Chin. Ann. of Math.
12B: 2(1991), 147—151.

ON THE FINITE GROUP WITH A T. I. SYLOW P-SUBGROUP

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Abstract

This paper studies the relations between T. I. conditions and cyclic conditions on the Sylow p-subgroups of a finite group G. As examples, the following two results are proved.

1. Let G be a finite group with a T. I. Sylow p-subgroup P. If $p=3$ or 5, we suppose G contains no composition factors isomorphic to the simple group $L_2(2^3)$ or $Sz(2^5)$ respectively. If G has a normal subgroup W such that $p|(|W|, |G/W|)$, then G is p-solvable.

2. Let G be a finite group with a T. I. Sylow p-subgroup P. Suppose $p>11$ and P is not normal in G. Then P is cyclic if and only if G has no composition factors $L_2(p^n)(n>1)$ and $U_3(p^m)(m\geqslant 1)$.

The finite groups with cyclic Sylow p-subgroups is a very important class of finite groups. It contains all simple groups of finite order. The following theorem plays a fundamental role in the theory of finite groups with cyclic Sylow p-subgroups.

Theorem 1. *Let G be a finite group with cyclic Sylow p-subgroups. If there exists a normal subgroup N in G such that $p|(|N|, |G/N|)$, then G is p-solvable.*

W. Feit first proved this theorem[3], employing the abstract group theoretic method. R. Brauer gave another elegant proof with the modular representation theory. In 1985, Blau[2] proved that the cyclic Sylow subgroups of any simple finite groups are T. I. sets. If a non-cyclic Sylow p-subgroup P of a finite group is a T. I. set, then the structure of G is much restricted as shown in [4]. In this paper, we will study when a T. I. Sylow p-subgroup is cyclic.

In this paper, all groups are assumed to be finite. The notation throughout this paper is standard and follows that of [3] and [7].

Lemma 2. *Let G be a finite group with a T. I. set H. If $M\leqslant G$, then $H\cap M$ is a T. I. set of M.*

Proof Let $x\in M$. If $(H\cap M)^x\cap(H\cap M)\neq 1$, then $H^x\cap H\neq 1$. Since H is a T. I. set of G, $H^x=H$, $(H\cap M)^x=H^x\cap M^x=H\cap M$. So $H\cap M$ is a T. I. set of M.

Manuscript received March 14, 1988.

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Lemma 3. *Suppose H is a T. I. subgroup of a finite group G. Let N be a normal subgroup of G such that $(|N|, |H|)=1$, Then HN/N is a T. I. set of G/N.*

Proof Set $\bar{G}=G/N$. If $\bar{H}^x \cap \bar{H} \neq 1$ for an element $x \in G$, then there exist h_1, $h_2 \in H$ such that $h_1^x N = h_2 N$. $\langle h_1^x \rangle N = \langle h_2 \rangle N$. Since $(|H|, |N|)=1$, there exists $y \in N$ such that $\langle h_1^x \rangle = \langle h_2^y \rangle$, $h_1^x = (h_2^y)^i$ for some integer i. $H^{xy^{-1}} \cap H \geq \langle h_2^i \rangle \neq 1$, $xy^{-1} \in N_G(H)$, $\bar{x} \in N_{\bar{G}}(\bar{H})$. Hence $\bar{H}=HN/N$ is a T. I. set in $\bar{G}$.

Theorem 4. *Let G be a finite group with a T. I. Sylow p-subgroup P. If $p=3$ or 5, we assume that G contains no compositions factor isomorphic to the simple group $L_2(2^3)$ or $Sz\ (2^5)$ respectively. If there exists a normal subgroup W in G such that $p \mid (|W|, |G/W|)$, then G is p-solvable.*

Proof Suppose the theorem is not true and let G be a minimal counterexample. Then

(I). $O_{p'}(G)=1$. If $O_{p'}(G) \neq 1$, by Lemma 3, it is easy to see that $G/O_{p'}(G)$ satisfies the conditions in the theorem. By the minimum of G, $G/O_{p'}(G)$ is p-solvable, so is G, contrary to the assumption on G.

(II). Let H be a minimal normal subgroup of G such that $H \leq W$. Then H is simple and $p \mid |H|$.

By (I), $p \mid |H|$. If H is p-solvable, by (I), $O_p(H) \neq 1$, $O_p(G) \neq 1$. Since P is a T. I. set of G, P is normal in G. G is of course p-solvable; It is absurd. So H is not p-solvable. If H is not simple, then $H=S_1 \times S_2 \times \cdots \times S_m$, where S_i is isomorphic to S_1 for $i=1, 2, \cdots, m$, S_1 is non-abelian simple, $m>1$. Let P_i be a Sylow p-subgroup of S_i such that $P_i \leq P \cap H \in \mathrm{Syl}_p(H)$. By Lemma 2, $P \cap H$ is a T. I. set of H. Then $N_H(P_1) \leq N_H(P \cap H)$. Clearly, $S_2 \leq N_H(P_1)$. Then $S_2 \leq N_H(P_2)$, P_2 is normal in S_2, contrary to the simplicity of S_2. Therefore, H, is simple.

(III). $G=PH$. By Frattini argument, $G=N_G(P \cap H)H$. Since P is a T. I. set of G, $N_G(P \cap H) \leq N_G(P)$, PH is normal in G. It is easy to see that PH satisfies the conditions of Theorem 4. If PH is a proper subgroup of G, by the minimum of G, PH is p-solvable, so is H, contrary to the simplicity of H. Hence $G=PH$.

(IV). $G/H \leq \mathrm{Out}(H)$. If there exists an element $x \in P$ such that $x \notin H$ and x induces an inner automorphism of H, then there exists a p-element $y \in H$ such that $xy \in C_G(H)$. Clearly, $(P \cap H)^y = P \cap H$, $(P \cap H)^x = P \cap H$. Since $P \cap H$ is a Sylow p-subgroup of H, $y \in P \cap H$. $xy \in P^h \cap P$ for any $h \in H$. Since P is a T. I. set, $h \in N_G(P)$. So P is normal in $PH=G$, $P \cap H$ is normal in H, contrary to the simplicity of H.

(V). Last contradiction. By Theorem 1, P is not cyclic. Clearly $N_G(P)$ is a strongly p-embedded subgroup of G. By [5, Theorems 4.29 and 4.249], we have

(a). $p=2$, $H \approx L_2(2^n)$, $n \geq 2$, $Sz(2^{2n+1})$, $n \geq 1$ or $U_3(2^n)$, $n \geq 2$.

(b). $p\neq 2$, $H\approx L_2(p^n)$, $U_3(p^n)$ or A_{2p}.

(c). $p=3$, $H\approx L_3(4)$, ${}^2G_2(3^n)$, M_{11} or $G\approx \mathrm{Aut}(L_2(2^3))$.

(d). $p=5$, $H\approx {}^2F_4(2)'$, Mc, $M(22)$ or $G\approx \mathrm{Aut}\ (Sz(32))$.

(e). $p=11$, $H\approx J_4$.

It is well-known that $|\mathrm{Out}\ (S)|\,|4$ for any sporadic simple groups and alternative groups S. So H is not a sporadic simple group or alternative group. If H is a simple group of Lie type of characteristic p, then H has only two p-blocks, one is the principal p-block, the other of defect zero. So there exists only one irreducible ordinary chraacter X of H such that $p^a|X(1)$, where p^a is the order of $P\cap H$. By [8], G has a p-block of defect zero. Let Y be an irreducible ordinary character of G such that $|P|\,|Y(1)$. By [6, Corollaries 11.29 and 6.19], $Y|_H=e\sum_{i=1}^{t}X_i$, where $t>1$, $et\,||G:H|$, $X_i\neq X_j$ $X_i(1)=X_j(1)$, $i\neq j$. $Y(1)=etX_i(1)$, $p^a|X_i(1)$, $i=1, 2, \cdots, t$. By the property of H, $X_i=X_j$, $t=1$, a contradiction. By the assmuption on G, we have $p=3$ and $H\approx L_3(4)$ or $p=5$ and $H\approx {}^2F_4(2)'$. But the order of the out automorphism group of ${}^2F_4(2)'$ is 2, it leads to that $p=3$. So $G\leqslant \mathrm{Aut}(L_3(4))$. By the character table of $L_3(4)$ (see [7]), the Sylow 3-subgroup of G is not a T. I. set, contrary to the assumption on G. The contradiction proves the theorem.

Corollary 5. *Let G be a finite group with a T. I, Sylow p-subgroup P. Suppose $p\neq 3$, 5. If there exists a normal subgroup N of G such that $p|(|N|, |G/N|)$, then G is p-solvable.*

Corollary 6. *Let G be a finite group with a T. I. Sylow p-subgroup P. Suppose there exists a normal subgroup N in G such that $p|(|N|, |G/N|)$. If $|P|\neq p^3$, then G is p-solvable.*

Proof Suppose the corollary is not true. Let G be a minimal counterexample. Mimic the proof of Theorem 4. Then $G=PN$ and N is a simple group. By Theorem 4, we have $p=3$ and $N\approx L_2(2^3)$ or $p=5$ and $N\approx Sz(2^5)$. Since $G/N\leqslant \mathrm{Out}(N)$, $G\leqslant \mathrm{Aut}(N)$. The Sylow 3-subgroup of $\mathrm{Aut}(L_2(2^3))$ is of order 27 and the Sylow 5-subgroup of $\mathrm{Aut}(Sz(2^5))$ is of order 5^3, so P is of order p^3, contrary to the assumption on G. Hence G is p-solvable.

Theorem 7. *Let G be a p-solvable group with a non-normal T. I. Sylow p-subgroup P. If $p\neq 2$, then P is cyclic. If $p=2$ and G is quaternion-free, then P is cyclic.*

Proof Since G is p-solvable and P is a T. I. set, by Lemma 3, $PO_{p'}(G)/O_{p'}(G)$ is a T. I. set in $G/O_{p'}(G)$, $PO_{p'}(G)$ is a normal subgroup of G. Let P_i be a Sylow p_i-subgroup of $O_{p'}(G)$ such that $P\leqslant N_G(P_i)$ and $P_1P_2\cdots P_m=O_{p'}(G)$. If $O_p(PP_i)\neq 1$ for $i=1, 2, \cdots, m$, since P is a T. I. set in PP_i, then P is normal in PP_i. It follows that P is normal in $PO_{p'}(G)$. Then P is normal in G, contrary to the assumption on P. So there exists a j such that $O_p(PP_j)=1$. By Lemma 3, $PP_j/\mathrm{Frt}(P_j)$ has a T. I. Sylow p-subgroup $\bar{P}$, where $\mathrm{Frt}(P_j)$ is the Frattini subgroup of P_j. If $O_p(\bar{P}\bar{P}_j)\neq 1$, then $\bar{P}$ is normal in $\bar{P}\bar{P}_j$, P acts trivially on $\bar{P}_j$, so does on P_j. It is absurd. If $PP_j\nleqslant G$ or $\mathrm{Frt}(P_j)\neq 1$, by induction, P is cyclic. If $PP_j=G$ and $\mathrm{Frt}(P_j)=1$, P_j is elementary abelian, $P_j=A\times B$, where A and B are P-invariant subgroup such that $PA=P\times A$ and P acts on B with no fixed points. Clearly, $B\neq 1$, PB is a Frobenius group with kernel B. By the assumption on G, we see that P is cyclic.

Theorem 8. *Let G be a finite group with a T. I. Sylow p-subgroup P. Suppose $p>11$ and P is not normal in G. Then P is cyclic if and only if G contains no composition factors $L_2(p^n)$ $(n>1)$ and $U_3(p^m)$ $(m\geqslant 1)$.*

Proof We only need to prove the "if" part. If the result is not true, let G be a minimal counterexample. Suppose N is a minimal normal subgroup of G. If $p\mid(|N|, |G/N|)$, by Theorem 4, G is p-solvable. By Theorem 7, P is cyclic, contrary to the assumption on G. So $p\nmid(|N|, |G/N|)$. If $p\nmid|N|$, then $\bar{G}=G/N$ share the same properties as G. By the minimum assumption on G, $\bar{G}$ has a cyclic Sylow p-subgroup, so does G, contrary to the assumption. Therefore $p\mid|N|$, $p\nmid|G/N|$. N is simple as shown in the proof of Theorem 4. By the minimum of G, $G=N$. Since $p>11$, by [4], $N\approx L_2(p^n)$ or $U_3(p^m)$ ($L_2(p)$ has a cyclic Sylow p-subgroup, so $n\geqslant 2$), contrary to the assumption on G. The contradiction proves the theorem.

Corollary 9. *Let G be a finite group with a T. I. Sylow p-subgroup P. Suppose $p>11$ and G is $L_2(p)$-free. Then P is either cyclic or normal in G.*

Proof Since $L_2(p)$ is a section of $L_2(p^n)$ and $U_3(p^m)$, the corollary follows Theorem 8 immediately.

Theorem 10. *Let G be a finite group with a nonnormal T. I. Sylow p-subgroup P. Suppose $p>11$. Then P is abelian if and only if G has no composition factors $U_3(p^m)$ $(m\geqslant 1)$.*

Proof Mimic the proof of Theorem 8.

The author would like to thank Professor Tuan Hsiofu and professor Shi Sheng Ming for their helpful comments.

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