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DYNAMICAL BEHAVIOR IN THE EQUATION OF J-J TYPE WITH LARGE DC-CURRENT

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Abstract

This paper studies the dynamical behavior in the equation of J-J type with large dc-current by using rigorous mathematical analysis. The problem is completly solved. The conclusion is no chaotic motion takes place, every trajectory is asymptotically periodic or quasi-periodic.

§1. Introduction

Recently there has been considerable interest in the study of chaos in nonlinear dynamical systems. One of such systems which can exhibit a wide variety of chaotic phenomena is that of the Josephson Junction[1]. Now, much progress on the dynamical behavior of the J-J system has been made by numerical calculations and by experiments. It is interesting to note that various routes to chaos have been observed, e. g. via (i) periodic-doubling bifurcation[2], (ii) intermittency[3], and (iii) quasiperiodicity[4].

In this paper, we will study the dynamical behavior in the equation:

$$\ddot{x}+\beta(1+\varepsilon f'(x))\dot{x}+f(x)=\rho+bg(t), \tag{1.1}$$

where $f(x)$, $g(x)$ satisfy the following conditions: $f,\ g\in C^2$;

$$f(-x)=-f(x),\ f(x+2\pi)=f(x),\ f(x+\pi)=-f(x),\ f(\pi-x)=f(x),$$

$$f'(x)>0,\ x\in\left[0,\frac{\pi}{2}\right),\ |f(x)|\leqslant 1,\ |f'(x)|\leqslant\frac{2}{\pi}.$$

$$f(x)=\begin{cases}\dfrac{2}{\pi}x, & x\in\left[-\dfrac{\pi}{2}+\dfrac{1}{\rho},\ \dfrac{\pi}{2}-\dfrac{1}{\rho}\right],\\[2mm] 2-\dfrac{2}{\pi}x, & x\in\left[\dfrac{\pi}{2}+\dfrac{1}{\rho},\ \dfrac{3}{2}\pi-\dfrac{1}{\rho}\right],\end{cases} \tag{1.2}$$

$$g(-t)<-g(t),\ g(t+T)=g(t),\ g\left(t+\frac{T}{2}\right)=-g(t),\ g\left(\frac{T}{2}-t\right)=g(t),$$

$$g'(t)\geqslant 0,\ t\in\left[0,\frac{T}{4}\right],\ |g(t)|\leqslant 1. \tag{1.3}$$

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$\rho>\max\left\{1+b, \frac{2}{\pi}\right\}$, $\beta>0$, $b\geqslant 0$, s is arbitrary. We say (1.1) is of tho J-J type.

For ρ not sufficiently large, but $\beta>\frac{2}{1+s}$, $|s|<1$, we have studied in [5].

For β, ρ, b small, $g(t)=\sin\omega t$, we have proved that chaotic behavior actually takes place by using Melnikov's method[6].

For $0<\beta<\frac{2}{1+s}$ and moderate ρ and b, through the numerical computation of the Lyapunov exponents, power spectrum as well as the phase curves, we found two kinds of chaotic behavior:

a. The periodic-doubling chaotic behavior (with a fixed voltage).

b. Intermittent chaotic behavior.

Feigenbaum constant appears in case a[10].

The idea we adopted to establish our main result in this paper is the followin g: for a two dimensional non-autonomous periodic system, the existence of an invariant closed curve of the Poincare mapping can be proved by an argument similar to Poincare-Bendixon Theorem supplemented by a kind of apriori-estimation concerning the derivative mapping. The latter can be worked out for equation (1.1), actually for its variational equation. The invariant curve is also unique and attracting globally. The method we use in this paper can be applied to study many other two dimensional non-autonomous periodic system, e. g. for the system with tangent discriminator and frequency modulation input: $\ddot{x}+(\alpha+\beta\sec^2 x)\dot{x}+\gamma\,\mathrm{tg}\,y=A\sin\omega t$, we have proved that when $0\leqslant\mu\leqslant\beta$, $(\alpha+\beta)^2\geqslant 4\gamma>0$, there exists a unique periodic solution with period $T=\frac{2\pi}{\omega}$[11].

§ 2. Properties of the Poincare Map

Instead of considering equation (1.1), we consider the following equations:

$$\begin{cases}\dot{x}=y-\beta(x+sf(x)),\\ \dot{y}=-f(x)+\rho+bg(t),\end{cases}\tag{2.1}$$

where f, $g\in C^2$ and satisfy (1.2), (1.3), $\rho>\max\left\{1+b, \frac{2}{\pi}\right\}$, $\beta>0$, $b\geqslant 0$, s is arbitrary.

System (2.1) is periodic with period $(2\pi, 2\pi\beta)$, i. e. let $P(x, y, t)=y-\beta(x+sf(x))$, $Q(x, y, t)=-f(x)+\rho+bg(t)$, then $P(x+2\pi, y+2\pi\beta, t)=P(x, y, t)$, $Q(x+2\pi, y+2\pi\beta, t)=Q(x, y, t)$. So we can consider this system on a cylinder.

Let $0<k<\frac{\beta(1+|s|)+\sqrt{\beta^2(1+|s|)^2+8/\pi}}{2}$ be a constant. $(x(t, z_0, 0), y(t, z_0, 0))$ be the solution of (2.1) with initial condition $t=0$, $x=x_0$, $y=y_0$, where $z_0=$

(x_0, y_0).

Definition 2.1. *The map M_ρ: $R^2 \to R^2$, $z_0 \to Z(T, z_0, 0)$, $z_0 \in R^2$, $z(T, z_0, 0) = (x(T, z_0, 0), y(T, z_0, 0))$ is called the Poincare map of (2.1). Also we define M^t: $R^2 \to R^2$, $z_0 \to z(t, z_0, 0)$, $t \in R'$.*

We will denote M_ρ by M for simplicity.

If $(x(t+pT, z_0, 0), y(t+pT, z_0, 0)) = (x(t, z_0, 0), y(t, z_0, 0)) + (2\pi q, 2\pi\beta q)$ for any $t \in R^1$, where p, q are the smallest non-negative intogers satisfying the above equality, then $(x(t, z_0, 0), y(t, z_0, 0))$ is called a periodic solution of (2.1) of (p, q) type.

Let l_1: $y = \beta x + \beta|s| + \frac{\rho+b+1}{\beta}$, l_2: $y = \beta x - \beta|s| + \frac{\rho-b-1}{\beta}$, and $S \subset R^2$ be a closed region bounded by l_1 and l_2.

Lemma 2.1. *For any $z_0 \in R^2$, there is a $t_0 > 0$, such that $z(t_0, z_0, 0) \in S$. For $z_0 \in S$, $t_0 \in [0, T)$, and $\forall t \geqslant t_0$, we have $z(t, z_0, t_0) \in S$, where $z(t, z_0, t_0) = (x(t, z_0, t_0), y(t, z_0, t_0))$, $z_0 = (x_0, y_0)$.*

Proof First assume z_0 to be above l_1. For an arbitrary point (x, y) above l_1, we have $\dot{x} = y - \beta(x + sf(x)) > \frac{\rho+b+1}{\beta} > 0$, i. e., x increases as t increases and

$$\left|\frac{dy}{dx}\right| = \left|\frac{-f(x)+\rho+bg(t)}{y-\beta(x+sf(x))}\right| \leqslant \frac{\rho+b+1}{y-\beta(x+sf(x))} < \frac{\beta(\rho+b+1)}{\rho+b+2} = \beta$$

(β is the slope of l_1). So there must exist a $t_0 > 0$ such that $z(t_0, z_0, 0) \in S$.

Next consider a point (x, y) below l_2. Since $\rho > b+1$, for an arbitrary point (x, y) below l_2, we have $\dot{y} = -f(x) + \rho + bg(t) \geqslant \rho - b - 1 > 0$, i. e. y increases as t increaes. If $\dot{x} = y - \beta(x + sf(x)) \leqslant 0$, obviously, the direction of the flow below l_2 is toward S. If $\dot{x} = y - \beta(x + sf(x)) > 0$, then $\dot{x} = y(x + sf(x)) \leqslant \beta x - \beta|s| + \frac{\rho-b-1}{\beta} - \beta(x+sf(x)) < \frac{\rho-b-1}{\beta}$ and

$$\frac{dy}{dx} = \frac{-f(x)+\rho+bg(t)}{y-\beta(x+sf(x))} \geqslant \frac{\rho-b-1}{y-\beta(x+sf(x))} > \frac{\beta(\rho-b-1)}{\rho-b-1} = \beta,$$

which implies the direction of the flow below l_2 is also toward S. So there exists a $t_0 > 0$ such that $z(t_0, z_0, 0) \in S$.

Similarly, it can be shown that on the straight lines l_1 and l_2, the directions of the flows are toward the inside of S. So for any $z_0 \in S$, $t_0 \in [0, T]$ and $t \geqslant t_0$, we have $z(t, z_0, t_0) \in S$.

Definition 2. 2. *A curve $y = h(x)$ in S is called a k-horizontal curve, if h is continuous, $h(x+2\pi) = h(x) + 2\pi\beta$ and*

$$k(x_2 - x_1) \leqslant h(x_2) - h(x_1) \leqslant \frac{\beta(1+|s|) + \sqrt{\beta^2(1+|s|)^2 + 8/\pi}}{2}(x_2 - x_1)$$

$(x_2 \geqslant x_1)$.

The distance between a point $z\in R^2$ and the k-horizontal curve l: $y=h(x)$ is defined by $f(z, l)=\min_{x\in R'} d(z, (x, h(x)))$.

The distance between two k-horizontal curves l_1: $y=h_1(x)$, l_2: $y=h_2(x)$ is defined by $d(l_1, l_2) \max_{R^1\in x}|h_2(x)-h_1(x)|$. Obviously, $d(l_1, l_2)=\max_{x\in[0,2\pi]}|h_2(x)-h_1(x)|$.

A set $H\subset S$ is called a k-horizontal strip if $H=\{(x, y)\,|\,h_1(x)\leqslant y\leqslant h_2(x),\ x\in R^1\}$ and $h_1(x)\neq h_2(x)$, where $y=h_1(x)$, $y=h_2(x)$ are k-horizontal curves.

$\widetilde{H}=\{(x, y)\,|\,h_1(x)\leqslant y\leqslant h_2(x),\ x\in[0, 2\pi]\}$ is called a generating element of H. The width and the area of a generating element of $H=\{(x, y)\,|\,h_1(x)\leqslant y\leqslant h_2(x),\ x\in R^1\}$ denoted by $\|\widetilde{H}\|$ and $S_{\widetilde{H}}$ are defined by $\|\widetilde{H}\|=\max_{x\in[0,2\pi]}|h_2(x)-h_1(x)|$, and $S_{\widetilde{H}}=\int_0^{2\pi}|h_1(x)-h_2(x)|dx$.

Lemma 2.2. *Suppose $\rho>b+1+2\beta^2|\varepsilon|$. For any $z_0\in S$, $t_0\in[0, T)$, $z(t, z_0, t_0)$ is the solution of (2.1) with initial condition $z(t, z_0, t_0)=z_0$, Where $z(t, z_0, t_0)=(x(t, z_0, t_0), y(t, z_0, t_0))$. If $|x(t, z_0, t_0)-x_0|=d$ where $t>t_0$, then*

$$\frac{\beta d}{\rho+b+1+2\beta^2|\varepsilon|}\leqslant t-t_0\leqslant\frac{\beta d}{\rho-b-2-2\beta^2|\varepsilon|}.$$

Proof By equation (2.1), $\dot{x}=y-\beta(x+\varepsilon f(x))$. For $(x, y)\in S$, we have

$$\frac{\rho-b-1-2\beta^2|\varepsilon|}{\beta}\leqslant y-\beta(x+\varepsilon f(x))\leqslant\frac{\rho+b+1+2\beta^2|\varepsilon|}{\beta}.$$

So $\frac{\rho-b-1-2\beta^2|\varepsilon|}{\beta}\leqslant\dot{x}\leqslant\frac{\rho+b-1+2\beta^2|\varepsilon|}{\beta}$. By Lemma 2.1, for $z_0\in S$, $z(t, z_0, t_0)\in S$, any $t\geqslant t_0$, we have

$$\frac{\rho-b-1-2\beta^2|\varepsilon|}{\beta}(t-t_0)\leqslant x(t, z_0, t_0)-x_0\leqslant\frac{\rho+b+1+2\beta^2|\varepsilon|}{\beta}(t-t_0)$$

for any $t\geqslant t_0$. Since $\rho-b-1-2\beta^2|\varepsilon|>0$, we get

$$\frac{\beta d}{\rho+b+1+2\beta^2|\varepsilon|}\leqslant t-t_0\leqslant\frac{\beta d}{\rho-b-1-2\beta^2|\varepsilon|}$$

for t satisfying $x(t, z_0, t_0)-x_0=d$.

Lemma 2.3. *Suppose H is the k-horizontal strip in S, S_{MH} denotes the area of the region: $\{M(x, y)\,|\,(x, y)\in H,\ x\in[0, 2\pi]\}$. For any α_0, $0<\alpha_0<\beta T$, there exists a $\rho_0(\beta, \varepsilon, b, \alpha_0)>0$, such that as $\rho>\rho_0$, we have $S_{MH}\leqslant e^{-\alpha_0}S_{\widetilde{H}}$.*

Proof Let $D=\{(x, y)\,|\,(x, y)\in H,\ x\in[0, 2\pi]\}$. Obviously, $S_{\widetilde{H}}=\iint_D dxdy$, $S_{MH}=\iint_D|\det(dM)|dxdy$, where $dM=dM^t$, and dM^t satisfies the equation:

$$\frac{d}{dt}(dM^t)=\begin{pmatrix}-\beta(1+\varepsilon f'(x(t, z))) & 1\\ -f'(x(t, z)) & 0\end{pmatrix}dM^t.$$

By the theorem about homogeneous linear system, we get

$$\det(dM|_{z_0})=\exp\left(-\int_0^T\beta(1+\varepsilon f'(x(t, z_0, 0)))dt\right)$$

$$\leq \exp(-\beta T)\exp\left(\beta|\varepsilon|\cdot\left|\int_0^T f'(x(t, z_0, 0))dt\right|\right).$$

Now, we estimate $\int_0^T f'(x(t, z_0, 0))dt$.

First, we assume $0=t_0\leq t_1<t_2<\cdots<t_{2n-1}\leq t_{2m}=T$ satisfies that for $t\in[t_{2i}, t_{2i-1}]$, $f'(x(t, z_0, 0))=\frac{2}{\pi}$; for $t\in(t_{2i+1}, t_{2i+2})$, $f'(x(t, z_0, 0))<\frac{2}{\pi}$. By Lemma 2.2 $t_{2i+1}-t_{2i}\geq\frac{\beta\left(\pi-\frac{2}{\rho}\right)}{\rho+b+1+2\beta^2|\varepsilon|}$ as $1\leq i\leq m-1$; $t_{2i+2}-t_{2i+1}\leq\frac{\beta\left(\pi+\frac{2}{\rho}\right)}{\rho-b-1-2\beta^2|\varepsilon|}$ as $0\leq i\leq m-2$.

Thus we get the estimation:

$$\int_0^T f'(x(t, z_0, 0))dt$$

$$\geq(m-1)\left[\frac{\frac{2}{\pi}\beta\left(\pi-\frac{2}{\rho}\right)}{\rho+b+1+2\rho^2|\varepsilon|}-\frac{\frac{2}{\pi}\beta\left(\pi+\frac{2}{\rho}\right)}{\rho-b-1-2\beta^2|\varepsilon|}\right]-\frac{2}{\pi}\frac{\beta\left(\pi+\frac{2}{\rho}\right)}{\rho-b-1-2\beta^2|\varepsilon|}$$

$$\geq\frac{T(\rho+b+1+2\beta^2|\varepsilon|)}{2\beta\left(\pi-\frac{2}{\rho}\right)}\left[\frac{\frac{2}{\pi}\beta\left(\pi-\frac{2}{\rho}\right)}{\rho+b+1+2\beta^2|\varepsilon|}-\frac{\frac{2}{\pi}\beta\left(\pi+\frac{2}{\rho}\right)}{\rho-b-1-2\beta^2|\varepsilon|}\right]$$

$$-\frac{2}{\pi}\frac{\beta\left(\pi+\frac{2}{\rho}\right)}{\rho-b-1-2\beta^2|\varepsilon|}$$

$$=\frac{T}{\pi}-\frac{T}{\pi}\frac{\pi+\frac{2}{\rho}}{\pi-\frac{2}{\rho}}\frac{\rho+b+1+2\beta^2|\varepsilon|}{\rho-b-1-2\beta^2|\varepsilon|}-\frac{2}{\pi}\frac{\beta\left(\pi+\frac{2}{\rho}\right)}{\rho-b-1-2\beta^2|\varepsilon|}.$$

Next, assume $0=t'_0<t'_1<t'_2<\cdots\leq t'_{2m-1}\leq t'_{2m}=T$ satisfies that for $t\in[t'_{2i}, t'_{2i+1}]$, $f'(x(t, z_0, 0))=-\frac{2}{\pi}$; for $t\in(t'_{2i+1}, t'_{2i+2})$, $f'(x(t, z_0, 0))>-\frac{2}{\pi}$. By Lemma 2.2, $t'_{2i+1}-t'_{2i}\geq\frac{\beta\left(\pi-\frac{2}{\rho}\right)}{\rho+b+1+2\beta^2|\varepsilon|}$ as $1\leq i\leq m-1$; $t'_{2i+2}-t'_{2i+1}\leq\frac{\beta\left(\pi+\frac{2}{\rho}\right)}{\rho-b-1-2\beta^2|\varepsilon|}$, as $0\leq i\leq m-2$.

Thus we get the following estimation:

$$\int_0^T f'(x(t, z_0, 0))dt$$

$$\leq(m-1)\left[\frac{\frac{2}{\pi}\beta\left(\pi+\frac{2}{\rho}\right)}{\rho-b-1-2\beta^2|\varepsilon|}-\frac{\frac{2}{\pi}\beta\left(\pi-\frac{2}{\rho}\right)}{\rho+b+1+2\beta^2|\varepsilon|}\right]+\frac{2}{\pi}\frac{\beta\left(\pi+\frac{2}{\rho}\right)}{\rho-b-1-2\beta^2|\varepsilon|}$$

$$\leq\frac{T}{\pi}\frac{\pi+\frac{2}{\rho}}{\pi-\frac{2}{\rho}}\frac{\rho+b+1+2\beta^2|\varepsilon|}{\rho-b-1-2\beta^2|\varepsilon|}-\frac{T}{\pi}+\frac{2}{\rho}\frac{\beta\left(\pi+\frac{2}{\rho}\right)}{\rho-b-1-2\beta^2|\varepsilon|}.$$

By the above two estamations, we known that there exists a number $\rho_0(\beta, \varepsilon, b, \alpha_0)>0$, such that as $\rho>\rho_0$, $|\det(dM)|_{z_0}|\leqslant e^{-\alpha_0}$ holds for any $z_0\in S$ So. when $\rho>\rho_0(\beta, \varepsilon, b, \alpha_0)$, $S_{MH}\leqslant e^{-\alpha_0}S_H$.

Lemma 2.4. *For fixed $\beta>0$, b, ε and $0<k<\beta$, there exists $\rho_1(\beta, \varepsilon, b, k)>0$, such that for $\rho>\rho_1$, M maps a k-horizontal curve in S to a k-horizontal curve in S.*

Proof We only prove that Poincare map M maps a smooth k-horizontal curve in S to a smooth k-horizontal curve in S. Let l: $y=h(x)$ be a smooth k-horizontal curve, $\bar{l}=Ml$. By Lemma 2.1 $\bar{l}$ is a smooth curve in S. We have only to prove $\bar{l}$ is a k-horizontal Curve in S.

First of all, we express l in parametric form: $x=s$, $y=h(s)$, $s\in R^1$. Let $(x(t, s), y(t, s))$ be the solution of (2.1) with initial conditions $t_0=0$, $x_0=s$, $y_0=h(s)$. If we denote $\xi(t)=\frac{dx(t, s)}{ds}$, $\eta(t)=\frac{dy(t, s)}{ds}$, then $(\xi(t), \eta(t))$ satisfy the system:

$$\begin{cases}\dot{\xi}=\eta-\beta(1+\varepsilon f'(x(t, s)))\xi, \\ \dot{\eta}=-f(x(t, s))\xi,\end{cases} \tag{2.2}$$

and the initial conditions $\xi(0)=1$, $\eta(0)=h'(s)$. Because of the periodicity of (2.1) and l, we know that $\bar{l}$ is also periodic. So what we have to prove is $k\leqslant\frac{\eta(T)}{\xi(T)}\leqslant\frac{\beta(1+|\varepsilon|)+\sqrt{\beta^2(1+|\varepsilon|)^2+8/\pi}}{2}$, $\xi(T)>0$. For this purpose, we consider the following two equations:

$$\begin{cases}\dot{\xi}=\eta-\beta\left(1+\frac{2}{\pi}\varepsilon\right)\xi, \\ \dot{\eta}=-\frac{2}{\pi}\xi,\end{cases} \tag{2.3}$$

and

$$\begin{cases}\dot{\xi}=\eta-\beta\left(1-\frac{2}{\pi}\varepsilon\right)\xi, \\ \dot{\eta}=\frac{2}{\pi}\xi.\end{cases} \tag{2.4}$$

Let $k'=k+\frac{\beta k(\beta-k)\pi}{\rho}$, $(\bar{\xi}_0, \bar{\eta}_0)=(1, k)$, (ξ_1, η_1) be the value of the solution of (2.3) with initial condition $t_0=0$, $\xi_0=\bar{\xi}_0$, $\eta_0=\bar{\eta}_0$ at $t=\frac{\beta\pi}{\rho-b-1-2\beta^2|\varepsilon|}$ (suppose $\rho>b+1+2\beta^2|\varepsilon|$), (ξ_2, η_2) be the value of the solution of (2.4) with initial condition $t_0=0$, $\xi_0=\xi_1$, $\eta_0=\eta_1$, at $t=\frac{\beta\pi}{\rho+b+1+2\beta^2|\varepsilon|}$. By the properties of $f(x)$ and Lemma 2.2 as well as the properties of homogenous linear system, it is obvious that we need only to porve $\xi_2>0$ and $\frac{\eta_2}{\xi_2}\geqslant k'$.

By solving equations (2.3) and (2.4), we have

$$\xi_1=\exp\left(\frac{-\beta\left(1+\frac{2}{\pi}\varepsilon\right)}{2}t_1\right)\cdot\left[1+\left(k-\frac{\beta\left(1+\frac{2}{\pi}\varepsilon\right)}{2}\right)t_1\right]+o\left(\frac{1}{\rho}\right),$$

$$\left(t_1=\frac{\beta\pi}{\rho-b-1-2\beta^2|\varepsilon|}\right),$$

$$\eta_1=\exp\left(\frac{-\beta\left(1+\frac{2}{\pi}\varepsilon\right)}{2}t_1\right)\cdot\left[k+\left(\frac{k\beta\left(1+\frac{2}{\pi}\varepsilon\right)}{2}-\frac{2}{\pi}\right)t_1\right]+o\left(\frac{1}{\rho}\right).$$

$$\xi_2=\exp\left(\frac{-\beta\left(1-\frac{2}{\pi}\varepsilon\right)}{2}t_2\right)\cdot\left(\xi_1+\left(\eta_1-\frac{\beta\left(1-\frac{\pi}{\pi}\varepsilon\right)}{2}\xi_1\right)t_2\right)+o\left(\frac{1}{\rho}\right),$$

$$\left(t_2=\frac{\beta\pi}{\rho+b+1+2\beta^2|\varepsilon|}\right),$$

$$\eta_2=\exp\left(\frac{-\beta\left(1-\frac{2}{\pi}\varepsilon\right)}{2}t_2\right)\cdot\left(\eta_1+\left(\frac{2}{\pi}\xi_1+\frac{\beta\left(1-\frac{2}{\pi}\varepsilon\right)}{2}\eta_1\right)t_2\right)+o\left(\frac{1}{\rho}\right).$$

So we have

$$\xi_2=\exp\left(\frac{-\beta(t_1+t_2)}{2}\right)\cdot\left(1+\left(k-\frac{\beta\left(1+\frac{2}{\pi}\varepsilon\right)}{2}\right)t_1+\left(k-\frac{\beta\left(1-\frac{2}{\pi}\varepsilon\right)}{2}\right)t_2\right)+o\left(\frac{1}{\rho}\right)$$

$$=\exp\left(\frac{-\beta(t_1+t_2)}{2}\right)\cdot\left(k+\left(\frac{\beta k\left(1+\frac{2}{\pi}\varepsilon\right)}{2}-\frac{2}{\pi}\right)t_1\right.$$

$$\left.+\left(\frac{2}{\pi}+\frac{\beta k\left(1-\frac{2}{\pi}\varepsilon\right)}{2}\right)t_2\right)+o\left(\frac{1}{\rho}\right).$$

Thus, as ρ is large enough, we have

$$\frac{\eta_2}{\xi_2}=\frac{k+\left(\frac{\beta k\left(1+\frac{2}{\pi}\varepsilon\right)}{2}-\frac{2}{\pi}\right)t_1+\left(\frac{2}{\pi}+\frac{\beta k\left(1-\frac{2}{\pi}\varepsilon\right)}{2}\right)t_2+o\left(\frac{1}{\rho}\right)}{1+\left(k-\frac{\beta\left(1+\frac{2}{\pi}\varepsilon\right)}{2}\right)t_1+\left(k-\frac{\beta\left(1-\frac{2}{\pi}\varepsilon\right)}{2}\right)t_2+o\left(\frac{1}{\rho}\right)}$$

$$=\frac{k+\beta k t_1+o\left(\frac{1}{\rho}\right)}{1+(2k-\beta)t_1+o\left(\frac{1}{\rho}\right)}=\frac{k(1+\beta t_1)}{1+(2k-\beta)t_1}+o\left(\frac{1}{\rho}\right)$$

$$=k(1+\beta t_1)(1-(2k-\beta)t_1+o\left(\frac{1}{\rho}\right)=k(1+(2\beta-2k)t_1)+o\left(\frac{1}{\rho}\right)$$

$$=k+\frac{2k(\beta-k)}{\rho-b-1-2\beta^2|\varepsilon|}+o\left(\frac{1}{\rho}\right),\quad \text{and } \xi_2>0.$$

Obviously, the direction of the flow of (2.1) on

$$\eta=\frac{\beta(1+|\varepsilon|)+\sqrt{\beta^2(1+|\varepsilon|)^2+8/\pi}}{2}\xi,$$

$\xi>0$, is toward to the line $\eta=\beta\xi$, $\xi>0$. Thus we know that there exists a number $\rho_1(\pi, \varepsilon, b, k)>0$, such that $\rho>\rho_1$, $\frac{\eta_2}{\xi_2}\geqslant k'$, $\xi_2>0$. We have proved the result.

Let $\bar{\rho}=\max\{\rho_0(\beta, \varepsilon, b, \alpha_0), \rho_1(\beta, \varepsilon, b, k)\}$.

Lemma 2. 5. *There exists a number $\alpha>0$, such that for every k-horizontal strip H in S, the following inequalities $\alpha\|\tilde{H}\|^2\leqslant S_{\tilde{H}}\leqslant 2\pi\|\tilde{H}\|$ hold.*

The proof is omitted. Readers can refer to [5].

Corollary 2.5.1. *For any $n\in N$, $\|M^nH\|^2\leqslant\frac{2\pi e^{-n\alpha_0}}{\alpha}\|\tilde{H}\|$, where $\rho>\bar{\rho}$.*

Theorem 2.1. *For fixed $\beta>0$, b, ε and $0<k<\rho$, $\rho>\bar{\rho}$, there exists a unique k-horizontal curve l: $y=h(x)$ in S such that $Ml=l$ and for any $z_0\in R^2$, $d(M^nz_0, l)\to 0$ when $n\to\infty$*

Proof Denote the straight lines $y=\beta x+\beta|\varepsilon|+\frac{\rho+b+1}{\beta}$, $y=\beta x-\beta|\varepsilon|+\frac{\rho-b-1}{\beta}$ by l_1, l_2. Obviously l_1, l_2 are k-horizontal curves in S. By Lemma 2.4, for $\rho>\bar{\rho}$, M^nl_1, M^nl_2 are also k-horizontal curves in S. On the interval $[0, 2\pi]$, $\{M^nl_1\}$, $\{M^nl_2\}$ form a sequence of uniform bounded and equi-continuous functions. By compact argument and the periodicity of M^nl_1, M^nl_2, there exists a sequence of natural number n_k and k-horizontal curves $\bar{l}_1$, $\bar{l}_2$ such that $d(M^{n_k}l_1, \bar{l}_1)\to 0$, $d(M^{n_k}l_2, \bar{l}_2)\to 0$ $(n_k\to\infty)$. Let H be the k-horizontal strip bounded by l_1, l_2. Obviously $H\supset MH\supset M^2H\supset\cdots$. By Corollary 2.5.1, $\|M^nH\|\to 0$. Therefore $\bar{l}_1=\bar{l}_2$ and $d(M^nl_1, \bar{l}_1)\to 0$, $d(M^nl_2, \bar{l}_2)\to 0$ when $n\to\infty$. Let $l=\bar{l}_1=\bar{l}_2$. l is obviously the unique invariant curve of map M, i. e. $Ml=l$ and for any $z_0\in R^2$, $d(M^nz_0, \rho)\to 0$, when $n\to\infty$. The proof is completed.

For $\rho>\bar{\rho}$, let l_ρ: $y=h_\rho(x)$ be the unique k-horizontal curve, invariant under the Poincare map M_ρ. Now we consider the relationship between l_ρ and ρ.

Theorem 2.2. *l_ρ depends continuously on ρ, i. e. $d(l_\rho, l_{\rho_0})\to 0$ when $\rho\to\rho_0$.*

Proof Let l_1: $y=\beta x+\beta|\varepsilon|+\frac{\rho+b+1}{\beta}$, l_2: $y=\beta x-\beta|\varepsilon|+\frac{\rho-b-1}{\beta}$. According to the proof of Theorem 2.1, for any $\sigma>0$, $\exists N$ such that $d(M^N_{\rho_0}l_i, l_{\rho_0})<\sigma$ $(i=1, 2)$. Since M^N_ρ depends continuously on ρ and the initial point (x, y)[7], there exists a $\delta>0$, such that when $|\rho-\rho_0|<\delta$, $d(M^N_\rho l_i, M^N_{\rho_0}l_i)<\sigma$ $(i=1, 2)$. Therefore $d(M^N_\rho l_i, l_{\rho_0})<2\sigma$, when $|\rho-\rho_0|<\delta$. Since l_ρ is between $M^N_\rho l_1$ and $M^N_\rho l_2$, we have $d(l_\rho, l_{\rho_0})<2\sigma$, when $|\rho-\rho_0|<\delta$. We have proved Theorem 2.2.

§ 3. Dynamical Behavior of the System and Characteristic of the ρ–V Curve

In this section, we always assume $\rho>\bar{\rho}$. Let $z_0=(x, y_0)\in R^2$ and $z(t, z_0, t_0)=$

$(x(t, z_0, t_0), y(t, z_0, t_0))$ be the solution of (2.1) with initial condition $t=t_0$, $x=x_0$, $y=y_0$. If $\lim\limits_{t\to+\infty}\frac{x(t, x_0, y_0, 0)}{t}$ exists, then we define

$$V(x_0, y_0, \rho)=\frac{T}{2\pi}\lim_{t\to+\infty}\frac{x(t, x_0, y_0, 0)}{t}$$

as the average voltage and denote $V(\rho)=\{V(x_0, y_0, \rho)\mid(x_0, y_0)\in R^2, V(x_0, y_0, \rho)$ exists$\}$.

Consider the $\rho-V(x_0, y_0, \rho)$ $(\rho-V)$ curve. The horizontal segement $V(x_0, y_0, \rho)=\frac{q}{p}$ for any ρ, $|\rho-\rho_0|<\delta$, where ρ_0 is fixed, is called a staircase of (p, q) type.

In the following, we will prove that $V(x_0, y_0, \rho)$ exists for any $(x_0, y_0)\in R^n$ and is independent of (x_0, y_0). So $V(\rho)$ is a single value function of ρ. We shall prove that $V(\rho)$ is a continuous function. The dynamical property of the system is determined by $V(\rho)$: when $V(\rho)$ is a rational number, the attractor is composed of periodic solutions; when $V(\rho)$ is an irrational number, the attractor is comopsed of quasi-periodic motions. There is no chaos.

Let M_ρ be the Poincare map of system (2.1), l_ρ: $y=h_\rho(x)$ $(\rho>\bar{\rho})$ the unique k-horizontal curve invariant under map M_ρ.

Definition 3.1. *E_ρ: $R^1\to R^1$ is the map $x\to\Pi_x M_\rho(x, h_\rho(x))$ where Π_x is the projection operator toward x-axis.*

Lemma 3.1. *$F_\rho(x+2\pi)=F_\rho(x)+2\pi$ and F_ρ is a single value strictly increasing continuous function of x.*

Definition 3.2. *Let $\bar{F}_\rho$: $R^1\to R^1$ be the map $x\to\frac{1}{2\pi}F_\rho(2\pi x)$ and f^*_ρ: $S^1\to S^1$, $e^{2\pi ix}\to e^{2\pi i\bar{F}_\rho(x)}$, $0\leqslant x\leqslant 1$.*

Obviously, $\bar{F}_\rho(x+1)=\bar{F}_\rho(x)+1$, and $\bar{F}_\rho$ is a single value strictly increasing continuous function. f^*_ρ ie called an orientation preserving homeomorphism and $\bar{F}_\rho$ is called a lift of f^*_ρ[8, 9].

Theorem 3.1. *For any $x\in R^1$. $V(x, h(x), \rho)$ exists and $V(x, h(x), \rho)=\lim\limits_{n\to\infty}\frac{\bar{F}^n\left(\frac{x}{2\pi}\right)}{n}$. This impiles that $V(x, h(x), \rho)$ is indepenent of x.*

Lemma 3.2. *For any $\rho\geqslant 0$, $\left|\frac{F^n_\rho(0)}{2\pi n}-\frac{F^m_\rho(0)}{2\pi n}\right|\leqslant\frac{1}{n}+\frac{1}{m}$, $n, m\in N$.*

Theorem 3.2. *For any $(x_0, y_0)\in S$, $V(x_0, y_0, \rho)$ exists and is independent of (x_0, y_0).*

Theorem 3.3. *$V(\rho)$ is a continuous function of ρ $(\rho>\bar{\rho})$.*

Theorem 3.4. *If ρ makes $V(\rho)=\frac{p}{q}$ a rational number, (p, q are relatively prime), then f^*_ρ has only p-periodic points and M_ρ has only (p, q) periodic points, i. e. for some $(x, h_\rho(x))$, $M^q_\rho(x, h_\rho(x))=(x, h_\rho(x))+(2\pi q, 2\pi\beta q)$. For thhs kind of ρ,* *system (2.1) has its attractor composed of (p, q) subharmonics.*

Theorem 3.5. *If $V(\rho)$ is an irrational number, $\Omega(f_\rho^*)$ is either a nowhere dense complete set of S^1 or $\Omega(f_\rho^*)=S^1$, there do not exist periodic points of any kind for M_ρ. System (2.1) has its attractor composed of quasi-periodic motions.*

By the above theorems, we know for $\rho>\bar{\rho}$, $\beta>0$, that system (2.1) behaves either periodically or quasi-periodically; hence the absence of chaotic motion in this case.

Finally, we shall estimate the range of ρ for which a staircase of (p, q) type exists.

Theorem 3.6. *A staircase of (p, q) type can only exists for $\frac{q}{p}\beta-1\leqslant p\leqslant\frac{q}{p}\beta+1$, where $T=2\pi$.*

Remark The proofs of the lemmas and theorems in this section are all omitted. Readers can refer to [5].

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