)\Big)+P(I_n), \tag{13}$$

where c_n is the bound on the jump of the martingale in E_n, which is less then

$$2\overline{F}^0(a_n)/n\overline{F}(a_n)=o(1)/n(\overline{F}(a_n))^{1/2}=o(1)\,(\log n)^{-\beta/2}/n^{1-\alpha/2}$$

since condition (2) implies $\overline{F}^0(t)/\overline{H}(t)\to 0$ as $t\to T_{F^0}$ and $\overline{F}=\overline{F}^0\overline{H}$. Notice that on the set

$$\|m/n\overline{F}-1\|_0^{a_n}\leq\sqrt{2(1+\delta)\log n/n\overline{F}(a_n)}=d_n,$$

for some $\delta>0$, then

$$\int_0^{a_n}\frac{(m(s)-n\overline{F}(s))^2}{m(s)n^2\overline{F}(s)}\,\frac{dF^0(s)}{\overline{H}(s)}\leq M\,\frac{d_n^2}{n(1-d_n)},$$

where $d_n\to 0$ as $n\to\infty$ and $M=\int_0^{T_F}dF^0/\overline{H}$. Take

$$\tau_n=2M(1+\delta)^2\,\frac{\log n}{n^2\overline{F}(a_n)},$$

then conditional event I_n is empty for large n. In other words, (13) is bounded by

$$2\exp\Big(-\frac{\lambda_n^2(\log n)^{\beta}n^{2-\alpha}}{16M(1+\delta)^2\log n}\,\psi\Big(o(1)\frac{\lambda_n n^{1-\alpha/2}}{4M(1+\delta)^2(\log n)^{1-\beta/2}}\Big)\Big)$$
$$+P(\|m;\,n\bar{F}-1\|_0^{a_n}>d_n).$$

Take

$$\lambda_n=4M^{1/2}(1+\delta)^2(\log n)^{1-\beta/2}n^{1-\alpha/2},$$

then (14) is dominated by $3n^{-(1+\delta)}$ for large n by Lemma 4 and the fact that the quantity inside ψ tend to zero as $n\to\infty$.

Before we prove Theorem 4, we state some important lemmas.

Lemma 6.[8] *Let $\hat{F}_1^0$, F_2^0 and $\hat{F}_3^0$ stand for any of the estimates described at the beginning of this paper. Then for any t, if $m(t)>1$,*

$$|\hat{F}_1^0(t)-\hat{F}_2^0(t)|\leqslant(1-\hat{F}_3^0)\frac{4}{m(t)}.$$

Note: Cuzick's original lemma is not of this form, but this form is an immediate consequence of his original form.

Lemma 7. *Let a_n be that of Theorem 3, then*

$$\|A_n-A\|_{-\infty}^{a_n}=O\Big(\Big(\frac{\log n}{n\bar{F}(a_n)}\log\frac{1}{\bar{F}^0(a_n)}\Big)^{1/2}\Big)\quad a.s. \tag{15}$$

Proof Just as in the proof of Theorem 3, we may replace A_n-A by $\tilde{B}_n$. $\tilde{B}_n$ is a martingale with covariate process

$$\int_{-\infty}^{a_n}\frac{1(m>n\bar{F}/2)\,dA(s)}{m(s)}=O\Big(\frac{1}{n\bar{F}(a_n)}\log\frac{1}{\bar{F}^0(a_n)}\Big),$$

and jump size $(n\bar{F}(a_n))^{-1}$. We apply Lemma 3 and Lemma 4 and use a similar argument as in the proof of Theorem 3 to get Lemma 7. Details are omitted.

Lemma 8. *Under condition (2), $\mathscr{G}$ is relatively compact set in the supremum norm of function over $[-\infty, T_F]$, where $\mathscr{G}$ is defined in the statement of Theorem 1.*

Lemma 9. *Under condition (2), $\{(n/2\log\log n)^{1/2}\bar{F}^0B_n(\cdot\wedge b_n)\}$ is almost surely relatively compact in the supremum norm of function over $[-\infty, T_F]$, and its limit point is g.*

The proofs of Lemma 8 and Lemma 9 are standard with the help of Lemma 2, so we will put them in the Appendix.

Proof of Theorem 4 For $t>b_n$, according to Csörgö and Horváth[7]

$$|\hat{F}_n^0(t)-F_n^0(t)|\leqslant|\hat{F}_n^0(b_n)-F_n^0(b_n)|+\bar{F}^0(b_n). \tag{16}$$

So to prove Theorem 4, we need to prove almost surely

$$\|\hat{F}_{n,A}^0-F^0-\bar{F}^0B_n\|_0^{b_n}=o(n^{-1/2}), \tag{17}$$

$$\|\hat{F}_n^0-\hat{F}_{n,A}^0\|_0^{b_n}=o(n^{-1/2}), \tag{18}$$

$$|\bar{F}^0(b_n)B_n(b_n)|=o(n^{-1/2}), \tag{19}$$

since $\bar{F}^0(b_n)$ is $o(n^{-1/2})$.

First we notice that

$$\bar{F}(b_n)\geqslant k(\bar{F}^0(b_n))^{1+\alpha}=kn^{-(1+\alpha')/2}$$

for n large and $\alpha<\alpha'<1$. Denote $R_n(t)=A_n(t)-A(t)-B_n(t)$. By Theorem 3,

$$\|\bar{F}^0 R_n\|_0^{b_n}=o(n^{-(1-\alpha'/2)}) \quad \text{a.s.} \tag{20}$$

Since $\bar{F}^0=\exp(-A)$ and $1-\hat{F}^0_{n,A}=\exp(-A_n)$, we have

$$\begin{aligned}\|\hat{F}^0_{n,A}-F^0-\bar{F}^0B_n\|_0^{b_n}&\leqslant\|\bar{F}^0(1-\exp(A-A_n)+A-A_n)\|_0^{b_n}+\|\bar{F}^0R_n\|_0^{b_n}\\ &\leqslant\|\bar{F}^0(A-A_n)^2\exp(|A-A_n|)\|_0^{b_n}+o(n^{-1/2}) \quad \text{a.s.}\end{aligned} \tag{21}$$

by two terms Taylor expansion and (20). Also by (20) and Lemma 9, we have $\|\bar{F}^0(A-A_n)\|_0^{b_n}$ in the order of $O((\log\log n/n)^{1/2})$ almost surely. By Lemma 7, we have $\|A-A_n\|_0^{b_n}=O(n^{-(1-\alpha')/2}\log n)$ almost surely. Therefore the first term of (21) is also $o(n^{-1/2})$ and we get (17).

To prove (18), by Lemma 6 the left-side of (18) is less than

$$4\|\exp(-A_n)/m(\cdot)\|_0^{b_n} \tag{22}$$

if $m(b_n)>1$. By Lemma 5, we have almost surely when n large $m(t)\geqslant(1-\delta)n\bar{F}(t)$ for all $t\in(-\infty, b_n]$. Also notice that

$$\|\exp(-A_n)-\bar{F}^0\|_0^{b_n}=O(\log\log n/n)^{1/2}) \quad \text{a. s.}$$

by (17) and Lemma 9. So (22) is bounded by

$$\begin{aligned}&8\|\bar{F}^0/n\bar{F}\|_0^{b_n}+8\|(\exp(-A_n)-\bar{F}^0)/n\bar{F}\|_0^{b_n}\\ &=O((\bar{F}^0(b_n))^{-\alpha}/n)+O((\log\log n/n)^{1/2}(\bar{F}^0(b_n))^{-1-\alpha}/n)\\ &=O(n^{-(1-\alpha/2)}(\log n)^{\alpha})+O(n^{-(1-\alpha/2)}(\log\log n)^{1/2}(\log n)^{1+\alpha})=o(n^{-1/2}) \quad \text{a.s.}\end{aligned}$$

since $\alpha<1$.

For (19), let a_n be such that $\bar{F}^0(a_n)=n^{-1/2}$. Then

$$\begin{aligned}|\bar{F}^0(b_n)B_n(b_n)|&\leqslant|\bar{F}^0(b_n)/\bar{F}^0(a_n)|\,|\bar{F}_0(a_n)B_n(a_n)|+\left|\bar{F}^0(b_n)\int_{a_n}^{b_n}\frac{dM_n(s)}{n\bar{F}(s)}\right|\\ &\leqslant\frac{1}{\log n}\left(\frac{\log\log n}{n}\right)^{1/2}+2\sup_{a_n\leqslant t\leqslant b_n}\left|\int_{a_n}^{t}\bar{F}^0(s)\frac{dM_n(s)}{n\bar{F}(s)}\right|,\end{aligned} \tag{23}$$

by Lemma 9 and Lemma 2. To see that second term is almost surely $o(n^{-1/2})$ we notice that it is a martingale with the quadratic variation

$$\begin{aligned}\int_{a_n}^{b_n}\frac{m(s)}{n^2\bar{F}(s)\bar{H}(s)}dF^0(s)&\leqslant\frac{1}{k(1-\delta)}\int_{a_n}^{b_n}\frac{1}{n(\bar{F}^0(s))^{\alpha}}dF^0(s)\\ &\leqslant C(\bar{F}^0(a_n))^{1-\alpha}/n\leqslant Cn^{-(1+(1-\alpha)/2)},\end{aligned}$$

where the first inequality is by Lemma 4. The jump size is in the order of

$$\frac{1}{n\bar{H}(b_n)}\leqslant\frac{1}{kn(\bar{F}^0(b_n))^{\alpha}}=O(n^{-(1-\alpha/2)}(\log n)^{\alpha}).$$

By Lemma 3 and an argument similar to that in the proof of Theorem 3, we see that the second term of (23) is $O(n^{-1/2}(\log n)^{-2})=o(n^{-1/2})$.

Proof of Theorem 1 A simple consequence of Lemma 9 and Thorem 4.

Proof of Theorem 2 A simple consequence of Theorem 1.

Appendix

Proof of Lemma 8 To see Lemma 8 is true, we only have to see that for any ε

>0 there exist $T<T_F$ such that

$$|g(t)|\leqslant\sqrt{\epsilon} \text{ for all } g\in\mathscr{G} \text{ and } t\geqslant T, \tag{A.1}$$

since the relative compactness with retrictions to $[-\infty, T]$ is obvious. By the Schwatz inequality, we have

$$|h(t)|\leqslant\sqrt{t} \text{ for all } h\in\mathscr{S}. \tag{A.2}$$

On the other hand, it is not difficult to see that $(\bar{F}^0(t))^2d(t)\to0$ as $t\to T_F$ since $\int dF^0/\bar{H}<\infty$. So for any $\epsilon>0$ there exists $T<T_F$ such that

$$|\bar{F}^0(t)^2d(t)|\leqslant\epsilon \text{ for all } t\geqslant T. \tag{A.3}$$

Now for any $g\in\mathscr{G}$, there exists $h\in\mathscr{S}$ and for all $t\geqslant T$,

$$|g(t)|=|\bar{F}^0(t)h\cdot d(t)|\leqslant|\bar{F}^0(t)\sqrt{d(t)}|\leqslant\sqrt{\epsilon}.$$

Proof of Lemma 9 Let $T_m=\{0=t_0<t_1<\cdots<t_m=T_F\}$ be such that

$$\int_{t_{i-1}}^{t_i}\frac{dF^0(s)}{\bar{H}(s)}=\frac{1}{m}\int_{-\infty}^{\infty}\frac{dF^0(s)}{\bar{H}(s)}.$$

According to [26] (page 76) we only have to check that for any fixed m the T_m-approximation B_{nm} of B_n satisfies

$$\limsup_{n\to\infty}\sqrt{\frac{n}{2\log\log n}}\|B_{nm}-B_n\|\leqslant a_m \text{ where } a_m\to0 \text{ as } m\to\infty, \tag{A.4}$$

and for $\mathscr{G}$ (which satisfies (16)—(18) of [26], page 76 by Lemma 8) and fixed m

$$\Pi_{T_m}\left(\sqrt{\frac{n}{2\log\log n}}B_n\right)\text{limit set } \Pi_{T_m}(\mathscr{G}) \text{ a.s. wrt } \|\,\| \text{on } R_m. \tag{A.5}$$

The proof of (A.5) is similar to that on page 76 of [25], so we omit it here. To prove (A.4), first we not that

$$\|B_{mn}-B_n\|\leqslant\max_{1\leqslant i\leqslant m}\|B_n-B_n(t_{i-1})\|_{t_{i-1}}^{t_i},$$

and

$$\|B_n-B_n(t_{i-1})\|_{t_{i-1}}^{t_i}\leqslant|\bar{F}^0(t_i)-\bar{F}^0(t_{i-1})|\left|\int_0^{t_{i-1}}\frac{dM_n(s)}{n\bar{F}(s)}\right|+\left\|\int_{t_{i-1}}^{t_i}\bar{F}^0(s)\frac{dM_n(s)}{n\bar{F}(s)}\right\|_{t_{i-1}}^{t_i} \tag{A.6}$$

by Lemma 2. Because of

$$E\left|\int_0^{t_{i-1}}\frac{dM_n(s)}{\sqrt{n}\bar{F}(s)}\right|^2=\int_0^{t_{i-1}}\frac{dF^0(s)}{(\bar{F}^0(s))^2\bar{H}(s)}$$

and

$$E\left|\int_{t_{i-1}}^{t_i}\bar{F}^0(s)\frac{dM_n(s)}{\sqrt{n}\bar{F}(s)}\right|^2=\int_{t_{i-1}}^{t_i}\frac{dF^0(s)}{\bar{H}(s)},$$

and the martingale inequality, Dudley and Philip's [9], Theorem 4.1 give us

$$\limsup_{n\to\infty}\sqrt{\frac{n}{2\log\log n}}\|B_{mn}-B_n\|\leqslant\max_{1\leqslant i\leqslant m}|\bar{F}^0(t_i)-\bar{F}^0(t_{i-1})|\sqrt{\int_0^{t_{i-1}}\frac{dF^0(s)}{(\bar{F}^0(s))^2\bar{H}(s)}}+C\sqrt{\int_{t_{i-1}}^{t_i}\frac{dF^0(s)}{\bar{H}(s)}}$$

for some constant C. (A. 4) is proved once we see that the right-hand side of the last expression tends to zero as $m\to\infty$.

Acknowledgement. The author is most grateful to Professor T. L. Lai, who directed this line of research. His comments improve the the condition from $\alpha<2/3$ to $\alpha<1$ in Theorems 1, 2 and 4 and also resulted in a much better proof.

References

[1] Aalen, O., Nonparametric estimation of partial transition probabilities in multiple decrement models, *Ann. Statist.*, 6, (1978a), 534—545.

[2] Aalen, O., Nonparametric inference for a family of counting processes, *An. Statist.* **6**(1978b), 701—726.

[3] Altshuler, B., Theory for the measurement of competing risk in animal experiments, *Math. Bioasc.* **6** (1970), 1—11.

[4] Breslow, N. E. & Crowley, J., A large sample study of the life table and product limit estimates under random censnorship, *Ann. Statist*, 6(1974), 437—453.

[5] Burke, M. D., Csörgö, S. & Horváth, L., Strorg approximations of some biometric estimates under random censorship, *Z. Wahrscheinlichkeitrtheorie und verw. Gebiete* **56**(1981), 87—112.

[6] Chung, K. L., An estimate concerning the Kolmogoroff limit distribution, *Trans. Amer. Math. Soc.* **67** (1949), 36—50.

[7] Csörgö, S. & Horváth, L., The rate of strong uniform consistency for the product-limit estimator, *Z. Wahrscheinlichkeitrtheorieund verw. Gebiete*, **62**(1983), 411—426.

[8] Cuzick, J., Asymptotic properties of censored linear rank tests, *Ann. Statis.*, **13**(1985), 133—141.

[9] Dudley R. M. & Philip, W., Invariance principles for sums of Banach space valued random elements and empirical processes, *Z. Wahrscheinlichkeitrtheorie und verw. Gebiete*, **62**(1983), 509—552.

[10] Efron, B., The two-sample problem with censored data, *Proc. Fifth Berkeley Symp. Math. Statist. Prob.*, **4**(1967), 831—854. Los Angeles: Univ. California Press.

[11] Feller, W., The general form of the so-called law of iterated logarithm, *Trans. Amer. Math. Soc.*, **54** (1943), 373—402.

[12] Földes, A., Strong uniform consistency of the product limit estimator under variable censoring, *Z. Wahrscheihlichkeitrtheorieund verw. Gebiete*, 58(1981), 95—107.

[13] Földes, A., & Rejtö, L., A remark on the convergence rate of the product-limit estimator, *Periodica Math. Hung.*, **11**(1980), 251—254.

[14] Földes, A., & Rejtö, L., Strong uniform consistency for nonparametric survival estimators from randomly censored data, *Ann. Statist.* **9**(1981), 122—129.

[15] Földes, A., Rejtö, L., & Winter, B. B., Strong consistency properties of nonparametric estimators for randomly censored data, I. The product limit estimator, *Periodica Math. Hung.*, **11**(1980), 233—250.

[16] Ghorai, J. K., Rejtö, L., & Susarla, V., LIL type properties (on the whole line) of the PL and the Bayesian estimator. *In: Proceedings on Nonparametric Statistical Inference (Colloquia Math. Soc. J. Bolyal 32)* (1982), Amsterdam: North-Holland.

[17] Gill, R., Censoring and Stochastic Integrals, *Mathematisch Centrum Tract*, **124**(1980), Amsterdam.

[18] Gill, R., Large sample behavior of the product-limit estimator on the whole line, *Ann. Statist.* **11** (1983), 49—58.

[19] Jacobsen, M., Statistical Analysis of Counting Processes. *Lecture Notes in Stctistics* **12**(1982), Springer-Verlag, Berlin-New York.

[20] Kaplan, E. L. & Meier, P., Nonparametric estimation from incomplete observations, *J. Amer. Statist. Assoc.*, **53**(1958), 457—481.

[21] Lai, T. L., On Chernoff-Savage statistics and sequential rank test, *Ann. Statist.* **3**(1975), 825—845.

[22] Lo, S. H., & Singh, K., The product-limit estimator and the bootstrap: Some asymptotic representation, *Probab. Theory Rel. Fields*, **71**(1986), 455—465.

[23] Nelson, W., Theory and applications of hazard plotting for censored failure data, *Technometrics*, **14**(1972), 945—966.

[24] Phadia, E. G., & Van Ryzin, J., A note on convergence rates for the product limit estimator, *Ann. Statists.* **8**(1980), 673—678.

[25] Prentice, R. L., Linear rank tests with right censored data, *Biometrika*, **65** (1978), 167—179.

[26] Shorack, G. R. & Wellner, J., Empirical Processes with Applications to Statistics. (1986), Wiley, New York.

[27] Susarla, V. & Van Ryzin, J., Large sample theory for a Bayesian non-parametric survival curve estimator based on censored samples, *Ann. Statist.*, **6** (1978), 755—768.

[28] Susarla, V. & Van Ryzin, J., Addendum to "Large sample theory for a Bayesian nonparametric survival curve estimator based on censored samples", *Ann. Statist.*, **8** (1980), 693.

[29] Wang, J. G., A note on the uniform consistency of the Kaplan-Meier estimator, *Ann. Statist.*, **15** (1987), 1313—1316.

[30] Wellner, J., Limit theorems for the ratio of the empirical distribution function to the true distribution function, *Z. Wahrscheinlichkeitrtheorie und verw. Gebiete*, **45** (1978), 73—88.