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ON THE QUOTIENT RING OF COMMUTATIVE RINGS WITH ACC$^{\perp}$ON ANNIHILATOR IDEALS

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Abstract

The author concludes that every commutative ring with ascending chain condition on annihilator ideals has a Kasch quotient ring, which generalizes the Theorem[1] that every commutative noetherian ring has a Kasch quosient ring. If follows that if R is a commutative ring with $acc^{\perp}$, then that $Q(R)$ is semiprimary is equivalent to that it is perfect, or to that R satisfies regular condition. Besides, that $Q(R)$ is quasi-frobenius equals that $Q(R)$ is FPF or PF, and that $Q(R)$ is artinian equals that R/N_i are of finite dimension, $i=1, 2, \cdots, n$. $N_i=J^i\cap R$.

In this paper we shall study the commutative rings with ascending chain condition on annihilator ideals. We conclude the following: If R is an above ring, then the classical quotient ring of R is Kasch and semilocal, which generalizes the theorem in [1] that says every commutative noetherian ring has a Kasch classical quotient ring. It follows that we can drop off the condition that R must have finite dimension in well-known Goldile theorem if R is a commutativo ring. In the final section we shall discuss when the classical quotient ring of R is artinian, perfect and quasi-frobenius. Throughout this paper R is a commutative ring with a regular element. $Q(R)$ denotes the classical quotient ring of R. $ACC^{\perp}$ means ascending chain condition on annihilator ideals.

Lemma 1. *$Q(R)$ is Kasch if and only if every faithful ideal of R contains a regular element of R.*

Proof We say a ring is Kasch if every dense (right) ideal is R itself. Here R is commutative, thus dense ideal means faithful ideal. Assume $Q(R)$ is Kasch, $\mathfrak{A}$ is a faithful ideal of R, $\mathfrak{A}Q(R)=Q(R)$ by assumption, thus $1=\sum_{i=1}^{n}a_iq_i$ where $a_i\in\mathfrak{A}$, $q_i\in Q(R)$. There exists a regular element s of R such that $q_is\in R$ and $s=\sum_{i=1}^{n}a_iq_is$, $i=1, 2, \cdots, n$. $s\in\mathfrak{A}$.

Conversely, for any faithful ideal $\mathfrak{B}$ of $Q(R)$, $R\cap\mathfrak{B}$ is a faithful ideal of R since $(R\cap\mathfrak{B})Q(R)=\mathfrak{B}$. There exists a regular element s of $R\cap\mathfrak{B}$. So $\mathfrak{B}$ contains a

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unit of $Q(R)$ and $\mathfrak{B}=Q(R)$. Thus $Q(R)$ is a Kasch ring.

Theorem 2. *If R satisfies $acc^{\perp}$, $Q(R)$ is a Kasch ring, and if R has a unit, $Q(R)=Q_{\max}(R)$.*

Proof Let Mass (R) be the set $\{P\in \mathrm{Spec}(R)\mid P$ is maximal annihilator prime ideal$\}$. Mass (R) is not empty since R satisfies $acc^{\perp}$. We claim that Mass(R) is a finite set. Let $U=\{$all finite intersections of elements of Mass $(R)\}$. Since every element of U is still an annihilator ideal, and R has $acc^{\perp}$ on annihilator ideal, there exsits a minimal member of U. We say, it is $P_1\cap P_2\cap\cdots\cap P_n$ for some integer n. Thus for any $P\in$ Mass(R), $P\cap P_1\cap P_2\cap\cdots\cap P_n=P_1\cap P_2\cap\cdots\cap P_n$. by our choice. It implies $P_1\cap\cdots\cap P_n\subseteq P$ and $P_1P_2\cdots P_n\subseteq P$. There exists $P_i\subseteq P$ for some i, but P_i is maximal w. r. t annihilator, so $P_i=P$. Thus we have proved that Mass (R) $=\{P_1, P_2, \cdots, P_n\}$ is a finite set. If $\mathfrak{A}$ is a faithful ideal of R, $\mathfrak{A}\not\subseteq\bigcup_{i=1}^{n}P_i$. Otherwise, $\mathfrak{A}\subseteq P_i$ for some i [1], and $\mathfrak{A}^{\perp}\supseteq P_i\neq 0$ which is a contradiction. Let $s\in\mathfrak{A}\setminus\bigcap_{i=1}^{n}P_i$. Then s is a regular element of R. If not so, there exists $b\in R$ such that $sb=0$ and $b\neq 0$. Thus $s\in(b)^{\perp}$. There exists a $P\in$ Mass(R) such that $P\supseteq(b)^{\perp}$ since R has $acc^{\perp}$, so s belongs to $P\subseteq\bigcup_{i=1}^{n}P_i$, which is impossible. Therefore every faithful ideal of R contains a regular element of R, $Q(R)$ is Kasch by Lemma 1. If R has a unit, $Q_{\max}(R)$ exists and $Q(R)$ has no proper rational extension, so $Q(R)=Q_{\max}(R)$.

Corollary 3. *Every commutative noetherian ring or Goldile ring is an order of a Kasch ring.*

Corollary 4. *Let R be a ring with $acc^{\perp}$. $N=N(R)$ is the nil-radical of R. Then $Q(R/N)$ is semisimlpe artinian.*

Proof Since R has $acc^{\perp}$, $Z(R)=N(R)$ [2] is a nilpotent and annihilator ideal. Thus R/N also satisfies $acc^{\perp}$ since for every ideal of $(R/N)\mathfrak{A}/N$, $(\mathfrak{A}/N)^{\perp}=(N:\mathfrak{A})/N=(N^{\perp}\mathfrak{A})^{\perp}/N$. But R/N is semiprime, hence nonsingular[2]. By the previous proof every essential idael of R/N contains a regulrr element. Therefore $Q(R/N)$ is semisimple artinian by Goldile's theorem.

Corollary 5. *Every semiprime ring with $acc^{\perp}$ is an order of a semisimple artinian ring.*

In the following, we shall discuss when $Q(R)$ is artinian, or quasi-frobenius. We say ring R satisfies regular condition if $C(N)\subseteq C(0)$, where $C(N)$ is the set of all regular elements of R modulo N.

Lemma 6. *If R has $acc^{\perp}$, then $Q(R)$ is a semilocal ring, that is, $Q(R)/J(Q(R))$ is semisimple artinian.*

Proof First, $Q(R)$ also satisfies $acc^{\perp}$. Let $(S_1)^{\perp}\subseteq(S_2)^{\perp}\subseteq\cdots\subseteq(S_n)^{\perp}\subseteq\cdots$ be an ascending chain of annihilator ideal of $Q(R)$. Then S_i's can be taken as ideal's of

R. For $(S_i)^\perp=(S_iQ(R))^\perp$ $(S_iQ(R)\cap R)^\perp$, we have another ascending chain of annihilator ideals of R; $\mathrm{ann}_R(S_1)\sqsubseteq\mathrm{ann}_R(S_2)\sqsubseteq\cdots\sqsubseteq\mathrm{ann}_R(S_n)\sqsubseteq\cdots$. Since $\mathrm{ann}_R(S_i)=\mathrm{ann}_Q(S_i)\cap R$. There exists an n such that $\mathrm{ann}_R(S_i)=\mathrm{ann}_R(S_n)$ for any $i\geqslant n$. $\mathrm{ann}_Q(S_i)=\mathrm{ann}_Q(S_n)$ for any $i\geqslant n$ since $\mathrm{ann}_R(S_i)Q(R)=\mathrm{ann}_Q(S_1)$. That is, the first ascending chain is stable. Now $Q(R)$ is Kasch, hence the Jacobson radical of $Q(R)$ is $J=(\mathrm{Soc}(Q))^\perp$. $J=(\{x_1, x_2, \cdots, x_n\})^\perp$ since $Q(R)$ satisfies $acc^\perp$ w. r. t. annihilator ideals. Thus $J=(x_1)^\perp\cap(x_2)^\perp\cap\cdots\cap(x_n)^\perp$, and there exists a monomorphism $Q(R)/J\hookrightarrow x_1Q+x_2Q+\cdots+x_nQ$, where x_i's are in Soc (Q). Since $\mathrm{Soc}(Q)$ is semisimple $Q(R)$-module, $Q(R)/J$ is semisimple artinian.

Theorem 7. *If R satisfies acc, the following are equivalent.*

(a) $Q(R)$ *is semiprimary.*

(b) $Q(R)$ *is perfect.*

(c) *Soc* (Q) *is essential.*

(d) R *satisfies regular condition.*

(e) $Q(R)$ *satisfies regular condition.*

(f) $Q(R)/\overline{N}$ *is regular or selfinjective,* $\overline{N}=N(Q)$.

(g) Krull-dim$(Q)=0$.

(h) *every non-faithful prime ideal of R is maximal annihilator ideal.*

Proof (a)⇒(b)⇒(c). Clearly (cf. [4]).

(c)⇒(a). Siince Q is Kasch, $J(Q)=(\mathrm{Soc}(Q))^\perp=Z(Q)$. $J(Q)=\overline{N}(Q)$ is nilpotent, and $Q(R)$ is semiprimary.

(a)⇒(f), Obviously.

(f)⇒(e). It suffices to show it for the first case, for if $Q(R)/\overline{N}$ is selfinjective, then $Q_{\max}(Q/\overline{N})=Q/\overline{N}$ is semisimple by Corollary 4. For any $c\in C(\overline{N})$, c is a unit of $Q/\overline{N}$. There exists a b of Q such that $cb-1\in \overline{N}\sqsubseteq J$. cb is a unit, hence c is a unit of Q. $C(\overline{N})=C(0)$ is obvious.

(e)⇒(d). Let N be the nil-radical of R, for any $x\in C(N)$, $x\in C(\overline{N})\cap R$, so $x\in C(0)$. Clearly $C(N)=C(0)$.

(d)⇒(a). Since $Q(R)/\overline{N}$ also satisfies $acc^\perp$, and is of finite rank by Corollary 4, $Q(R)$ is semiprimary by [1].

Finally, $Q(R)/\overline{N}$ is regular iff $Q(R)/P$ is regular for each prime ideal of $Q(R)$ since $Q(R)/\overline{N}$ is semiprime[5] iff every prime ideal of $Q(R)$ is maximal ideal iff kruli-dimension of $Q(R)$ is 0.

(g)⇔(h). This is because there is lattice isomorphism between $\mathrm{Spec}(Q(R))$ and the set $\{P\in\mathrm{Spec}(R)\mid P$ does not contain regular elements$\}$. That is, every non-faithful prime ideal of R is maximal annihilator ideal.

Theorem 8. *If R satisfies $acc^\perp$ and (R) is perfect, the following are equivalent.*

(a) $Q(R)$ *is artinian.*

(b) *R/T_k is a Goldile ring, where $T_k=(N(R)^k)^{\perp}\cap N(R)$, $k=0, 1, 2, \cdots, n-2$, n is the nilpotent index of $N(R)$.*

(c) *R/N_i is of finite dimension, where $N_i=J^i\cap R$, $i=2, 3, \cdots, n$, $J=J(Q)$.*

(d) *The reduced rank $\rho(R)$ of R is finite, i. e., $\sum_{i=1}^{n}\rho(N^{i-1}/N^i)$ is finite.*

Proof (a)$\Leftrightarrow$(d). Cf. [6], Theorem (4).

(a)$\Leftrightarrow$(c). Since Q/J is semisimple, J^{i-1}/J^i is semisimple module, $i=1, 2, \cdots, n$, $J^n=0$. Thus Soc $(Q/J^i)\supseteq J^{i-1}/J^i$. Now Q/J^i is of finite dimension since R/N_i has finite dimension and $R/N_i\cong(R+J^i)/J^i$ is essential in Q/J^i as a $(R+J^i)/J^i$ module, $i=1, 2, \cdots, n$. Therefore J^{i-1}/J^i is finite generated, and $Q(R)$ is artinian.

(a)$\Leftrightarrow$(b). Since $Q(R)$ is perfect, R satisfies regularity condition by Theorem 7. On the other hand, $N^n=0$, $R/T_{n-1}=R/T_n=R/N$ is a Goldile ring by Corollary 4. Therefore, R is a T-Goldile ring, and $Q(R)$ is artinian by [9] and [10] (Theorem C).

In the following, a ring is called PF ring if every faithful module in Mod-R is a generator. R. is (right) FPF if every (right) finitelly generated faithful module is a generator in Mod-R.

Theorem 9. *If R satisfies acc$^{\perp}$, the following are equivalent.*

(A) *$Q(R)$ is quasi-frobenius.*

(B) *$Q(R)$ is PF.*

(C) *$Q(R)$ is FPF.*

(D) *$Q(R)$ is a cogenerator in* Mod-$Q(R)$.

(E) *$Q(R)$ is selfinjective.*

If R has a unit, $Q(R)=Q_{\max}(R)$. (A) is equivalent to the following.

(F) *For each ideal $\mathfrak{A}\in\mathscr{H}$ of R, $\mathfrak{A}^{\perp}=0$, where $\mathscr{H}=\{$ideal $\mathfrak{A}\mid\exists f\in$ Hom$(\mathfrak{A}, R)$ such that f has no extension$\}$.*

(G) *For any ideal $\mathfrak{A}$ of R and homomorphism α: $\mathfrak{A}\rightarrow R$ there exists a faithful ideal $\mathfrak{B}\supseteq\mathfrak{A}$ and homomorphism β: $\mathfrak{B}\rightarrow R$ such that $\beta|_{\mathfrak{A}}=\alpha$.*

Proof (A)$\Rightarrow$(B)$\Rightarrow$(C). Clearly.

(C)$\Rightarrow$(D). Since $Q(R)$ is a commutative FPF ring, $Q(Q)=Q$ is selfinjective[7], and Q is Kasch, Q is PF[3], hence Q is a cogenerator in Mod-Q[3].

(D)$\Rightarrow$(E). Since Q/J is semisimple by Theorem 6 and Kasch, there are only finite non-isomorphic classes of simple Q-modules in Mod-Q. Let $S_1, S_2, \cdots, S_n$ be the representatives of n non-isomorphic classes, $S_i\sqsubseteq Q$ (by assumption). $E=E(S_i)$ $\leqslant Q$ since E_i is indecomposable, $i=1, 2, \cdots, n$. Thus $Q=E(S_1\oplus S_2\oplus\cdots\oplus S_n)\oplus X=E(S_1)\oplus\cdots\oplus E(S_n)\oplus X$ for some module X. Let $E=E(S_1)\oplus\cdots\oplus E(S_n)$. E is projective and injective. Since $E(S_i)$ is indecomposable, $R_i=\mathrm{End}_Q(E_i)$ is local, $i=1, \cdots, n$. By [8] propositions 22.5 and 22.6, $R_i/J(R_i)=\mathrm{End}(E_i/E_iJ)$, where J

$=J(Q)$, $i=1, 2, \cdots, n$. Since E_i/E_iJ is a semisimple Q/J-module, hence a semisimple Q-module whose endomorphism ring is division, E_i/E_iJ must be simple module. Since E_i is f, g, and $E_i/E_iJ \cong E_j/E_jJ$ iff $E_i \cong E_j$ iff $S_i \cong S_j$ iff $i=j$. E_i/E_iJ is also a non-isomorphic simple module. So each simple Q-module is an epic-imagerof E, and E is projective, E is a generator in Mod-Q by Azumaya's theorem. There is an integer n such that $E(n)=Q\oplus M$, for some module M, E is injective, therforee Q is selfinjective.

(E)$\Rightarrow$(A). Since R satisfies $acc^{\perp}$, (A) holds by [1].

If R has a unit, $Q_{\max}(R)$ exists, $Q=Q_{\max}(R)$ by Theorem 2. $Q_{\max}(R)$ is selfinjective iff (F) holds iff (G) holds [1].

Corollary 10. *If R is FPF ring with $acc_{\perp}$, then $Q(R)$ is quasifrobenius.*

Proof Since $Q(R)$ is also FPF, $Q(R)$ is quasi-frobenius by Theorem 9.

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