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HOW BIG ARE THE LAG INCREMENTS OF A TWO PARAMETER WIENER PROCESS?**

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Abstract

The author discusses how big the lag increments of a two-parameter Wiener process are and proves the same results as in [4]. These results extende and improve the results of the increments of a two-parameter Wiener process in [1—3].

§1. Introduction

Let $\{W(x, y), 0\leqslant x, y<\infty\}$ be a two-parameter Wiener process and let $0<a_T\leqslant\beta$, $b_T\geqslant T^{1/2}$ be non-decreasing function of T. Set $R=[x_1, x_2]\times[y_1, y_2]\in R_+^2$. $\lambda(R)=(x_2-x_1)(y_2-y_1)$ denotes the area of R. Let

$$W(R)\triangleq W(x_2, y_2)-W(x_1, y_2)-W(x_2, y_1)+W(x_1, y_1),$$
$$D_T\triangleq D_T(b_T)=\{(x, y): xy\leqslant T,\ 0\leqslant x,\ y\leqslant b_T\},$$
$$D_T^*\triangleq D_T^*(b_T)=\{(x, y): xy=T,\ 0\leqslant x,\ y\leqslant b_T\},$$
$$L_T\triangleq L_T(a_T, b_T)=\{R: R\subset D_T,\ \lambda(R)\leqslant a_T\},$$
$$L_T^*\triangleq L_T^*(a_T, b_T)=\{R: R\subset D_T,\ \lambda(R)=a_T\}.$$

Csörgö and Révész[1] have discussed how big the increments of a two-parameter Wiener process are. They proved

Theorem A. *Define*

$$\delta_T=\{2a_T(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)+\log\log T)\}^{-1/2}.$$

Suppose that

(i) *δ_T is a non-increasing function of T,*

(ii) *Ta_T^{-1} is a non-decreasing function of T,*

(iii) *for any $\varepsilon>0$ there exists a $\theta_0=\theta_0(\varepsilon)>1$ such that*

$$\limsup_{k\to\infty}\delta_{\theta^k}/\delta_{\theta^{k+1}}\leqslant 1+\varepsilon$$

if $1<\theta<\theta_0$. Then

$$\limsup_{T\to\infty}\sup_{R\in L_T}\delta_T|W(R)|=\limsup_{T\to\infty}\sup_{R\in L_T^*}\delta_T|W(R)|=1\quad a.s. \tag{1}$$

If we also have

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(iv) $\lim \frac{\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)}{\log\log T}=\infty$,

then limsup can be instead by lim in (1).

Lin Zhengyan[2] weakened (iv) to

(iv′) $\lim\limits_{T\to\infty} \frac{\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)}{\log\log\log T}=\infty$,

and defined

$$\lambda_T=\{2a_T(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1))\}^{-1/2}.$$

Suppose that λ_T is a non-increasing function of T and for any $\varepsilon>0$ there exists a $\theta_0=\theta_0(\varepsilon)>1$ such that

(iii′) $\limsup\limits_{k\to\infty} \lambda_{\theta^k}/\lambda_{\theta^{k+1}}\leqslant 1+\varepsilon$

if $1<\theta\leqslant\theta_0$. Then

$$\liminf_{T\uparrow\infty}\sup_{R\in\tilde{L}_T}\lambda_T|W(R)|+\liminf_{T\to\infty}\sup_{R\in L^*_H}\lambda_T|W(R)=1 \quad \text{a. s.} \tag{2}$$

Kong Fanchao[3] proved that if the conditions (i), (ii) and

(iv″) $\lim\limits_{T\to\infty} \frac{\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)}{\log\log T}=r,\quad 0\leqslant r<\infty$,

are satisfied, then

$$\liminf_{T\to\infty}\sup_{R\in\tilde{L}_T}\delta_T|W(R)|=\liminf_{T\to\infty}\sup_{R\in L^*_T}\delta_T|W(R)|=\sqrt{\frac{r}{r+1}} \tag{3}$$

$$\limsup_{T\to\infty}\sup_{R\in L_T}\delta_T|W(R)|=\limsup_{T\to\infty}\sup_{R\in L^*_T}\delta_T|W(R)|=1, \tag{4}$$

$$\limsup_{T\to\infty}\sup_{R\in\tilde{L}_T}\lambda_T|W(R)|=\limsup_{T\to\infty}\sup_{R\in\tilde{L}^*_T}\lambda_T|W(R)|=\sqrt{\frac{r+1}{r}}. \tag{5}$$

The lag increments of a one-parameter Wiener process have been discussed in [4] and [5]. In this paper, we discuss how big the lag increments of a two-parameter Wiener process are. Let

$$L^*_T(t)\triangleq L^T_*(t,\ b_T,\ T)=\{R:R\subset D_T,\ x_2y_2=T,\ \lambda(R)=t\},$$

$$L_T(t)\triangleq L_T(t,\ b_T,\ T)=\{R:R\subset D_T,\ x_2y_2=T,\ \lambda(R)\leqslant t\},$$

$$\tilde{L}_T(t)\triangleq\tilde{L}_T(t,\ b_T,\ T)=\{R:R\subset D_T,\ x_2y_2=t',\ \lambda(R)\leqslant t,\ t\leqslant t'\leqslant T\},$$

$$\beta_{T,t}\triangleq\beta(T,\ t)=\{2t(\log T t^{-1}+\log(\log b_Tt^{-\frac{1}{2}}+1)+\log\log t)\}^{-\frac{1}{2}}.$$

Here and in the sequel we shall define $\log t=\log(\max(t,\ 1))$,

$$\log\log t=\log\log(\max(t,\ e)).$$

It is easy to see that $L^*_T(t)\subset L_T(t)\cap\tilde{L}_T(t)$,

$$\bigcup_{0<t\leqslant T}L^*_T(t)=\bigcup_{0<t\leqslant T}L_T(t),\quad \bigcup_{0<t\leqslant T}\tilde{L}_T(t)=\{R:R\subset D_T\}.$$

Theorem 1. *Define*

$$\gamma_T=\beta_{T,T}=\{2T(\log(\log b_TT^{-1/2}+1)+\log\log T)\}^{-1/2}.$$

Suppose that

(a) *γ_T is a non-increasing function of T,*

(b) *for any $\varepsilon>0$ there exists a $\theta_0=\theta_0(\varepsilon)>1$ such that*

$$\limsup_{k\to\infty} \gamma_{\theta^k}/\gamma_{\theta^{k+1}} \leqslant 1+\varepsilon$$

if $1<\theta\leqslant\theta_0$. *Then*

$$\limsup_{T\to\infty} \sup_{0<t\leqslant T} \sup_{R\in L_T^*(t)} \beta(T,t)|W(R)|=1 \quad a.\ s. \tag{6}$$

$$\limsup_{T\to\infty} \sup_{0<t\leqslant T} \sup_{R\in L_T(t)} \beta(T,t)|W(R)|=1 \quad a.\ s. \tag{7}$$

$$\lim_{T\to\infty} \sup_{0<t\leqslant T} \sup_{R\in \tilde{L}_T(t)} \beta(T,t)|W(R)=1 \quad a.\ s. \tag{8}$$

Theorem 2. *Let* $a>0$, $t\geqslant0$. *Suppose that*

(a) $\beta(t+a, a)$ *is a non-increasing function of* a,

(b) *for any* $\varepsilon>0$ *there exists a* $\theta_0=\theta_0(\varepsilon)>1$ *such that*

$$\sup_{t\geqslant0} \frac{\beta(t+\theta^k, \theta^k)}{\beta(t+\theta^{k+1}, \theta^{k+1})} \leqslant \theta^{1/2}(1+\varepsilon), \tag{9}$$

$$\sup_{n\geqslant1} \frac{\beta(\theta^{(n-1)k}, \theta^k)}{\beta(\theta^{nk}, \theta^k)} \leqslant 1+\varepsilon, \tag{10}$$

if integer $n>0$ *and* $1<\theta\leqslant\theta_0$. *Then we have*

$$\limsup_{a\to\infty} \sup_{0\leqslant t} \sup_{R\in L^*_{t+a}(a)} \beta(t+a,a)|W(R)|=1 \quad a.\ s. \tag{11}$$

$$\limsup_{a\to\infty} \sup_{0\leqslant t} \sup_{R\in L_{t+a}(a)} \beta(t+a,a)|W(R)|=1 \quad a.\ s. \tag{12}$$

$$\limsup_{a\to\infty} \sup_{0\leqslant t} \sup_{R\in \tilde{L}_{t+a}(a)} \beta(t+a,a)|W(R)|=1 \quad a.\ s. \tag{13}$$

Theorem 3. *Let* $0<a_T\leqslant T$, $a_T\to\infty$, *and conditions* (*a*) *and* (*b*) *of Theorem* 2 *are satisfied. Then*

$$\limsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{R\in L^*_{t+a_T}(a_T)} \beta(t+a_T, a_T)|W(R)|\leqslant1 \quad a.\ s. \tag{14}$$

$$\limsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{R\in L_{t+a_T}(a_T)} \beta(t+a_T, a_T)|W(R)|\leqslant1 \quad a.\ s. \tag{15}$$

If a_T is onto and γ_T satisfies the conditions (a) and (b) in Theorem 1, then we have equality in (14) and (15)

Furthermore, if we also have

(c) $a_T\to\infty$ continuously as $T\to\infty$ and $\beta(T,t)$, a_T/T is a non-increasing function of T and t, and

(iv) $$\lim_{T\to\infty} \frac{\log Ta_T^{-1}+\log(\log b_T a_T^{-1/2}+1)}{\log\log a_T}=\infty,$$

then limsup can be instead by lim and we have equality in (14) and (15).

§ 2. Proof of Theorem 1

1°) At first, from [1, Theorem 1.12.4] we have

$$\text{the left hand side of (6)} \geqslant \limsup_{T\to\infty} \sup_{(x,y)\in L_T^*} \gamma_T|W(x,y)|=1 \quad \text{a. s.} \tag{16}$$

and it is easy to see that

$$\text{the left hand side of (6)} \leqslant \text{the left hand side of (7)}$$
$$\leqslant \limsup_{T\to\infty} \sup_{0<t\leqslant T} \sup_{R\in \tilde{L}_T(t)} \beta(T,t)|W(R)|. \tag{17}$$

2°) In order to prove that

$$\limsup_{T\to\infty} \sup_{0<t\leq T} \sup_{R\in\tilde{L}_T(t)} \beta(T, t)\,|W(R)| \leq 1 \quad \text{a. s.} \tag{18}$$

we take real numbers $\theta>1$ and $v>1$ such that $1<2(1+\varepsilon)^2/((2+\varepsilon)\theta)=1+2\varepsilon'$ $(\varepsilon'>0)$. For $n=1, 2, \cdots$: $k=\cdots, -1, 0, 1, \cdots, k_n$, denote $T_n=v^n$, $t_k=\theta^k$, where $k_n=[((n+1)\log v)/\log\theta]+1$, $\beta=2/\varepsilon'$, $k_\theta=[(\log\theta)^{-1}]$, $k_n'=[(\log\theta)^{-1}\log(T_{n+1}(\log T_n)^\beta)]$. Here $[\alpha]$ is the integer part of real number α.

When $T_n<T\leq T_{n+1}$, we have

$$\begin{aligned}&\sup_{0<t\leq T}\sup_{R\in\tilde{L}_T(t)}\beta(T, t)\,|W(R)|\\ &\leq \sup_{-\infty<k\leq k_n-1}\ \sup_{R\in\tilde{L}_{T_{n+1}}(t_k, t_{k+1})}[2t_k(\log T_n t_{k+1}^{-n}+\log(\log b_T t_{k+1}^{-1})+\\ &\quad \log\log t_k)\}^{-1/2}|W(R)|\\ &\triangleq \sup_{-\infty<k\leq k_n-1} A_{nk},\end{aligned} \tag{10}$$

where $\tilde{L}_{T_{n+1}}(t_k, t_{k+1})=\{R;\ R\in D_{t'}(b_T),\ t_k\leq\lambda(R)=t\leq t_{k+1},\ t\leq t'\leq T_{n+1}\}$.

Since $\tilde{L}_{T_{n+1}}(t_k, t_{k+1})\subset\hat{L}_{T_{n+1}}(t_{k+1}, b_T, T_{n+1})\triangleq\{R: R\subset D_{T_{n+1}}(b_T),\ \lambda(R)\leq t_{k+1}\}$, using [1, Theorem 1.12.6] for $-\infty<k\leq k_\theta$, we have

$$\begin{aligned}&P(A_{nk}\geq 1+\varepsilon)\\ &\quad\leq P\{\sup_{R\in\tilde{L}_{T_{n+1}}(t_k, t_{k+1})}|W(R)|\\ &\quad\geq(1+\varepsilon)\{2t_k(\log T_n t_{k-1}^{-1}+\log b_T t_{k+1}^{-1/2}+1)+\log\log t_k)\}^{1/2}\}\\ &\quad\leq P\{\sup_{R\in\hat{L}_{T_{n+1}}(t_{k+1}, b_T, T_{n+1})}|W(R)|\,t_{k+1}^{-1/2}\\ &\quad\geq(1+\varepsilon)(2\theta^{-1}(\log T_n t_{k+1}^{-1}+\log(\log b_T t_{k+1}^{-1/2}+1)))^{1/2}\}\\ &\quad\leq C\frac{T_{n+1}}{t_{k+1}}\Big(1+\log\frac{T_{n+1}}{t_{k+1}}\Big)\Big(1+\log\frac{b_T}{\sqrt{t_{k+1}}}\Big)\\ &\qquad\exp\Big\{-\frac{2(1+\varepsilon)^2}{(2+\varepsilon)\theta}\Big(\log\frac{T_{n+1}}{t_{k+1}}+\log\Big(\log\frac{b_T}{\sqrt{t_{k+1}}}+1\Big)\Big)\Big\}\\ &\quad\leq C(\theta^{k+1}v^{-n})^{2\varepsilon'}(1+\log v^{n+1}\theta^{-k-1}).\end{aligned}$$

Here and in the sequel C denotes a constant, and can be assumed to be different values on each of its appearance even within the same formula. Hence it follows that

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{-\infty<k\leq k_\theta}P(A_{nk}\geq 1+\varepsilon)\\ &\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\theta^{-k}v^{-n})^{2\varepsilon'}\log(v^{n+1}\theta^k)+C\sum_{n=1}^{\infty}(k_\theta+1)\frac{(\theta e)^{2\varepsilon'}}{v^{2n\varepsilon'}}<\infty.\end{aligned} \tag{20}$$

For the case $k_\theta<k\leq k_n-1$, from [1, Theorem 1.12.6], we have

$$\begin{aligned}&P(A_{nk}\geq 1+\varepsilon)\\ &\quad\leq C\frac{T_{n+1}}{t_{k+1}}\Big(1+\log\frac{T_{n+1}}{t_{k+1}}\Big)\Big(1+\log\frac{b_T}{\sqrt{t_{k+1}}}\Big)\exp\Big\{-\frac{2(1+\varepsilon)^2}{(2+\varepsilon)\theta}\Big(\log\frac{T_{n+1}}{t_{k+1}}\\ &\qquad+\log\Big(\log\frac{b_T}{\sqrt{t_{k+1}}}+1\Big)+\log\log t_k\Big)\Big\}\end{aligned}$$

$$\leqslant C\left(\frac{t_{k+1}}{T_n}\right)^{2\varepsilon'}(\log t_k)^{-1-2\varepsilon'}\left(1+\log\frac{T_{n+1}}{t_{k+1}}\right).$$

By the same discussion as [5], we have

$$\sum_{n=1}^{\infty}\sum_{k=k'_n+1}^{k_n'}P\{A_{nk}\geqslant 1+\varepsilon\}<\infty \tag{21}$$

and

$$\sum_{n=1}^{\infty}\sum_{k=k'_n+1}^{k_n-1}P\{A_{nk}\geqslant 1+\varepsilon\}<\infty, \tag{22}$$

so that we get

$$\sum_{n=1}^{\infty}P\{\sup_{-\infty<k\leqslant k_n-1}A_{nk}\geqslant 1+\varepsilon\}<\infty,$$

and (18) follows by the Borel-Cantelli lemma. Thus, merging (16), (17) and (18) together, we proved (6) and (7).

3°) In order to prove (8), it is enough to show that

$$I=\liminf_{T\to\infty}\sup_{0<t\leqslant T}\sup_{R\in\bar{L}_T(t)}\beta(T,t)|W(R)|\geqslant 1.\quad \text{a. s.} \tag{23}$$

Denote $n=[T]$, $\tilde{L}_n(1,b_T)=\{R: R\subset D_T,\ x_2y_2=t',\ \lambda(R)=1,\ 1\leqslant t'\leqslant n\}\subset\tilde{L}_T(1)$. Let

$$A_{i+1}=\left[\left(\frac{n-1}{n}\right)^{i+1}b_T,\left(\frac{n-1}{n}\right)^{i}b_T\right]\times\left[0,\frac{n^{i+1}}{(n-1)^{i}b_T}\right],\ i=0,1,\cdots,l,$$

where $l=\max\{i: n^{i+1}<(n-1)^i b_T^2\}$. It is easy to see that $A_i\in\tilde{L}_n(1,b_T)$ $(1<\leqslant i\leqslant l+1)$, $l\approx Cn\log(b_T^2n^{-1})$ and we have

$$\begin{aligned}I&\geqslant\liminf_{T\to\infty}\sup_{R\in\tilde{L}_T(1)}\beta(T,1)|W(R)|\\&\geqslant\liminf_{T\to\infty}\sup_{R\in\tilde{L}_n(1,b_T)}\{2(\log(n+1)+\log(\log b_T+1))\}^{-1/2}|W(R)|\\&\geqslant\liminf_{T\to\infty}\max_{1\leqslant i\leqslant l+1}\{2(\log(n+1)+\log(\log b_T+1))\}^{-1/2}|W(A_i)|.\end{aligned}$$

By using the wellknown estimate of the tail probability

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right)\leqslant 1-\Phi(x)\leqslant\frac{1}{x\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\quad(x>0),$$

it follows that

$$\begin{aligned}&\sum_{n=1}^{\infty}P\{\max_{1\leqslant i\leqslant l+1}|W(A_i)|\leqslant(2(1-\varepsilon)(\log(n+1)+\log(\log b_T+1)))^{\frac{1}{2}}\}\\&\quad\leqslant\sum_{n=1}^{\infty}\{1-\exp(-(1-\varepsilon)(\log(n+1)+\log(\log b_T+1)))\}^{cn\log(b_T^2n^{-1})}\\&\quad\leqslant\sum_{n=1}^{\infty}\exp\{-C(n\log b_n)^{\varepsilon}\}<\infty.\end{aligned}$$

Hence, by using the Borel-Cantelli lemma, it implies that (23) is true.

§ 3. Proof of Theorem 2

Let $1<\theta\leqslant\theta_0$ be as in (b), $a_k=\theta^k$,

$L_{k,n}=L_{a^{n_k}}(a_k,b_{a^{n_k}})$, $\bar{L}_{t+a_k}(a_k)\subset\{R: R\subset D_{t+a_k},\ \lambda(R)\leqslant a_k,\ x_2y_2=t'\leqslant t+a_k\}$.

we have $\bar{L}_{t+a_k}(a_k)\subset L_{k,n}$ for $a_k^{n-1}-a_k\leqslant t\leqslant a_k^n-a_k$. Denote

$$A_k=\sup_{2\leqslant n<\infty}\ \sup_{a^{n-1}_k-a_k\leqslant t\leqslant a^n_k-a_k}\ \sup_{R\in\bar{L}_{t+a_k}(a_k)}\beta(t+a_k,a_k)W(R).$$

From [1, Theorem 1.12.6] and (10), we get

$$\sum_{k=1}^{\infty}P(A_k\geqslant 1+2\varepsilon)\leqslant\sum_{k=1}^{\infty}\sum_{n=2}^{\infty}P\{\sup_{R\in L_{k,n}}\beta(a_k^n,a_k)|W(R)|\geqslant 1+\varepsilon\}$$

$$\leqslant\sum_{k=1}^{\infty}\sum_{n=2}^{\infty}Ca_k^{n-1}(1+\log a_k^{n-1})(1+\log b_{a^n_k}a_k^{-1/2})$$

$$\times\exp\left\{-\frac{2(1+\varepsilon)^2}{2+\varepsilon}(\log a_k^{n-1}+\log(\log b_{a^n_k}a_k^{-1/2}+1)+\log\log a_k)\right\}$$

$$\leqslant C\sum_{k=1}^{\infty}\sum_{n=2}^{\infty}\theta^{-\frac{3}{2}nk\varepsilon}(1+nk\log\theta)(\log k)^{-(1+\varepsilon)}(1+\log b_{a^n_k}a_k^{-1/2})^{-\varepsilon}<\infty.$$

By the Borel-Cantelli lemma that implies

$$\limsup_{k\to\infty}A_k\leqslant 1\quad\text{a. s.}$$

From (a), (b) and this inequality we have

$$\limsup_{a\to\infty}\sup_{0\leqslant t}\sup_{R\in\bar{L}_{t+a}(a)}\beta(t+a,a)|W(R)|$$

$$=\limsup_{k\to\infty}\sup_{a_k\leqslant a\leqslant a_{k+1}}\sup_{0\leqslant t}\sup_{R\in\bar{L}_{t+a}(a)}\beta(t+a,a)|W(R)|$$

$$\leqslant\limsup_{k\to\infty}\sup_{0\leqslant t}\sup_{R\in\bar{L}_{t+a_{k+1}}(a_{k+1})}\beta(t+a_{k+1},a_{k+1})|W(R)|\sup_{0\leqslant t}\frac{\beta(t+a_k,a_k)}{\beta(t+a_{k+1},a_{k+1})}$$

$$\leqslant\limsup_{k\to\infty}A_k\cdot\theta^{1/2}(1+\varepsilon)\leqslant\theta^{1/2}(1+\varepsilon)\quad\text{a. s.}$$

if $1<\theta\leqslant\theta_0$, so we prove

$$\limsup_{a\to\infty}\sup_{0\leqslant t}\sup_{R\in\bar{L}_{t+a}(a)}\beta(t+a,a)|W(R)|\leqslant 1\quad\text{a. s.}$$

In order to finish the proof of Theorem 2, we notice that

$$\liminf_{a\to\infty}\sup_{0\leqslant t}\sup_{R\in L^*_{t+a}(a)}\beta(t+a,a)|W(R)|$$

$$\geqslant\liminf_{a\to\infty}\sup_{R\in L^*_{a^2}(a)}\beta(a^2,a)|W(R)|=\liminf_{T\to\infty}\sup_{R\in L^*_T(T^{1/2})}\beta(T,\sqrt{T})|W(R)|.$$

By the same proof as in the step 3 of the proof of [1, Theorem 1.12.5], if we take $a_T=T$, we have

$$\liminf_{T\to\infty}\sup_{R\in L^*_T(\sqrt{T})}\beta(T,\sqrt{T})W(R)\geqslant 1\quad\text{a. s.}$$

§4. Proof of Theorem 3

1°) It is easy to see that (11) and (12) imply (14) and (15). If a_T is onto, from [1, Theorim 1.12.4] we have

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{R\in L^*_{t+a_T}(a_T)}\beta(t+a_T,a_T)|W(R)|$$

$$\geqslant\limsup_{T\to\infty}\sup_{R\in L^*_{t+a_T}(a_T)}\beta(a_T,a_T)|W(R)|=\limsup_{T\to\infty}\sup_{R\in L^*_T(T)}\beta(T,T)|W(R)|$$

$$\geqslant \limsup_{T\to\infty} \sup_{(x,y)\in D^*_T} \gamma_T |W(x, y)| = 1 \quad \text{a. s.}$$

2°) If (c) and (iv) is satisfied, let us now to prove that

$$\liminf_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{R\in L^*_{t+a_T}(T)} \beta(t+a_T, a_T)\,|W(R)| \geqslant 1 \quad \text{a. s.} \tag{24}$$

Not loss of generality, we can assumethat a_T is non-decreasing. In fact, let $a'_T = \sup_{0\leqslant t\leqslant T} a_T$, then $\{a'_T\}$ is non-decreasing, continuous and satisfies (iv), and there exists a T' such that $0\leqslant T'\leqslant T$, $a'_{T'} = a_T$ for large T. Therefore we also have

$$\text{the left hand side of (24)} \geqslant \liminf_{T\to\infty} \sup_{0\leqslant t\leqslant T-a'_T} \sup_{R\in L^*_{t+a_T}(a'_T)} \beta(t+a'_T, a)\,|W(R)|. \tag{25}$$

First we prove that there exists a sequence of integers $\{T_k\}$, $T_k \uparrow \infty$ $(k\to\infty)$, such that

$$\liminf_{k\to\infty} \sup_{0\leqslant t\leqslant T_k-a_{T_k}} \sup_{R\in L^*_{t+a_k}k(a_{T_k})} \beta(t+a_{T_k}, a_{T_k})\,|W(R)| \geqslant 1 \quad \text{a. s.} \tag{26}$$

In the case of $\lim_{T\to\infty} a_T/T = P < 1$, denote that $E_1 = \{T\colon\ b_T a_T^{-1/2} \geqslant T a_T^{-1}\}$, $E_2 = \{T\colon\ b_T a_T^{-1/2} < a_T^{-1}\}$, and we may as well assume that the two sets of positive real numbers are unbounded. Take

$$A'_i = \left[\left(\frac{T-a_T}{T}\right)^{i+1} b_T, \left(\frac{T-a_T}{T}\right)^{i} b_T\right] \times \left[0, \frac{T^{i+1}}{(T-a_T)^i b_T}\right], \quad i = 0, 1, \cdots, l,$$

where l is the greatest integer satisfying $T^{l+1} \leqslant (T-a_T)^l b_T^2$. Then we have $x_2 y_2 = T$, $\lambda(A'_i) = a_T$ and $l \approx C \dfrac{T}{a_T} \log \dfrac{b_T^2}{T}$. By the estimate of the tail probability, $1-\Phi(x)$, it follows that

$$\begin{aligned}
&P\{\max_{0\leqslant i\leqslant l} \beta(T, a_T)\,|W(A'_i)| \leqslant 1-\varepsilon\} \\
&\leqslant \left\{1 - \left(T a_T^{-1}\left(1+\log\frac{b_T}{\sqrt{a_T}}\right)\log a_T\right)^{-(1-\varepsilon)}\right\}^{C T a_T^{-1} \log b_T^2 T^{-1}} \\
&\leqslant \exp\left\{-\,(T a_T^{-1}(1+\log b_T a_T^{-1/2}))^{\varepsilon} (\log a_T)^{-(1-\varepsilon)} \times \frac{\log b^2 T^{-1}}{\log b_T a_T^{-1/2}}\right\}.
\end{aligned} \tag{27}$$

Since $(\log b_T^2 T^{-1})/(\log b_T a_T^{-1/2}) \geqslant 1$ for large $T\in E_1$ and from (iv) we have

$$(T a_T^{-1}(1+\log b_T a_T^{-1/2}))^{\varepsilon} (\log a_T)^{-(1-\varepsilon)} \geqslant (\log a_T)^2. \tag{28}$$

Let us take T_k such that $a_{T_k} = \theta^k$. Thus, using the Borel-Cantelli lemma together with (27) and (28), we get

$$\liminf_{k\to\infty} \max_{0\leqslant i\leqslant l} \beta(T_k, \theta^k)\,|W(A'_i)| \geqslant 1 \quad \text{a. s.}$$

So we have

$$\liminf_{k\to\infty} \sup_{R\in L^*_{T_k}(\theta^n)} \beta(T_k, \theta^k)\,|W(R)| \geqslant 1 \quad \text{a. s.}, \tag{29}$$

and then (26) is true for $T_k \in E_1$.

For the large $T\in E_2$, we take $0 = x_0 < x_1 < \cdots < x_m \leqslant b_T < x_{m+1}$ so that $(x_i - x_{i-1}) y_i = a_T$. Here $y_i = b_T$, if $0 < i < i_0$; $y_i = T/x_i$, if $i_0 < i \leqslant m+1$, $i_0 = [T a_T^{-1}] + 1$. It is easy to see that

$$m > [T a_T^{-}] - 1 + (\log b_T^2 T^{-1})/(\log T (T-a_T)^{-1}) > T a_T^{-1} - 1.$$

Take

$$A_i=[x_{i-1}, x_i]\times[0, y_i].$$

Since $\beta(t+a_T, a_T)$ is a non-increasing function of t, we have

$$\text{the left hand side of (24)}\geqslant\liminf_{T\to\infty}\max_{1\leqslant i\leqslant m}\beta(T, a_T)\,|W(A_i)|,$$

and $\log b_T a_T^{-1/2}<\log T a_T^{-1}=o(Ta_T^{-1})$ for large $T\in E_2$. Then by the estimate of the tail probability $1-\Phi(x)$ and (iv) we have

$$\begin{aligned}&P\{\max_{0\leqslant i\leqslant m}|W(A_i)|\leqslant(1-\varepsilon)\beta(T, a_T)^{-1}\}\\ &\leqslant\{1-C(Ta_T^{-1}(1+\log b_T a_T^{-1/2})\log a_T)^{-(1-\varepsilon)}\}^m\\ &\leqslant\exp\{-(\log a_T)^2\}.\end{aligned}$$

Let us take T_k such that $a_{T_k}=\theta^k$. Then by the same discussion as above, it implies that (29) is also true. So it proved that (26) holds true.

For the case of $\lim_{T\to\infty} a_T/T=1$, we can discuss it similarly as in [2].

The remainder of the proof consists of filling in the gaps of the sequence T_k For any given $T>0$ there is a k such that $T_k\leqslant T\leqslant T_{k+1}$. We have

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T-a_T}\ \sup_{R\in L^*_{t+a_T}(a_T)}\beta(t+a_T, a_T)\,|W(R)|\\ &\geqslant\sup_{0\leqslant t\leqslant T_k-a_{T_k}}\ \sup_{R\in L^*_{t+a_T}(a_{T_k})}\beta(t+a_T, a_T)\,|W(R)|\\ &\quad-4\sup_{0\leqslant t\leqslant T-a_T}\ \sup_{R\in L_{t+a_T}(a_T-a_{T_k})}\beta(t+a_T, a_T)\,|W(R)|\\ &\triangleq I_1+4I_2.\end{aligned}$$

From (26) and the conditions (a), (b) of Theorem 2, we get

$$\begin{aligned}\liminf_{k\to\infty} I_1\geqslant&\liminf_{k\to\infty}\sup_{0\leqslant t\leqslant T_k-a_{T_k}}\ \sup_{R\in L^*_{t+a_{T_k}}(a_{T_k})}\beta(t+a_{T_k}, a_{T_k})\,|W(R)|\\ &\times\frac{\beta(t+\theta^{k+1}, \theta^{k+1})}{\beta(t+\theta^k, \theta^k)}\geqslant 1.\end{aligned}$$

On the other hand, from the proof of Theorem 2 we have

$$\begin{aligned}\limsup_{T\to\infty} I_2\leqslant&\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-(a_T a_{T_k})}\ \sup_{R\in L_{t+a_k-a_{T_k}}(a_T-a_{T_k})}\beta(t+a_T-a_{T_k}, a_T-a_{T_k})\\ &\times|W(R)|\frac{\beta(t+a_{T_k}, a_{T_k})}{\beta(t+a_T-a_{T_k}, a_T-a_{T_k})}\\ \leqslant&\limsup_{k\to\infty}\sup_{0\leqslant t}\frac{\beta(t+\theta^k, \theta^k)}{\beta(t+\theta^k(\theta-1), \theta^k(\theta-1))}\to 0\end{aligned}$$

when $\theta\to 1$. So it proved that (24) is true.

Remark The corresponding conclusions of (2) and (3) can be reached, for example we have

Theorem 4. *Define*

$$\lambda(T, t)=\{2t(\log Tt^{-1}+\log(\log b_T t^{-1/2}+1))\}^{-1/2}.$$

Let $0<a_T\leqslant T$, $a_T\nearrow\infty$ *continuously and suppose*

(a′) $\lambda(t+a, a)$ *is a non-increasing function of* a,

(b′) *for any given* $\varepsilon>0$ *there exists a* $\theta_0=\theta_0(\varepsilon)>1$ *such that for any* $k>0$ *and*

$1<\theta\leqslant\theta_0$, *we have*

$$\sup_{n\geqslant 1}\lambda(\theta^{(n-1)k},\ \theta^k)/\lambda(\theta^{nk},\ \theta^k)\leqslant 1+\varepsilon.$$

and

$$\sup_{t\geqslant 0}\lambda(t+\theta^k,\ \theta^k)/\lambda(t+\theta^{k+1},\ \theta^{k+1})\leqslant\theta^{1/2}(1+\varepsilon),$$

if (iv′) *is satisfied. Then*

$$\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{R\in L^*_{t+a_T}(a_T)}\lambda(t+a_T,\ a_T)\,|W(R)|$$
$$=\liminf_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{R\in L_{t+a_T}(a_T)}\lambda(t+a_T,\ a_T)\,|W(R)|=1\quad \text{a. s.}$$

The proof of Theorem 4 is similar to the proof of Theorem 3 and [2], so it is omitted here.

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