Chin. Ann. of Math.
12B:1(1991), 131—136.

HORN-TYPE THEOREM IN FRÉCHET SPACES AND APPLICATION**

LIANG JIN(梁进)*   HUANG FALUN(黄发伦)*

Abstract

This paper establishes a Horn-type fixed point theorem in Fréchet spaces which is an answer to Burton's problem[6] in these spaces. As an application of the result, the authors obtain an existence theorem of periodic solutions for functional differential equations with infinite delay.

§1. Introduction

Horn has showed a fixed point theorem in Banach spaces in [1], it is the extension of Browder's Theorem 2 in [2]. This fixed point theorem has proved very useful in getting the existence theorem of periodic solutions for the functional differential equations with infinite delay (see [6—9]). In this paper, we shall set up this Horn-type fixed point theorem in Fréchet space which is an answer to Burton's problem[6] in these spaces, and as an application of the result, we obtain an existence theorem of periodic solutions for functional differential equations with infinite delay in Fréchet spaces.

§2. Some Lemma

Lemma 1[1]. *Given a complex K, let K_0 be a subcomplex and C_0 a closed, bounded, acyclic subset of K_0. Suppose that f is a simplicial mapping of the nth barycentric subdivison of K into K such that, for some positive integer m, $f^j(K_0)\subset C_0$ for $m\leqslant j\leqslant 2m-1$. Then f has a fixed point in C_0.*

Lemma 2[1]. *Let X be a finite-dimensional linear topological space and let $S_0\subset S_1\subset S_2$ be bounded convex sets of X such that S_0 and S_2 are closed and S_1 is a neighborhood of S_0, relative to S_2. Let f: $S_2\to X$ be a continuous map such that for some integer $m>0$ we have*

$$f^j(S_1)\subset S_2,\ 1\leqslant j\leqslant m-1,$$

Maunscript received February, 2, 1988. Revised March 26, 1990

* Department of Mathematics, Sichuan University, Chendu, Sichuan, China.

** Projects supported by National Natural Science Foundation of China.

and

$$f^j(S_1)\subset S_0., \ m\leqslant j\leqslant 2m-1.$$

Then f has a fixed point in S_0.

Lemma 3[3]. *Let X be any metrizable space and $A\subset X$ a closed subset. Let E be any locally convex linear space. Then any f: $A\to E$ has an extension F: $X\to E$ with $F(X)\subset conv\, f(A)$. In particular, there exists a retraction of a Fréchet space onto any compact, convex subset of itself.*

Lemma 4[5]. *Lex X be a Fréchet space, with topology τ. Then there exists a countable separating family of seminorms $\{p_i\}$ such that the metric defined by*

$$\rho(x, y)=\sum_{i=1}^{\infty}2^{-i}\frac{p_i(x-y)}{1+p_i(x-y)}, \text{ for any } x, y\in X$$

is compatible with τ.

Lemma 5. *Lex X be a Fréchet space and let $D\subset K$ be subset of X, let f be a uniformly continuous mapping of a set K into itself. For any given integer $m>0$ and any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that, if g: $D\to D$ is a selfmapping on D and $\|g(x)-f(x)\|<\delta$ on D, then $\|g^j(x)-f^j(x)\|<\varepsilon$ on D for $1\leqslant j\leqslant m$.*

Proof For $m=1$, the statement is true naturally. Assume it is true for 1, 2, $\cdots$, $m-1$. Then

$$\rho(g^m(x), f^m(x))\leqslant\rho(g(g^{m-1}(x)), f(g^{m-1}(x)))+\rho(f(g^{m-1}(x)), f(f^{m-1}(x))). \quad (1)$$

Since f is a uniformly continuous mapping on K, there exists an $\eta>0$ such that $\rho(f(x), f(y))<\varepsilon/2$ whenever $\rho(x, y)<\eta$. We choose $\delta>0$ so that $\rho(g(x), f(x))<\varepsilon/2$ on D, $\rho(g^j(x), f^j(x))<\varepsilon$ for $2\leqslant j\leqslant m-1$ and $\rho(g^{m-1}(x), f^{m-1}(x))<\eta$, then $\rho(g^m(x), f^m(x))<\varepsilon$ by (1). So the result is true.

§ 3. Fixed Point Theorem

Theorem 1. *Let $S_0\subset S_1\subset S_2$ be convex subsets of the Fréchet space X, with S_0 and S_2 compact and S_1 open relative to S_2. Let f: $S_2\to X$ be a continuous mapping such that, for some integer $m>0$,*

$$f^j(S_1)\subset S_2, \ 1\leqslant j\leqslant m-1, \quad (2)$$

and

$$f^j(S_1)\subset S_0, \ m\leqslant j\leqslant 2m-1. \quad (3)$$

Then f has a fixed point in S_0.

Proof According to Lemma 4, we may assume the metric in X just is

$$\rho(x, y)=\sum_{i=1}^{\infty}2^{-i}\frac{p_i(x-y)}{1+p_i(x-y)},$$

where $\{p_i\}$ is a countable separating family of seminorms on X.

We also may assume $f(S_2)\subset S_2$, since if this is not the case then by Lemma 3, there exists a retraction $r: X \to S_2$. We may then define a new map $\bar{f}=rf$ which has properties (2) and (3) and whose fixed points in S_0 are also fixed points of f.

Since S_1 is open in S_2 and S_0 is compact, there exists $\varepsilon>0$ such that $N_\varepsilon(S_0)\cap S_2 \cap S_1$. By Lemma 5 there exists $\eta>0$ such that for any map g defined on a subset D of S_2 we have $\rho(g^j(x), f^j(x))<\varepsilon/100$ for $1\leqslant j\leqslant 2m$ and for all $x\in D$ whenever $\rho(g(x), f(x))<\eta$ for all $x\in D$.

Since S_2 is compact, there exists a finite collection of points $\{x_i\}$ in S_2, such that for any $x\in S_2$ there exists an x_i with $\rho(x, x_i)<\eta/50$. Obviously, we may assume one point x_j in $\{x_i\}$ belongs to S_0. Let H be the finite-dimensional linear manifold generated by $\{x_i\}$. Let $R_0=S_0\cap H$, $R_1=S_1\cap H$, and $R_2=S_2\cap H$. Then R_0, R_1, and R_2 are nonempty convex sets in H. Therefore, there exists a triangulation $T: K\to R_2$ for some complex K. Since f is uniformly continuous on R_2, there exists $\delta>0$ such that $\rho(f(x), f(y))<\eta/50$ whenever $\rho(x, y)<\delta$. We may assume that the mesh of the triangulation T is less than δ, since a barycentric subdivision will give this.

Define a map $g: R_2\to R_2$ as follows. For each vertex $v\in R_2$ of the triangulation let $g(v)=x_i$ such that $\rho(x_i, f(v))<\eta/50$. For any $x\in R_2$, if $T^{-1}(x)=\sum\alpha_i T^{-1}(v_i)$, let $g(x)=\sum\alpha_i g(v_i)$, then g is defined for all of R_2.

Now if v_i and v_j are vertices of a common simplex in R_2, then we have

$$\begin{aligned}\rho(g(v_i), g(v_j))&\leqslant\rho(g(v_i), f(v_i))+\rho(f(v_i), f(v_j))+\rho(f(v_j), g(v_j))\\&\leqslant\eta/50+\eta/50+\eta/50=3\eta/50.\end{aligned}$$

For any $x\in R_2$, if $\{v_i\}$ are vertices of any simplex containing x in R_2, then

$$\rho(g(x), g(v_j))=\rho(\sum\alpha_i g(v_i), g(v_j))=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{p_n(\sum\alpha_i g(v_i)-g(v_j))}{1+p_n(\sum\alpha_i g(v_i)-g(v_j))}.$$

By the increasing property of function $f(x)=\dfrac{x}{1+x}$ and the property of seminorm, we have

$$\rho(g(x), g(v_j))\leqslant\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\sum\alpha_i p_n(g(v_i)-g(v_j))}{1+\sum\alpha_i p_n(g(v_i)-g(v_i))}.$$

Since $\rho(g(v_i), g(v_j))<3\eta/50$, that is,

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{p_n(g(v_i)-g(v_j))}{1+p_n(g(v_i)-g(v_j))}<\frac{3\eta}{50},$$

if $p_n(g(v_i)-g(v_j))\geqslant 1$, then

$$2p_n(g(v_i)-g(v_j))\geqslant 1+p_n(g(v_i)-g(v_j)),$$

that is,

$$\frac{1}{2}\leqslant\frac{p_n(g(v_i)-g(v_j))}{1+p_n(g(v_i)-g(v_j))}.$$

Hence

$$\frac{1}{2^{n+1}}\leqslant\frac{p_n(g(v_i)-g(v_j))}{2^n[1+p_n(g(v_i)-g(v_j))]}\leqslant\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{p_n(g(v_i)-g(v_j))}{1+p_n(g(v_i)-g(v_j))}<\frac{3\eta}{50}.$$

Obviously there exists an integer n_1 such that $\frac{1}{2^{n+1}}\geqslant\frac{3\eta}{50}$ whenever $n\leqslant n_1$ and $1/2^{n+1}<3\eta/50$ whenever $n>n_1$. So we have $p_n(g(v_i)-g(v_j))<1$ whenever $n\leqslant n_1$ (n_1 is not related to i). Then

$$1+p_n(g(v_i)-g(v_j))<2, \quad (n\leqslant n_1)$$

that is,

$$\frac{1}{1+p_n(g(v_i)-g(v_j))}>\frac{1}{2}.$$

Hence,

$$\sum_{n=1}^{n_1}\frac{1}{2^n}p_n(g(v_i)-g(v_j))\leqslant\sum_{n=1}^{n}\frac{2}{2^n}\frac{p_n(gv_i)-g(v_j))}{1+p_n(g(v_i)-g(v_j))}$$
$$\leqslant 2\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{p_n(g(v_i)-g(v_j))}{1+p_n(g(v_i)-g(v_j))}<\frac{3\eta}{25}.$$

In the other hand,

$$\sum_{n=n_1}^{\infty}\frac{1}{2^n}=\frac{1}{2^{n_1-1}}=\frac{2^3}{2^{n_1+2}}<\frac{12\eta}{25},$$

so

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\sum\alpha_ip_n(g(v_i)-g(v_j))}{1+\sum\alpha_ip_n(g(v_i)-g(v_j))}\leqslant\sum\alpha_i\sum_{n=1}^{n_1}\frac{1}{2^n}p_n(g(v_i)-g(v_j))+\sum_{n=n_1}^{\infty}\frac{1}{2^n}<\frac{3\eta}{25}+\frac{12\eta}{25}.$$

It follows that, for any $x\in R_2$,

$$\rho(f(x),g(x))\leqslant\rho(f(x),f(u))+\rho(f(u),g(u))+\rho(g(u),g(x))$$
$$<\eta/50+3\eta/5+\eta/50<\eta,$$

where u is any vertex of any simplex containing x.

By (3), we see that

$$g^j(R_1)\subset N_{\varepsilon/100}(f^j(R_1))\subset N_{\varepsilon/100}(f^j(S_1))\subset N_{\varepsilon/100}(S_0),\ m\leqslant j\leqslant 2m+1.$$

But g: $R_2\to R_2$, and so we have

$$g^j(R_1)\subset N_{\varepsilon/100}(S_0)\cap R_2=N_{\varepsilon/100}(R_0)\cap R_2\subset\mathrm{conv}(N_{\varepsilon/100}(R_0))\cap R_2,\ m\leqslant j\leqslant 2m+1.$$

Let $R_0'=\mathrm{conv}(N_{\varepsilon/100}(R_0))\cap R_2$. By the same arguments as above, we get $R_0'\subset N_{\varepsilon/2}(R_0)\cap R_2$. Hence $\bar{R}_0'$ is closed and $N_{\varepsilon/4}(\bar{R}_0')\cap H\subset R_1$. But $g^j(R_1)\subset\bar{R}_0'$ for $m\leqslant j\leqslant 2m-1$, and so by Lemma 2, g has a fixed point in $\bar{R}_0'$.

Let $\{\varepsilon_n\}$ be a null sequence of ε' s as considered above, and let $\{g_n\}$ be the corresponding maps with fixed points $\{x_n\}$ respectively. By the compactness of S_2 there exists a subsequence $x_{n_i}\to x_0$, so $f(x_0)=\lim g_{n_i}(x_{n_i})=\lim x_{n_i}=x_0$. Since $x_{n_i}\in\bar{R}'_{0(n_i)}=\overline{\mathrm{conv}}(N_{\varepsilon_{n_i}/100}(R_{0(n_i)}))\cap R_{2(n_i)}$, it follows that $x_0\in S_0$.

§4. Application

As an application of Theorem 1, we obtain Theorem 2 as follows, where the concept of continuous dependence in condition (iv) is initiated in [6]. So Theorem 2 is not the same as the result in [7, 8].

Theorem 2. *Let the system*

$$x'=h(t, x)+\int_{-\infty}^{t} q(t, s, x(s))ds, \tag{4}$$

where h: $R\times R^n\to R^n$, q: $R\times R\times R^n\to R^n$ *are continuous and* $h(t+T, x)=h(t, x)$, $q(t+T, s+T, x)=q(t, s, x)$, *satisfy*

(i) *Let* g: $(-\infty, 0]\to[1, \infty)$ *be a strictly decreasing continuous function,* $g(0)=1$, $g(r)\to\infty$ *as* $r\to-\infty$. *If* ϕ: $(-\infty, 0]\to R$ *is continuous and satisfies* $\|\phi(s)\|\leqslant rg(s)$ *for* $-\infty<s\leqslant0$ *and some* $r>0$, *then there is a unique solution* $x(t, 0, \phi)$ *of* (4) *on* $[0, \infty)$.

(ii) *Solutions of* (4) *are g-uniform bounded and g-uniform ultimate bounded, i. e., for each* $r>0$ *there exists* $B_1(r)>0$ *such that if* ϕ: $(-\infty, 0]\to R^n$ *is continuous and satisfies* $\|\phi(s)\|\leqslant rg(s)$ *for* $-\infty<s\leqslant0$, *then* $\|x(t, 0, \phi)\|\leqslant B_1$ *for* $t\geqslant0$, *and there is a number* $B>0$ *such that for each* $r>0$ *there exists* $K>0$ *such that if* $\|\phi(s)\|\leqslant rg(s)$ *for* $-\infty<s\leqslant0$, *then* $\|x(t, 0, \phi)\|\leqslant B$ *for* $t\geqslant K$.

(iii) *For each* $r>0$ *there is a continuous function of* t $L(t, r)>0$ *such that* $\|\phi(t)\|\leqslant r$ *on* $(-\infty, 0]$ *implies* $\|x'(t, 0, \phi)\|\leqslant L(t, r)$ *on* $[0, \infty)$.

(iv) *For each* $r>0$, *if* $U=\{\phi$: $(-\infty, 0]\to R^n|\phi$ *is continuous,* $\|\phi(t)\|\leqslant r\}$, *then solutions of* (4) *depend continuously on initial functions in* U *relative to* (X, ρ), *where* $X=\{\phi|\phi$: $(-\infty, 0]\to R^n$ *is continuous*$\}$ *and for any* $\phi_1, \phi_2\in X$,

$$\rho(\phi_1, \phi_2)=\sum_{n=1}^{\infty}2^{-n}\frac{\rho_n(\phi_1, \phi_2)}{1+\rho_n(\phi_1, \phi_2)},$$

$$(\rho_n(\phi_1, \phi_2)=\|\phi_1-\phi_2\|^{[-n,0]}=\sup_{-n\leqslant t\leqslant0}\|\phi_1(t)-\phi_2(t)\|).$$

Then (3) *has a periodic solution of period* T.

Proof Obviously X is a Fréchet space.

By (ii) there exist B, $B_1=B_1(B)$ and $K_1=K_1(B)$ such that $\|\phi(s)\|\leqslant Bg(s)$ for $-\infty<s\leqslant0$ implies

$$\|x(t, 0, \phi)\|\leqslant B \text{ for } t\geqslant0,\ \|x(t, 0, \phi)\|\leqslant B_1 \text{ for } t\geqslant K.$$

For the same reason, there exist B_2, K_2 such that

$$\|x(t, 0, \phi)\|\leqslant B_2 \text{ for } t\geqslant0;\ \|x(t, 0, \phi)\|\leqslant B \text{ for } t\geqslant K,$$

if $\|\phi(s)\|\leqslant(B_1+1)g(s)$ for $-\infty<s\leqslant0$.

Determine a number $H>0$ with $Bg(-H)=B_1+1$.

Construct a continuous strictly decreasing function r: $(-\infty, -H]\to[B_1+1, B_2+1]$ with $r(t)\leqslant Bg(t)$, $r(-H)=Bg(-H)$, $r(t)\to B_2+1$ as $t\to-\infty$, $r(t-H/2)<(B_1+1)g(t)$ for $t\leqslant-H/2$.

Chosen $J>0$ with $r(-J)=B_2$. By (iii), we have

$$\|x'(t, 0, \phi)\|\leqslant\max_{0\leqslant t\leqslant T}L(t, B_2+1)\overset{\text{def}}{=\!=}L^*$$

whenever $\|\phi(t)\|\leqslant B_2+1$. Find an integer m with $mT>K_2+J$.

Define

$$S_0=\{\phi \mid \phi\in X,\ \|\phi(t)\|\leqslant r(t) \text{ if } t\leqslant -H,\ \|\phi(t)\|\leqslant Bg(t) \text{ if } -H\leqslant t\leqslant 0,$$
$$\|\phi(u)-\phi(v)\|\leqslant L^*|u-v|\},$$

$$S_1=\{\phi \mid \phi\in X,\ \|\phi(t)\|<r(t-H/2) \text{ if } t\leqslant -H/2,\ \|\phi(t)\|<B_1+1$$
if $-H/2\leqslant t\leqslant 0,\ \|\phi(u)-\phi(v)\|\leqslant L^*|u-v|\}$,

$$S_2=\{\phi \mid \phi\in X,\ \sup_{-\infty<t\leqslant 0}\|\phi(t)\|\leqslant B_2+1,\ \|\phi(u)-\phi(v)\|\leqslant L^*|u-v|\}.$$

For any $\phi\in S_2$, define

$$P(\phi)=x(t+T,\ 0,\ \phi), \text{ for } -\infty<t\leqslant 0.$$

According to (iv), P: $S_2\to X$ is a continuous map. By the periodic property of h, q, we see that

$$P^2(\phi)(t)=P(P(\phi))(t)=x(t+T,\ 0,\ P(\phi))=x(t+2T,\ 0,\ \phi), \text{ for } -\infty<t\leqslant 0.$$

In general

$$P^k(\phi)(t)=x(t+kT,\ 0,\ \phi), \text{ for } -\infty<t\leqslant 0 \text{ and } k=1,\ 2,\ \cdots.$$

By the strictly decreasing property of g and r, we have $S_0\subset S_1\subset S_2$, S_0, S_1, S_2 are convex subsets of X, and S_1 is open relative to S_2. By the construction of S_0, S_1 and S_2, we have $P^j(S_1)\subset S_2$ for $j=1,\ 2,\ \cdots,\ m-1$ and $P^j(S_1)\subset S_0$ for $m\leqslant j\leqslant 2m-1$. S_0 and S_2 are compact subsets of X by the proof of Theorem 1 in [6]. Then according to Theorem 1 there exists a $\phi\in S_0$ with $P(\phi)=\phi$ such that $x(t,\ 0,\ \phi)=x(t,\ 0,\ P(\phi))=x(t+T,\ 0,\ \phi)$ and the result is proved.

References

[1] Horn, W. A., Some fixed point theorems for compact maps and flows in Banach spaces, *Tras. Amer. Math. Soc.*, **149**(1970), 391—404.

[2] Browder, F. E., On a generalization of the Schauder fixed point theorem, *Duke Math. J.*, **26**(1959), 291—303, MR21 4368.

[3] Dugundji, J. & Granas, A., Fixed Point Throry, V. I., Printed in Poland, 1982.

[4] Armstrong, M. A., Basic Topology, Springer-Verlag, 1983.

[5] Rudin, W., Functional Analysis, McGraw-Hill Book Company, 1973.

[6] Burton, T. A., Periodic solutions of nonlinear Volterra equations, *Funkcialaj Ekvac.*, **27**(1984), 301—317.

[7] Arino, O. & Burton, T. A. & Haddock, J., Periodic solutions of functional differential equations, *Proc. Roy. Soc. Edinburgh*, **A101** (1985), 253—271.

[8] Burton, T. A. & Zhang, Shunian, Unified boundedness, periodicity, and stability in ordinary and functional differential equations, *ANN. MATH. Serie Quarta, Tomo*, **CXLV** (1986), 129—258.

[9] Wang Ke & Huang Qichang, C_h space and boundedness and periodicity of solutions for functional differential equations, *Scientia Sinica (Series A)*, **3**(1987), 243—252.