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A BOUNDARY VALUE PROBLEM FOR A NONLINEAR ORDINARY DIFFERENTIAL EQUATION INVOLVING A SMALL PARAMETER——

The Riemann problem for a Generalized Diffusion Equation

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Abstract

This paper studies the boundary value problem involving a small parameter

$$((k(V(t))+\varepsilon)|V'(s)|^{N-1}V'(s))'+(sg(V(s))+f(V(s)))V'(s)=0 \text{ for } s\in \mathbf{R},$$

$$V(-\infty)=A,\ V(+\infty)=B;\ A<B,$$

which originates from the Riemann problem for a generalized diffusion equntion

$$g(U)D_tU=p'(t)p^N(t)D_x((k(U)+\varepsilon)|D_xU|^{N-1}D_xU)+p'(t)f(U)D_xU \text{ for } x\in \mathbf{R},\ t>0,$$

$$U(x,0)=A \text{ for } x<0,\ U(x,0)=B \text{ for } x>0,$$

under the hypotheses H_1—H_4. The author's aim is not only to determine explicitly the discontinuous solution $U_0(x,t)=V_0(s)$, $s=x/p(t)$, to the reduced problem, and the form and the number of its curves of discontinuity, but also to present, in an extremely natural way, the jump conditions which it must satisfy on each of its curves of discontinuity. It is proved that the problem has a unique solution $U_\varepsilon(x,t)=V_\varepsilon(s)$, $s=x/p(t)$, $\varepsilon\geqslant 0$, $V_\varepsilon(s)$ pointwise converges to $V_0(s)$ as $\varepsilon\downarrow 0$, $V_0(s)$ has at least one jump point if and only if $k(y)$ possesses at least one interval of degeneracy in $[A,B]$, and there exists a one-to-one correspondence between the collection of all intervals of degeneracy of $k(y)$ in $[A,B]$ and the set of all jump points of $V_0(s)$.

§ 1. Introduction

In this paper we study a boundary value problem for a second order nonlinear ordinary differential equation involving a small parameter, $\varepsilon\geqslant 0$, of the form

$$((k(V(s))+\varepsilon)|V'(s)|^{N-1}V'(s))'+(sg(V(s))+f(V(s)))V'(s)=0 \qquad (1)$$

on the whole real axis $\mathbf{R}$ with the boundary conditions

$$V(-\infty)=A \text{ and } V(+\infty)=B, \qquad (2)$$

which originates from the Riemann problem for a generalized diffusion equation with convection

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$$g(U)D_tU = p'(t)p^N(t)D_x((k(U)+\varepsilon)\,|D_xU|^{N-1}D_xU) + p'(t)f(U)D_xU \text{ for } x\in\mathbf{R},\ t>0, \tag{3}$$

$$U(x,\ 0) = A + (B-A)H(x) \text{ for } x\in\mathbf{R}, \tag{4}$$

where $H(x)=0$ for $x<0$, $H(x)=1$ for $x>0$, and $H(0)=[0,\ 1]$, D_x and D_t respectively represent differentiation with respect to the independent variables x and t, and $p'(t)f(U)D_xU$ stands for convection. When $f(U)\equiv 0$, $g(U)\equiv 1$, and $p(t)=t^{1/(N+1)}$, equation $(3)_\varepsilon$ has been suggested as a model for certain generalized diffusion processes by Philip [1]; when $k(U)+\varepsilon\equiv 0$ and $p(t)=t$, problem $(3)_0$-(4) is well known as the Riemann problem for a scalar conservation law. From the form of the two problems, it can be seen that if $V_\varepsilon(s)$ is a solution to the boundary value problem $(1)_\varepsilon$-(2), then $U_\varepsilon(x,\ t) \overset{\text{def.}}{=\!=} V_\varepsilon(x/p(t))$ is a similarity solution to the Riemann problem $(3)_\varepsilon$-(4), and vise versa.

Unless otherwise indicated, we always make the following four hypotheses:

H_1. A, B, $A<B$, and $N>0$ are given constants.

H_2. $p(t)$ is an increasing, locally absolutely continuous (being abbreviated a. c. later) function defined on $[0,\ +\infty)$ with $p(0)=0$ and $p(t)>0$ for $t>0$.

H_3. $F(y) \overset{\text{def.}}{=\!=} \int_A^y f(s)\,ds$ and $G(y) \overset{\text{def.}}{=\!=} \int_A^y g(s)\,ds$ are a. c. function defined on $[A,\ B]$ such that $G(y)$ is strictly increasing and $\frac{F(B)}{G(B)}G(y)-F(y)$ nonnegative on $[A,\ B]$.

H_4. $k(y)$ is a nonnegative measurable function defined on $[A,\ B]$ such that $G(y)(G(B)-G(y))k^{1/N}(y)$ is Lebesgue integrable on $[A, B]$.

To investigate the behaviour of the solution $V_\varepsilon(s)$ at the minus and plus infinity, we need a special case of hypothesis H_3, namely

H_3^s. $f(y)$ and $g(y)$ are nonnegative measurable function defined on $[A,\ B]$ such that $\frac{f(y)}{g(y)}$ is equivalent to an increasing, a. c. function defined on $[A, B]$.

"In essence, a perturbation procedure consists of constructing the solution for a problem involving a small parameter ε, either in the differential equation or the boundary conditions or both, when the solution for the limiting case $\varepsilon=0$ is known." (Quoted from the Preface to the book [2]). Our problem is in itself different from the case mentioned above; this perturbation procedure consists of determining the solution for the limiting case $\varepsilon=0$, because the structure of the solution we want to find is not known a priori and depends upon the perturbation procedure. As we can see later, our problem is a singular perturbation problem under hypotheses H_1-H_4, provided that the function $k(y)$ has at least one interval of degeneracy in $[A,\ B]$. What is termed an interval of degeneracy of $k(y)$ in $[A,\ B]$ is a closed subinterval on which $k(y)=0$ a. e., and furthermore, $k(y)>0$ a. e. in the complement of the union of all intervals of degeneracy in $[A, B]$.

When the function $k(y)$ has at least one interval of degeneracy in $[A, B]$, equation $(3)_0$ must be of degenerate parabolic type. It has long been found that discontinuities may occur in generalized solutions to a second order quasilinear degenerate parabolic equation, just as they do in those to a first order quasilinear hyperbolic equation——an equation which is itself a degenerate parabolic equation where no second derivatives of the unknown function appear at all. For this reason, there should be certain jump conditions which a discontinuous solution must satisfy on the set of its jump points; by a discontinuous solution we mean a generalized solution in which discontinuities have already arisen. As early as 1969, Vol'pert and Hudjaev[3] first presented such jump conditions; it is a pity that the jump conditions were not completely correct. It was not until 1985 that Wu Zhuoqun[4] pointed out that one of the jump conditions was false and gave it a correct expression. As far as we know, the jump conditions presented in [3] were only consequences of a definition of generalized solution and the definition came into being on the analogy of a definition of generalized solution to a first order quasilinear hyperbolic equation; besides, in [3] there were no examples which can exhibit a discontinuous solution and the form of the set of its jump points. All the facts mentioned above were the reasons why the mistake was made in [3]. Our purpose of studyihg the Riemann problem is not only to determine explicitly the discontinuous solution $U_0(x,t) \overset{\text{def.}}{=\!=} V_0(s)$, $s=x/p(t)$, to the reduced problem $(3)_0$-(4), and the form and the number of its curves of discontinuity, but also to present, in an extremely natural way, the jump conditions which $U_0(x,t)$ must satisfy on each of its curves of discontinuity.

The organization of this paper is as follows. In Section 2 we announce our main reeults. In Section 3 we convert the problem of determining similarity solutions to the Riemann problem $(3)_\varepsilon$-(4) into the boundary value problem $(1)_\varepsilon$-(2) and then transform the latter into a two-point boundary value problem of the form

$$-\left(\frac{w'(y)+f(y)}{g(y)}\right)=\left(\frac{k(y)+\varepsilon}{w(y)}\right)^{1/N} \quad \text{for } y\in(A,B), \tag{5$_\varepsilon$}$$

$$w(A)=0,\ w(B)=0. \tag{6$_0$}$$

Section 4 is devoted to the two-point boundary value problem. In the last section we construct the solution $V_\varepsilon(s)$ to the boundary value problem $(1)_\varepsilon$-(2), making use of the unique solution $w_\varepsilon(y)$ to the two-point boundary value problem.

§ 2. Main Results

In this paper we establish the following main results.

Theorem 1 (Existence and Uniqueness Theorem). *Under hypothese* H_1-H_4, *the boundary value problem* $(1)_\varepsilon$-(2), $\varepsilon\geqslant 0$, *has a unique solution* $V_\varepsilon(s)$. *Moreover, the solution* $V_\varepsilon(s)$ *converges to the solution* $V_0(s)$ *pointwise on* $\mathbf{R}$ *as* $\varepsilon\downarrow 0$.

Theorem 2 *Assume further that hypothesis* H_3^s *is valid. Then there is a finite number* s_A (s_B) *such that* $V_\varepsilon(s)\equiv A$ *for* $s\leqslant s_A$ ($V_\varepsilon(s)\equiv B$ *for* $s\geqslant s_B$), *if and only if*

$$\int_A^{(A+B)/2}\left(\frac{k(s)+\varepsilon}{G(s)}\right)^{1/N}ds<+\infty\left(\int_{(A+B)/2}^B\left(\frac{k(s)+\varepsilon}{G(B)-G(s)}\right)^{1/N}ds<+\infty.\right.$$

Theorem 3 (Structure Theorem). *The solution* $V_0(s)$ *can be represented by*

$$V_0(s)=A+\sum_j(b_j-a_j)H(s-s_j)+\int_{-\infty}^s V_0'(t)\,dt \text{ for all } s\in\mathbf{R},$$

where $V_0'(s)$, *a derivative of* $V_0(s)$, *is nonnegative and integrable on* $\mathbf{R}$, $\{s=s_j;\ j=1, 2,\cdots\}$ *the set of all jump points of* $V_0(s)$, *and* $\{[a_j,b_j];\ j=1,2,\cdots\}$ *the collection of all intervals of degeneracy of* $k(y)$ *in* $[A,B]$. *Moreover, in each connected component of the open set* $\mathbf{R}\setminus\bigcup_j\{s=s_j\}$, $V_0(s)$ *is a. c. and satisfies equation* $(1)_0$, *while at each jump point* $s=s_j$, $j=1, 2,\cdots$, $V_0(s)$ *must satisfy the following jump conditions:*

$$\left(k(V_0(s))\,|V_0'(s)|^{N-1}V_0'(s)+sG(V_0(s))+F(V_0(s))\right)\Big|_{s=s_j-0}^{s=s_j+0}=0, \tag{7}$$

$$V_0(s_j-0)=a_j,\ V_0(s_j+0)=b_j, \text{ and } V_0(s_j)=[a_j, b_j]. \tag{8}$$

Theorem 4 (Comparison Theorem). *To express explicitly the dependence of the solution* $V_0(s)$ *and its jump point* $s=s_j$, $j=1, 2,\cdots$, *upon both* A *and* B, *we denote them by* $V_0(s;A,B)$ *and* $s_j(A,B)$, *respectively. If* $A_1\leqslant A_2$ *and* $B_1\leqslant B_2$, *then*

$$V_0(s;A_1,B_1)\leqslant V_0(s;A_2,B_2) \text{ for } s\in\mathbf{R} \text{ and } s_j(A_1,B_1)\geqslant s_j(A_2,B_2),\ j=1, 2,\cdots.$$

Shifting the results mentioned above to the Riemann problem $(3)_\varepsilon$-(4), we can obtain the following theorems.

Theorem 1′ (Existence and Uniqueness Theorem). *Under hypotheses* H_1-H_4 *the Riemann problem* $(3)_\varepsilon$-(4), $\varepsilon\geqslant 0$, *has a unique solution* $U_\varepsilon(x,t)\overset{\text{def.}}{=\!=}V_\varepsilon(s)$, $s=x/p(t)$. *Moreover, the solution* $U_\varepsilon(x,t)$ *converges to the solution* $U_0(x,t)$ *pointwise on the domain* $\mathbf{R}\times(0, +\infty)$ *as* $\varepsilon\downarrow 0$.

Theorem 2′ *Assume further that hypothesis* H_3^s *is valid. Then there is a finite numcer* s_A (s_B) *such that* $U_\varepsilon(x,t)\equiv A$ *for* $x\leqslant s_A p(t)$, $t>0$ ($U_\varepsilon(x,t)\equiv B$ *for* $x\geqslant s_B p(t)$, $t>0$), *if and only if*

$$\int_A^{(A+B)/2}\left(\frac{k(s)+\varepsilon}{G(s)}\right)^{1/N}ds<+\infty\left(\int_{(A+B)/2}^B\left(\frac{k(s)+\varepsilon}{G(B)-G(s)}\right)^{1/N}ds<+\infty\right).$$

Theorem 3' (Structure Theorem). *The solution $U_0(x, t)$ can be represented by*

$$U_0(x,t) = A+\sum_j (b_j-a_j)\, H\Big(\frac{x}{p(t)}-s_j\Big)+\int_{-\infty}^{x/p(t)} V_0'(s)ds \text{ for } x\in\mathbf{R},\ t>0,$$

where $V_0'(s)$ is a nonnegative integrable function defined on $\mathbf{R}$, $\{x=\phi_j(t) \overset{\text{def.}}{=\!=} s_j p(t);\ j=1, 2,\cdots\}$ the set of all curves of discontinuity of $U_0(x, t)$, and $\{[a_j, b_j];\ j=1, 2, \cdots\}$ the collection of all intervals of degeneracy of the function $k(y)$ in $[A, B]$. Moreover, in each connected component of the open set $\mathbf{R}\times(0, +\infty)\setminus\bigcup\{x=\phi_j(t)\}$, $U_0(x,t)$ is a. c. and a. e. satisfies equation $(3)_0$, while on each curve of discontinuity, $x=\phi_j(t)$, $j=1, 2,\cdots$, $U_0(x, t)$ must satisfy thefollowing jump conditions:

$$\begin{aligned}(p'(t)p^N(t)k(U_0(x,t))\,|D_xU_0(x,t)|^{N-1}D_xU_0(x,t)+\phi_j'(t)G(U_0(x,t))\\ +p'(t)F(U_0(x,\ t)))\Big|_{x=\phi_j(t)-0}^{x=\phi_j(t)+0}=0 \text{ for } t>0,\end{aligned}\qquad(9)$$

$$U_0(\phi_j(t)-0,t)=a_j,\ U_0(\phi_j(t)+0,\ t)=b_j, \text{ and } U_0(\phi_j(t),t)=[a_j,b_j] \text{ for } t>0.\qquad(10)$$

Theorem 4' (Comparison Theorem). *To express explicitly the dependence of the solution $U_0(x,t)$ and its curve of discontinuity, $x=\phi_j(t) \overset{\text{def.}}{=\!=} s_j p(t)$, $j=1, 2, \cdots$, upon both A and B, we denote them by $U_0(x, t; A, B)$ and $\phi_j(t; A, B)$. If $A_1\leqslant A_2$ and $B_1\leqslant B_2$, then*

$$U_0(x,t;A_1,B_1)\leqslant U_0(x,t;A_2,B_2) \text{ for all } x\in\mathbf{R},\ t>0$$

and

$$\phi_j(t;A_1,B_1)\geqslant\phi_j(t;A_2,B_2) \text{ for all } t>0,\ j=1,2,\cdots.$$

Here it must be pointed out that when $g(y)\equiv1$ and $p(t)=t$, the jump condition (9) is exactly what is known as the Rankine-Hugoniot condition in gas dynamics, while the hypothesis that $\frac{y-A}{B-A}F(B)-F(y)$ is nonnegative on $[A, B]$ is exactly what is called the entropy inequality (see [5, p. 246-254]).

From the theorems mentioned above, we can draw the following conclusions:

1°. The solution $V_0(s)$ is a. c. on $\mathbf{R}$ and the solution $V_\varepsilon(s)$ uniformly converges to $V_0(s)$, if and only if the function $k(y)>0$ a. e. on $[A,B]$.

2°. When $k(y)$ possesses at least one interval of degeneracy in $[A, B]$, $V_0(s)$ has at least one jump point; when $k(y)$ possesses infinite many intervals of degeneracy in $[A,B]$, $V_0(s)$ has infinitely many jump points, because there exists a one-to-one correspondence between the collection of all intervals of degeneracy of $k(y)$ in $[A.B]$ and the set of all jump points of $V_0(s)$.

3°. As for as the discontinuous solution $U_0(x, t)$ is concerned, its curve of discontinuity, $x=s_jp(t)$, $j=1, 2, \cdots$, possibly has only the locally absolute continuity, and on its curve of discontinuity there are possibly some segments parallel to the t-axis, which symbolize waiting time.

§ 3. Converting into a Two-Point Boundary Value Problem

In this section we transform the Riemann problem $(3)_\varepsilon$-(4) into the boundary value problem $(1)_\varepsilon$-(2) and then convert the later into the two-point boundary value problem $(5)_\varepsilon$-$(6)_0$.

We shall call the function $U_\varepsilon(x,t)$ a solution to the Riemann problem $(3)_\varepsilon$-(4), $\varepsilon>0$, if it satisfies the following conditions:

(a) $U_\varepsilon(x, t)$ is defined and continuous on the domain $\mathbf{R}\times(0, +\infty)$,

(b) $\lim\limits_{t\to 0} U_\varepsilon(x,t)=A+(B-A)H(x)$ for almost all $x\in\mathbf{R}$,

(c) $D_tU_\varepsilon(x, t)$, $D_xU_\varepsilon(x, t)$, and $D_x((k(U_\varepsilon(x, t))+\varepsilon)|D_xU_\varepsilon(x, t)|^{N-1}D_xU_\varepsilon(x, t)$ exist a. e. on the domain $\mathbf{R}\times(0, +\infty)$, and

(d) $D_tU_\varepsilon(x,t)=p'(t)h^N(t)D_x((k(U_\varepsilon(x, t))+\varepsilon)|D_xU_\varepsilon(x, t)|^{N-1}D_xU_\varepsilon(x, t))+p'(t)D_xF(U_\varepsilon(x,t))$ holds a. e. on $\mathbf{R}\times(0, +\infty)$.

If the solution $U_\varepsilon(x,t)$ converges to a limit, denoted by $U_0(x,t)$, pointwise on the domain $\mathbf{R}\times(0,+\infty)$ as $\varepsilon\downarrow 0$, then $U_0(x,t)$ is called a generalized solution to the Riemann problem $(3)_0$-(4).

We seek similarity solutions of the form

$$U(x,t)\overset{\text{def.}}{=\!=}V(s),\quad s=x/p(t),$$

then we arrive at the boundary value problem $(1)_\varepsilon$-(2). Conversely if $V(s)$ is a solution to the the boundary value problem $(1)_\varepsilon$-(2), then $U(x, t)\overset{\text{def.}}{=\!=}V\left(\frac{x}{p(t)}\right)$ must be a solution to the Riemann problem $(3)_\varepsilon$-(4). So we shall consider only the boundary value problem $(1)_\varepsilon$-(2) in what follows.

By a solution to the boundary value problem $(1)_\varepsilon$-(2), $\varepsilon>0$, we mean the function $V_\varepsilon(s)$ satisfying the following conditions:

(a) $V_\varepsilon(s)$ is defined and a. c. on the whole real axis $\mathbf{R}$,

(b) $V_\varepsilon(-\infty)=A$ and $V_\varepsilon(+\infty)=B$,

(c) $(k(V_\varepsilon(s))+\varepsilon)|V_\varepsilon'(s)|^{N-1}V_\varepsilon'(s)$ is equivalent to an a. c. function defined on $\mathbf{R}$, and

(d) equality $(1)_\varepsilon$ holds a. e. on $\mathbf{R}$.

If the function $V_\varepsilon(s)$ converges to a limit, denoted by $V_0(s)$, pointwise on $\mathbf{R}$ as $\varepsilon\downarrow 0$, then $V_0(s)$ is called a generalized solution to the Riemann problem $(3)_0$-(4).

The following lemma is a starting point for dealing with our problem.

Lemma 1. *The solution $V_\varepsilon(s)$ to problem $(1)_\varepsilon$-(2), $\varepsilon>0$, is increasing.*

Proof Clearly, it is enough to show that $V_\varepsilon(s)$ is monotone on $\mathbf{R}$. If this is not the case, then there will be numbers a, $b\in\mathbf{R}$, $a<b$, and C such that $V_\varepsilon(a)=V_\varepsilon(b)=C$ and (say) $V_\varepsilon(s)>C$ in (a, b). Integrating equation $(1)_\varepsilon$ with $V=V_\varepsilon(s)$ over

(a,b) yields

$$0 \geqslant (k(V_\varepsilon(s))+\varepsilon)\,|V_\varepsilon'(s)|^{N-1}V_\varepsilon'(s)\Big|_{s=a}^{s=b} = \int_a^b (G(V_\varepsilon(s)) - G_\varepsilon(C))\,ds > 0,$$

which is a contradiction, by the assumption that $V_\varepsilon(s) > C$ in (a,b). This shows that $V_\varepsilon(s)$ is monotone.

Let $V(s)$ be a solution to the boundary value problem $(1)_\varepsilon$-(2). Then it is certainly increasing on $\mathbf{R}$ and $V'(-\infty)=V'(+\infty)=0$. Further, if it is strictly increasing on $\mathbf{R}$, then the function $s=Z(y)$, inverse to $y=V(s)$, exists, $s=Z(V(s))$ on $\mathbf{R}$, $y=V(Z(y))$ in (A,B), and $V'(Z(y))=1/Z'(y)$ holds a. e. in (A,B). Substituting $s=Z(y)$ into equation $(1)_\varepsilon$ and then putting $W(y) \overset{\text{def.}}{=\!=} (k(y)+\varepsilon)/(Z'(y))^N$, we formally obtain the two-point boundary value problem $(5)_\varepsilon$-$(6)_0$ and the equality

$$Z(y) = -(W'(y)+f(y))/g(y) \text{ for almost all } y \in (A,B). \tag{11}$$

§ 4 The Two-Point Boundary Value Problem

In this section we explore the two-point boundary value problem $(5)_\varepsilon$-$(6)_0$. As the two points $y=A$ and $y=B$ are singular in the problem, we need to consider the two-point boundary value problem without singular endpoints

$$-\left(\frac{W'(y)+f(y)}{g(y)}\right)' = \left(\frac{k(y)+\varepsilon}{W(y)}\right)^{1/N} \quad \text{for } y \in (A,B), \tag{5_ε}$$

$$W(A)=h,\ W(B)=h;\ h>0. \tag{6_h}$$

By a solution to the two-point boundary value problem $(5)_\varepsilon$-$(6)_h$, $\varepsilon \geqslant 0$, $h \geqslant 0$, we mean the function $W_\varepsilon(y;h)$ satisfying the following conditions:

(a) $W_\varepsilon(y;h)$ is a.c. and nonnegative on $[A, B]$, $W_\varepsilon(A;h)=W_\varepsilon(B;h)=h \geqslant 0$, and

(b) $Z_\varepsilon(y;h) \overset{\text{def.}}{=\!=} -(W'(y;\ h)+f(y))/g(y)$ is equivalent to a locally a.c. function defined in (A,B) such that equality $(5)_\varepsilon$ holds a.e. in (A,B).

Let $W(y)$ be a solution to problem $(5)_\varepsilon$-$(6)_h$. Then it can be represented by the formula

$$W(y) = \int_a^b J_{ab}(y,s)Q_\varepsilon(s,W(s))\,ds + E(y) - F(y) \text{ for all } y \in [a,b], \tag{12_{ab}}$$

where $[a,b]$ is any subinterval of $[A,B]$,

$$Q_\varepsilon(y,w) \overset{\text{def.}}{=\!=} ((k(y)+\varepsilon)/w)^{1/N} \text{ for } y \in [A,B],\ w \in (0,+\infty),$$

$$J_{ab}(y,s) \overset{\text{def.}}{=\!=} \begin{cases} (G(b)-G(y))(G(s)-G(a))/(G(b)-G(a)) & \text{for } a \leqslant s \leqslant y, \\ (G(b)-G(s))(G(y)-G(a))/(G(b)-G(a)) & \text{for } y \leqslant s \leqslant b, \end{cases}$$

is continuous and positive in $(a,b)\times(a,b)$, and

$$E(y)\overset{\text{def.}}{=\!=}((W(a)+F(a))(G(b)-G(y))+(W(b)+F(b))(G(y)-G(a)))/(G(b)-G(a)) \text{ on } [a,b].$$

Moreover, the function $Z(y)\overset{\text{def.}}{=\!=}-(W'(y)+f(y))/g(y)$ is given by the formula

$$Z(y)=\int_a^y \frac{G(s)-G(a)}{G(b)-G(a)}Q_\varepsilon(s,W(s))ds-\int_y^b \frac{G(b)-G(s)}{G(b)-G(a)}Q_\varepsilon(s,W(s))ds$$
$$+(W(a)-W(b)+F(a)-F(b))/(G(b)-G(a)) \text{ for almost all } y\in(A,B). \tag{13$_{ab}$}$$

For preciseness' sake, when $a=A$, $b=B$, and $W(A)=W(B)=h\geqslant 0$, the two formulae are labelled with $(12)^h_{AB}$ and $(13)^h_{AB}$, respectively. Next, when $k(y)=0$ a. e. on $[A,B]$, equation $(5)_0$ is linear and the formulae are still valid, in this case

$$W_0(y;h)=G(y)F(B)/G(B)-F(y) \text{ and } Z_0(y;h)=-F(B)/G(B) \text{ on } [A,B]. \tag{14}$$

Conversely, it is esaily seen that a nonnegative a. e. solution $W_\varepsilon(y;h)$ to the integral equation $(12)^h_{AB}$ is a solution to problem $(5)_\varepsilon$-$(6)_h$.

We now prove the existence and uniqueness of the solution $W_\varepsilon(y;h)$ and expose some of its properties we need later.

Lemma 2. *If* $\varepsilon_1\geqslant\varepsilon_2\geqslant 0$, *then for each fixed* $h>0$

$$0\geqslant W_{\varepsilon_1}(y;h)-W_{\varepsilon_2}(y;h)\leqslant\int_A^B J_{AB}(y,s)((k(s)+\varepsilon_1)^{1/N}-(k(s)+\varepsilon_2)^{1/N})h^{-\frac{1}{N}}ds \text{ on } [A,B].$$

Proof We first show the left inequality. If this is not the case, then there will be a point $y=Y$ where $W_{\varepsilon_1}(y;h)-W_{\varepsilon_2}(Y;h)<0$. Whence it follows that there exists a maximal interval (a,b) containing the point $y=Y$ such that

$$W_{\varepsilon_1}(y;h)-W_{\varepsilon_2}(y;h)<0 \text{ in } (a,b) \text{ and } W_{\varepsilon_1}(a;h)-W_{\varepsilon_2}(a;h)=W_{\varepsilon_1}(b;h)-W_{\varepsilon_2}(b;h)=0.$$

By the aid of formula $(12)_{ab}$, we come to a contradiction

$$0<W_{\varepsilon_1}(y;h)-W_{\varepsilon_2}(y;h)=\int_a^b J_{ab}(y,s)(Q_{\varepsilon_1}(s,W_{\varepsilon_1}(s;h))-Q_{\varepsilon_2}(s,W_{\varepsilon_2}(s;h)))ds\leqslant 0.$$

This shows that the left inequality holds.

From the left inequality and the formula $(12)^h_{AB}$ it follows that

$$0\leqslant W_{\varepsilon_1}(y;h)-W_{\varepsilon_2}(y;h)\leqslant\int_A^B J_{AB}(y,s)(Q_{\varepsilon_1}(s,W_{\varepsilon_2}(s;h))-Q_{\varepsilon_2}(s,W_{\varepsilon_2}(s;h)))ds$$
$$\leqslant\int_A^B J_{AB}(y,s)(Q_{\varepsilon_1}(s,h)-Q_{\varepsilon_2}(s,h))ds \text{ for all } y\in[A,B].$$

Here we have used the fact that $W_\varepsilon(y;h)\geqslant h$ for all $y\in[A,B]$.

In very much the same way, we can prove the following two lemmas.

Lemma 3. *If* $h_1\geqslant h_2>0$, *then* $0\leqslant W_\varepsilon(y;h_1)-W_\varepsilon(y;h_2)\leqslant h_1-h_2$ *for all* $y\in[A,B]$.

Lemma 4. *Let* $W_1(y)$ *and* $W_2(y)$ *be solutions to equation* $(5)_\varepsilon$ *on* $[A,B]$. *If* $W_1(a)=W_2(a)$ *and* $W_1(b)=W_2(b)$, $A\leqslant a<b\leqslant B$, *then* $W_1(y)\equiv W_2(y)$ *on* $[A,B]$. *In particular, the problem* $(5)_\varepsilon$-$(6)_h$, $\varepsilon\geqslant 0$, $h\geqslant 0$, *has at most one solution* $W_\varepsilon(t;h)$.

To prove the existence of the solution $W_s(y;h)$, we define a mapping M by the right hand side of the integral equation $(12)^h_{AB}$ with $h>0$

$$(Mw)(y) \overset{\text{def.}}{=\!=} \int_A^B J_{AB}(y, s)Q_s(s, w(s))ds + \frac{F(B)}{G(B)}G(y) - F(y) + h \text{ for all } w(s) \in X,$$

where $X \overset{\text{def.}}{=\!=} \{w(s) \in C[A,B];\ h \leqslant w(s) \leqslant (Mh)(s)\}$, and $C[A, B]$ is the set of all continuous functions defined on $[A, B]$. By hypotheses H_1-H_4, it is easy to show that M is a compactly continuous mapping from X into X. The Schauder Fixed Point Theorem tells us that the mapping M has at least one fixed point, say $W_s(y;h)$, in the set X. Obviously, the fixed point $W_s(y;h) \geqslant h$ is an a. c. solution to the integral equation $(12)^h_{AB}$ with $h>0$, namely, a solution to the two-point boundary value problem $(5)_s$-$(6)_h$ with $h>0$.

Next, Lemma 3 shows that the solution $W_s(y;h)$ converges to a limit, denoted by $W_s(y;0)$, uniformly on $[A, B]$, as $h\downarrow 0$. Inserting $W_s(y;h)$, $h>0$, into the integral equation $(12)^h_{AB}$ and then letting $h\downarrow 0$, we obtain the equality $(12)^0_{AB}$, by the Monotone Convergence Theorem. This shows that the limit function $W_s(y;0)$ is a solution to the two-point boundary value problem $(5)_s$-$(6)_0$.

We summarize the above results in the following statement.

Theorem 5. *Under hypotheses* H_1-H_4, *the two-point boundary value probem* $(5)_s$-$(6)_h$, $s\geqslant 0$, $h\geqslant 0$, *has a unique solution* $W_s(y;h)$. *Moreover, the function* $Z_s(y;h) \overset{\text{def}}{=\!=} -(W'(y;h)+f(y))/g(y)$ *is equivalent to an increasing, locally a. c. function defined in* (A,B).

Theorem 6. *As* $s\downarrow 0$, *the solution* $W_s(y;0)$ *converges to the solution* $W_0(y;0)$ *uniformly on* $[A, B]$, *and the function* $Z_s(y;0)$ *converges to the function* $Z_0(y;0)$ *uniformly on* $[A+\delta, B-\delta]$ *whenever* $2\delta \in (0, B-A)$.

Proof We first prove the first assertion. According to Lemma3, for $h>0$

$$0 \leqslant W_0(y;0) \leqslant W_0(y;h) \leqslant W_0(y;0)+h \text{ for all } y \in [A,B].$$

In terms of Lemma 2, for each fixed $h>0$, $W_s(y;h)$ converges to $W_0(y;h)$ uniformly on $[A,B]$, as $s\downarrow 0$. Hence, there is a positive number s_h such that

$$W_{s_h}(y;h) \leqslant W_0(y;h)+h \leqslant W_0(y;0)+2h \text{ for all } y \in [A,B].$$

Note that $W_s(y;h)$ is increasing in both s and h. Consequently, for all $s \in (0, s_h)$

$$0 \leqslant W_0(y;0) \leqslant W_s(y;0) \leqslant W_{ss}(y;h) \leqslant W_0(y;0)+2h \text{ for all } y \in [A,B],$$

which shows that $W_0(y;0)$ is the uniform limit of $W_s(y;0)$ as $s\downarrow 0$.

We now prove the second assertion. In terms of formulae $(12)^0_{AB}$ and $(13)^0_{AB}$, we have

$$|Z_s(y;0) - Z_0(y;0)| \leqslant (W_s(y;0) - W_0(y;0))/m_\delta \text{ for all } y \in [A+\delta, B-\delta],$$

where $m_\delta \overset{\text{def.}}{=\!=} \min\ \{G(A+\delta),\ G(B)-G(B-\delta)\}$. This shows that $Z_s(y;0)$ converges to $Z_0(y;0)$ uniformly on $[A+\delta, B-\delta]$ as $s\downarrow 0$, whenever $2\delta \in (0, B-A)$.

Theorems 5 and 6 point out that under hypotheses H_1-H_4 the two-point boundary value problem involving two small parameters ε and h is a regular perturbation problem.

We now inquire into the behaviour of the function $Z_\varepsilon(y) \overset{\text{def.}}{=\!=} Z_\varepsilon(y;\,0)$ at the two endpoints $y=A$ and $y=B$. For this purpose, we introduce hypothesis H_3^s. In fact, the hypothesis H_3^s is a special case of hypothesis H_3, because

$$\frac{F(B)}{G(B)}G(y)-F(y)=\int_A^B J_{AB}(y,s)\left(\frac{f(s)}{g(s)}\right)' ds \geqslant 0 \text{ for all } y\in[A,B].$$

Theorem 7. *Suppose that hypotheses* H_1, H_2, H_3^s, *and* H_4 *hold. If and only if*

$$\int_A^{(A+B)/2}\left(\frac{k(s)+\varepsilon}{G(s)}\right)^{1/N} ds<+\infty \quad \left(\int_{(A+B)/2}^{B}\left(\frac{k(s)+\varepsilon}{G(B)-G(s)}\right)^{1/N} ds<+\infty\right),$$

$Z_\varepsilon(A+0)$ $(Z_\varepsilon(B-0))$ *is finite.*

Proof We first conclude that $-W_\varepsilon(T(r))$, as a function of the variable r, is an a. c. convex function defined on $[0,\,G(B)]$, where

$$W_\varepsilon(y) \overset{\text{def.}}{=\!=} W_\varepsilon(y;0),$$

and the function $y=T(r)$ is inverse to the function $r=G(y)$. In fact

$$-\frac{d^2}{dr^2}W(T(r))=\frac{d^2}{dr^2}F(T(r))+\frac{1}{g(T(r))}Q_\varepsilon(T(r),W_\varepsilon(T(r)))\geqslant 0 \text{ a. e. in } (0,G(B)).$$

Here we have used Propositions 1 and 3, which will be given in the next section.

If $Z_\varepsilon(A+0)$ is finite, then $\theta_0 \overset{\text{def.}}{=\!=} \frac{d}{dr}W_\varepsilon(T(r))\,|_{r=+0}$ is positive, by $(13)^0_{AB}$. From tne convexity of $-W_\varepsilon(T(r))$, it follows that $W_\varepsilon(T(r))\leqslant\theta_0 r$ on $[0,\,G(B)]$, namely, $W_\varepsilon(y)\leqslant\theta_0 G(y)$ on $[A,B]$. By the aid of formula $(13)^0_{AB}$, we obtain

$$\begin{aligned}-Z_\varepsilon(A+0)-\frac{F(B)}{G(B)}&=\int_A^B\frac{G(B)-G(s)}{G(B)}Q_\varepsilon(s,W_\varepsilon(s))ds\\&\geqslant\left(1-\frac{G((A+B)/2)}{G(B)}\right)\int_A^{(A+B)/2}\left(\frac{k(s)+\varepsilon}{\theta_0 G(s)}\right)^{1/N}ds\geqslant 0,\end{aligned}$$

which shows that

$$\int_A^{(A+B)/2}\left(\frac{k(s)+\varepsilon}{G(s)}\right)^{1/N}ds<+\infty. \tag{15}$$

Conversely, assume that condition (15) holds. From the convexity of $-W_\varepsilon(T(r))$, it follows that $W_\varepsilon(T(r))\geqslant\theta_1 r$ on $[0,\,G((A+B)/2)]$, namely, $W_\varepsilon(y)\geqslant\theta_1 G(y)$ on $[A,(A+B)/2]$, where $\theta_1 \overset{\text{def.}}{=\!=} W_\varepsilon((A+B)/2)/G((A+B)/2)$. In terms of the representation $(13)^0_{AB}$, we obtain

$$\begin{aligned}\frac{F(B)}{G(B)}\leqslant -Z\,(A+0)&=\frac{F(B)}{G(B)}+\left(\int_A^{(A+B)/2}+\int_{(A+B)/2}^B\right)\frac{G(B)-G(s)}{G(B)}Q_\varepsilon(s,W_\varepsilon(s))ds\\&\leqslant\frac{F((A+B)/2)}{G((A+B)/2)}+\int^{(A+B)/2}\left(\frac{k(s)+\varepsilon}{\theta_1 G(s)}\right)^{1/N}ds+\theta_1<+\infty,\end{aligned}$$

which shows that $Z_\varepsilon(A+0)$ is finite. Here we have used formula $(12)^0_{AB}$, to estimate the second integral. The proof is complete.

Theorem 8. *We denote t e solution $W_s(y)$ and the function $Z_s(y)$ by $W_s(y; A, B)$ and $Z_s(y;A,B)$, respectively, to investigate their dependence upon A and B.*

If $A_1\leqslant A_2<B$, then the following two statements hold:

(a) $0\leqslant W_s(y;A_1,B)-W_s(y;A_2,B)\leqslant W_s(A_2;A_1,B)$ *for all* $y\in[A_2,B]$, *and*

(b) $0\leqslant Z_s(y;A_1,B)-W_s(y;A_2,B)\leqslant 2W_s(A_2;A_1,B)/(G(y)-G(A_2))$ *in* (A_2,B).

If $A<B_1\leqslant B_2$, then the following two statements hold:

(c) $0\leqslant W_s(y;A,B_2)-W_s(y;A,B_1)\leqslant W_s(B_1;A,B_2)$ *for all* $y\in[A,B_1]$, *and*

(d) $0\leqslant Z_s(y;A, B_1)-Z_s(y A,B_2)\leqslant 2W_s(B_1;A,B_2)/(G(B_1)-G(y))$ *in* (A,B_1).

Proof. The assertion that $W_s(y;A_1,B)-W_s(y;A_2, B)\geqslant 0$ directly follows from the fact that $W_s(A_2;A_1,B)\geqslant 0=W_s(A_2;A_2,B)$, by Lemma 4.

By representation $(12)_{A_2B}$, it follows from the assertion proved above that

$$\begin{aligned}0\leqslant W_s(y;A_1,B)-W_s(y;A_2,B)&=\int_{A_2}^{B} J_{A_2B}(y,s)\,(Q_s(s,W_s(s;A_1,B))\\&\quad-Q_s(s,W_s(s;A_2,B)))ds+\frac{G(B)-G(y)}{G(B)-G(A_2)}W_s(A_2;A_1,B)\\&\leqslant W_s(A_2;A_1,B) \text{ for all } [A_2,B].\end{aligned}\tag{16}$$

The statement has been proven.

Next, from formula $(13)_{aB}, A_2\leqslant a<B$, we have

$$\begin{aligned}Z_s(y;A_1,B)-Z_s(y;A_2, B)&=\int_y^B\frac{G(B)-G(s)}{G(B)-G(a)}(Q_s(s,W_s(s;A_2,B))\\&\quad-Q_s(s,W_s(s;A_1,B)))ds\\&\quad-\int_a^y\frac{G(s)-G(a)}{G(B)-G(a)}(Q_s(s,W_s(s;A_2,B))\\&\quad-Q_s(s,W_s(s;A_1,B)))ds+(W_s(a;A_1,B)\\&\quad-W_s(a;A_2,B))/(G(B)-G(a) \text{ for } y\in(a,B).\end{aligned}$$

Putting $a=y$ in the above gives

$$Z_s(y;A_1,B)-Z_s(y;A_2,B)\geqslant 0 \text{ in } (A_2,B),$$

and then putting $a=A_2$ yields

$$0\leqslant Z_s(y;A_1,B)-Z_s(y;A_2,B)=\left(\frac{1}{G(y)-G(A_2)}+\frac{1}{G(B)-G(A_2)}\right)W_s(A_2;A_1,B)$$

in (A_2,B),

by inequality (16). This shows that the statement (b) is valid.

In very much the same way, we can prove the statements (c) and (d).

§ 5. The Boundary Value Problem $(1)_s$-(2)

In the last section we construct the solution $V_s(s)$ to the boundary value problem $(1)_s$-(2), utilizing the unique solution $W_s(y)$ to the two-point boundary value problem $(5)_\varepsilon$-$(6)_0$.

We first introduce several propositions and definitions.

Proposition 1. *If $V(s)$ is an increasing function defined on $[a, b]$ and if $W(y)$ is an a. c. function defined on $[V(a), V(b)]$, then $W(V(s))$ has a finite derivative a. e. on $[a,b]$ and the chain rule*

$$\frac{d}{ds}W(V(s))=W(V(s))V'(s)$$

holds a. e. on $[a,b]$, where $w(y)$ is any function equivalent to $W'(y)$.

Proposition 2. *If $V(s)$ is an increasing a. c. function defined on $[a, b]$ and if $w(y)$ is a Lebesgue integrable function defined on $[V(a), V(b)]$, then $w(V(s))V'(s)$ is integrable on $[a,b]$ and the change of variables formula*

$$\int_a^b w(V(s))V'(s)\,ds=\int_{V(a)}^{V(b)} W'(y)dy=W(V(b))-W(V(a))$$

holds, where $W(y)$ is an indefinite integral of $w(y)$.

As Propositions 1 and 2 are Corollaries 4 and 6 in [6], respectively, we omit their proofs here.

Proposition 3. *If (a, b) is a finite open interval and if $Z(y)$ is a strictly increasing, locally a. c. function defined in $(a. b)$, then the function $y=V(s)$, inverse to $s=Z(y)$, is a strictly increasing, a. c. function defined in (s_a, s_b), where $s_a \stackrel{\text{def.}}{=\!=} Z(a+0)$ and $s_b \stackrel{\text{def.}}{=\!=} Z(b-0)$. Moreover, $\lim\limits_{s\downarrow s_a} V(s)=a$ and $\lim\limits_{s\uparrow s} V(s)=b$.*

Proof Clearly, it suffices to show the absolute continuity of $V(s)$. By definition, $y=V(Z(y))$ for $y\in(a, b)$ and $s=Z(V(s))$ for $s\in(s_a, s_b)$. Hence, the equality $V'(s)=1/Z'(V(s))$ holds a. e. in (s_a,s_b), by Proposition 1.

As $V(s)$ is (strictly) increasing in (s_a, s_b), $V'(s)$ is integrable on any closed subinterval $[\alpha,\beta]$. Integrating $V^p(s)$ over (α,β) yields

$$\int_\alpha^\beta V'(s)ds=\int_\alpha^\beta \frac{ds}{Z'(V(s))}=\int_{V(\alpha)}^{V(\beta)} \frac{Z'(y)dy}{Z'(V(Z(y)))}=V(\beta)-V(\alpha),$$

by the change of variables formula. And then letting $\alpha\downarrow s_a$ and $\beta\uparrow s_b$ gives

$$\int_{s_a}^{s_b} V'(s)ds=V(s_b)-V(s_a)=b-a,$$

which shows that $V(s)$ is a. c. in (s_a,s_b).

Definition 1. *Let (A,B) be a finite open interval and $s=Z(y)$ an increasing, locally a. c. function defined in (A,B). The function $y=V(s)$ is said to be generalized inverse to $s=Z(y)$, if it is defined on the whole s-axis, increasing, and possibly multiple-valued, so that its total variation is equal to $B-A$ and its graph is the same locally rectifiable continuous curve as the graph of $s=Z(y)$, provided that at each endpoint of the latter, if necessary, a half-line parallel to the s-axis is joined on to it.*

By the definition, $\lim\limits_{s\downarrow s_A} V(s)=A$ and $\lim\limits_{s\uparrow s_B} V(s)=B$, where $s_A \stackrel{\text{def.}}{=\!=} Z(A+0)$ and

$s_B \stackrel{\text{def.}}{=\!=} Z(B-0)$. Obviously, $V(s)=A$ for $s\leqslant s_A$ or $V(s)=B$ for $s\geqslant s_B$ when s_A or s_B is finite. For example, when $Z(y)\equiv 0$ in (A,B), $V(s)=A+(B-A)H(s)$ on $\mathbb{R}$.

Definition 2. *Let $y=V_\varepsilon(s)$, $\varepsilon>0$, and $y=V(s)$ be locally rectifiable continuous curves on the s-y plane. If for any $\delta>0$ there exists an $\varepsilon_\delta>0$ such that the maximal distance between the two curves is not greater than δ, whenever $\varepsilon<\varepsilon_\delta$, then we say that the curve $y=V_\varepsilon(s)$ uniformly approximates to the curve $y=V_0(s)$ as ε tends to zero.*

By the definition, if the curve $y=V_\varepsilon(s)$ uniformly approximates to the curve $y=V_0(s)$ as $\varepsilon\downarrow 0$, then the function $V_\varepsilon(s)$ pointwise converges to the function $V_0(s)$ as $\varepsilon\downarrow 0$.

Definition 3. *Let $k(y)$ be a nonnegative measurable function defined on $[A, B]$ such that $k^{1/N}(y)$ is Lebesgue integrable on $[A, B]$, where $N>0$. A closed subinterval $[a,b]$ is called an interval of degeneracy of the function $k(y)$ in $[A, B]$, if the following two conditions hold:*

(i) *$k(y)=0$ a. e. on $[a,b]$, and*

(ii) *for any $\delta>0$, when $a-\delta\leqslant A$ or $b+\delta\leqslant B$, $\int_{a-\delta}^{a} k^{1/N}(s)ds>0$ or $\int_{b}^{b+\delta} k^{1/N}(s)\,ds>0$.*

By the definition, between any two adjacent intervals of degeneracy of $k(y)$ in $[A, B]$ there exists one and only one open interval; otherwise, they would combine into one. Consequently, the complement of the union of all intervals of degeneracy of the function $k(y)$ in $[A,B]$ must be open.

We now construct the solution $V_\varepsilon(s)$ to the boundary value problem $(1)_\varepsilon$-(2). Let $W_\varepsilon(y)$ be the unique solution to the two-point boundary value problem$(5)_\varepsilon$-$(6)_0$. Theorem 5 tells us that the function

$$Z_\varepsilon(y) \stackrel{\text{def.}}{=\!=} -(W'_\varepsilon(y)+f(y))/g(y) \quad \text{for } y\in(A,B) \tag{11}$$

is equivalent to an increasing, locally a. c. function defined in (A, B). In what follows we always regard $Z_\varepsilon(y)$ as an increasing, locally a. c. function defined in (A,B). Hence, the function $y=V_\varepsilon(s)$, generalized inverse to $s=Z_\varepsilon(y)$, exists and satisfies the boundary conditions in (2). We assert that the function $V_\varepsilon(s)$ ($V_0(s)$) is a unique (generalized) solution to the boundary value problem $(1)_\varepsilon$-(2) (the reduced problem $(1)_0$-(2)).

We first prove the following statement.

Lemma 5. *If $\varepsilon<0$ or $k(y)>0$ a. e. on $[A,B]$, then $V_\varepsilon(s)$ is a solution.*

Proof From (11) and $(5)_\varepsilon$ it follows that

$$Z'_\varepsilon(y)=((k(y)+\varepsilon)/W_\varepsilon(y))^{1/N} \quad \text{for almost all } y\in(A,B). \tag{17}$$

Hence, $Z_\varepsilon(y)$ is strictly increasing in (A,B) under the considered condition, and the restriction of the generalized inverse function $V_\varepsilon(s)$ to (s_A, s_B) is exactly the inverse function of $Z_\varepsilon(y)$, where $s_A \overset{\text{def.}}{=\!=} Z_\varepsilon\ (A+0)$ and $s_B \overset{\text{def.}}{=\!=} Z_\varepsilon(B-0)$. Proposition 3 tells us that the function $V_\varepsilon(s)$ is strictly increasing and a. c. in (s_A, s_B), and further is a. c. on the whole s-axis.

From (17) it follows that

$$W_\varepsilon(y)=(k(y)+\varepsilon)/(Z'_\varepsilon(y))^N \text{ for almost all } y\in(A,B), \tag{18}$$

because $Z'_\varepsilon(y)>0$ a. e. in (A,B). Inserting $y=V_\varepsilon(s)$ into (18) yields

$$W_\varepsilon(V_\varepsilon(s))=(k(V_\varepsilon(s))+\varepsilon)(V'_\varepsilon(s))^N \text{ for almost all } s\in(s_A,s_B), \tag{19}$$

here we have used the fact that $1/Z'_\varepsilon(V_\varepsilon(s))=V'_\varepsilon(s)$ a. e. in (s_A,s_B). Moreover,

$$\lim_{s\to s_A,s_B}(k(V_\varepsilon(s))+\varepsilon)(V'_\varepsilon(s))^N=\lim_{s\to s_A,s_B}W_\varepsilon(V_\varepsilon(s))=0.$$

This shows that $(k(V_\varepsilon(s))+\varepsilon)(V'_\varepsilon(s))^N$ is (equivalent to) a continuous function defined on $\mathbf{R}$. By Proposition 1, it follows from (19) that

$$((k(V_\varepsilon(s))+\varepsilon)|V'_\varepsilon(s)|^{N-1}V'_\varepsilon(s))'=W'_\varepsilon(V_\varepsilon(s))V'_\varepsilon(s) \text{ a. e. in } (s_A,s_B). \tag{20}$$

Hence, for any closed subinterval $[a,b]$ of (s_A,s_B)

$$\int_a^b((k(V_\varepsilon(s))+\varepsilon)|V'_\varepsilon(s)|^{N-1}V'_\varepsilon(s))'ds=\int_a^b W'_\varepsilon(V_\varepsilon(s))V'_\varepsilon(s)ds$$

$$=W_\varepsilon(V_\varepsilon(b))-W_\varepsilon(V_\varepsilon(a))=(k(V_\varepsilon(s))+\varepsilon)|V'_\varepsilon(s)|^{N-1}V'_\varepsilon(s)\Big|_{s=b}^{s=a},$$

by Proposition 2. Letting $a\downarrow s_A$ and $b\uparrow s_B$ in the above gives

$$\int_{s_A}^{t_B}((k(V_\varepsilon(s))+\varepsilon)|V'_\varepsilon(s)|^{N-1}V'_\varepsilon(s))'ds=W_\varepsilon(B)-W_\varepsilon(A)=0;$$

and further, for all $s\in\mathbf{R}$

$$(k(V_\varepsilon(s))+\varepsilon)|V'_\varepsilon(s)|^{N-1}V'_\varepsilon(s)=\int_{-\infty}^{s}((k(V_\varepsilon(s))+\varepsilon)|V'_\varepsilon(t)|^{N-1}V'_\varepsilon(t))'dt$$

because $((k(V_\varepsilon(s))+\varepsilon)|V'_\varepsilon(s)|^{N-1}V'_\varepsilon(s)\equiv0$ on $(-\infty, s_A]$ or $[s_B,+\infty)$ when s_A or s_B is finite. This shows that $((k(V_\varepsilon(s))+\varepsilon)|V'_\varepsilon(s)|^{N-1}V'_\varepsilon(s)$ is equivalent to a nonnegative a. c. function defined on $\mathbf{R}$ with the value 0 at $s=-\infty$ and $s=+\infty$.

From (11), we obtain

$$W'_\varepsilon(y)=-(Z_\varepsilon(y)g(y)+f(y)) \text{ for almost all } y\in(A,B). \tag{21}$$

Substituting $y=V_\varepsilon(s)$ into (21) and then inserting the resultant into (20) yields

$$((k(V_\varepsilon(s))+\varepsilon)|V'_\varepsilon(s)|^{N-1}V'_\varepsilon(s))'=-(sg(V_\varepsilon(s))+f(V_\varepsilon(s)))V'_\varepsilon(s) \tag{1$_\varepsilon$}$$

for almost all $s\in(s_A,s_B)$; when s_A or s_B is finite, the equality reads $0=0$ in $(-\infty, s_A)$ or $(s_B,+\infty)$. All the facts proven above show that $V_\varepsilon(s)$ is a solution when $\varepsilon>0$ or $k(y)>0$ a. e.

An immediate consequence of Theorem 6 is the following statement.

Lemma 6. *Let $V_\varepsilon(s)$, $\varepsilon\geqslant0$, be generalized inverse to $Z_\varepsilon(y)$. Then $V_\varepsilon(s)$ converges to $V_0(s)$ pointwise $\mathbf{R}$, as $\varepsilon\downarrow0$.*

Lemmas 5 and 6 point out that the function $V_0(s)$, generalized inverse to $Z_0(y)$, must be a generalized solution to the reduced problem $(1)_0$-(2); moreover,

$V_0(s)$ is also a solution when $k(y)>0$ a. e. on $[A, B]$. We now turn to the case that the function $k(y)$ has at least one interval of degeneracy in $[A, B]$.

Let $\{[a_j, b_j]; j=1, 2, \cdots\}$ be the collection of all intervals of degeneracy of $k(y)$ in $[A, B]$. It follows from (17) that $Z_0'(y)$ has the same intervals of degeneracy in $[A, B]$ as $k(y)$. As $Z_0'(y)>0$ a. e. in the open set $(A, B)\setminus\bigcup_j[a_j, b_j]$, $Z_0(y)$ is strictly increasing in the open set. Obviously, on each interval of degeneracy, $[a_j, b_j]$, $j=1, 2,\cdots$, $Z_0(y)=s_j=$constant, and $Z_0(a_j-0)=Z_0(b_j+0)=s_j$. Hence, $s=s_j$, $j=1, 2,\cdots$, is a jump point of $V_0(s)$, at which

$$V_0(s_j-0)=s_j,\ V_0(s_j+0)=b_j, \text{ and } V_0(s_j)=[a_j, b_j]. \tag{8}$$

Because $Z_0(y)$ is strictly increasing and locally a. c. in each connected component of the open set $(A, B)\setminus\bigcup_j[a_j, b_j]$, $V_0(s)$ is strictly increasing and a. c. in each connected component of the open set $(s_A, s_B)\setminus\bigcup_j\{s=s_j\}$, by Proposition 3. Here we have used the fact that $Z_0((A, B)\setminus\bigcup_j[a_j, b_j])=(s_A, s_B)\setminus\bigcup_j\{s=s_j\}$. Repeating the arguments of Lemma 5, we reach the following conclusion: in each connected component of the open set $(s_A, s_B)\setminus\bigcup_j\{s=s_j\}$, $k(V_0(s))|V_0'(s)|^{N-1}V_0'(s)$ is equivalent to a strictly increasing, a. c. function so that equlity $(1)_0$ a. e. holds.

Integrating equlity (21) over $[a_j,b_j]$, $j=1,2,\cdots$, yields

$$(k(V_0(s))|V_0'(s)|^{N-1}V_0'(s)+sG(V_0(s))+F(V_0(s)))\Big|_{s=s_j-0}^{s=s_j+0}=0, \tag{7}$$

by equality (19) and the jump condition (8).

As $V_0(s)$ is the pointwise limit of the absolute continuous solution $V_\varepsilon(s)$ to problem $(1)_\varepsilon$-(2), the jump conditions (7) and (8) are essential, natural properties which the discontinuous solution $V_0(s)$ of the reduced problem possesses.

From the above mentioned, it is easy to see that $V_0(s)$ can be represented by

$$V_0(s)=A+\sum_j(b_j-a_j)H(s-s_j)+\int_{-\infty}^{s}V_0'(t)dt \text{ for all } s\in\mathbf{R},$$

which is what is known as the Lebesgue decomposition of the function $V_0(s)$ relative to the Lebesgue measure.

Furthermore, Theorems 2 and 4 are respectively direct consequences of Theorems 7 and 8, and the uniqueness of the solution $V_0(s)$, as the pointwise limit of the solution $V(s)$, is an immediate consequence of Theorem 4. Up to now we have completed the proofs of Theorems 1-4.

Remark 1. From the jump condition (7), it can be seen that the jump point $s=s_j$, $j=1, 2,\cdots$, of the solution $V_0(s)$ is not known a priori, unless $k(y)\equiv 0$.

Remark 2. The solution $V_0(s)$ to the reduced problem $(1)_0$-(2) depends upon its perturbation procedure. For example, a boundary value problem of the form

$$(sg(V(s))+f(V(s)))V'(s)=0 \text{ for } s\in\mathbf{R},$$

$$V(-\infty)=0,\ V(+\infty)=1$$

can be regarded as a reduced problem of the boundary value problem

$$\varepsilon V''(s)+(sg(V(s))+f(V(s)))h(V(s))V'(s)=0 \text{ for } s\in\mathbf{R},$$
$$V(-\infty)=0,\ V(+\infty)=1,$$

where $f(y)$ and $g(y)$ are functions satisfying hypothesis H_3^*, and $h(s)$ is a positive continuous function defined on [0, 1]. In terms of formulae (22) and (14), we obtain

$$V_0(s)=H(s-s_1) \text{ for } s\in\mathbf{R},\ s_1 \overset{\text{def.}}{=\!=} -\int_0^1 h(s)f(s)\,ds \Big/ \int_0^1 h(s)g(s)ds.$$

This shows that the unique jump point $s=s_1$ explicitly dependes on the values of the function $h(y)$ which can be chosen arbitrarily.

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